JEE - 2025

MATHS - MODULES

THEORY

EXERCISE

SOLUTIONS

- Useful for JEE MAINS and ADVANCED Exams
- Each topic contains Detailed Theory with images
- Every topic contains Exercises and Detailed solutions

CIRCLE

DEFINITION

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point (in the same given plane) remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

EQUATION OF A CIRCLE

The curve traced by the moving point is called its circumference i.e. the equation of any circle is satisfied by co-ordinates of all points on its circumference.

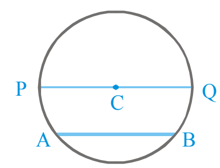
or

The equation of the circle means the equation of its circumference.

or

It is the set of all points lying on the circumference of the circle.

Chord and diameter - the line joining any two points on the circumference is called a chord. If any chord passing through its centre is called its diameter.



AB = chord, PQ = diameter

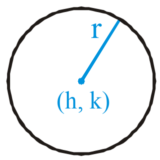
C = centre

Standard Equations of the Circle

(a) Central Form :

If (h, k) is the centre and r is the radius of the circle then its equation is

$(x-h)^2 + (y-k)^2 = r^2$

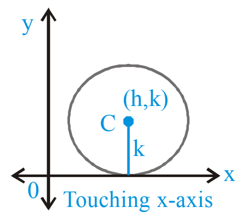


Special Cases

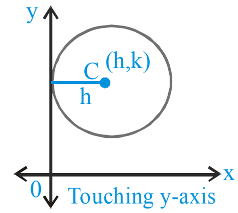
(i) If centre is origin (0,0) and radius is 'r' then equation of circle is $x^2 + y^2 = r^2$ and this is called the standard form.

(ii) If radius of circle is zero then equation of circle is $(x - h)^2 + (y - k)^2 = 0$. Such circle is called zero circle or **point circle.**

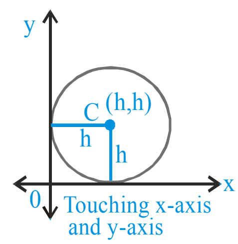
(iii) When circle touches x-axis then equation of the circle is $(x-h)^2 + (y-k)^2 = k^2$.



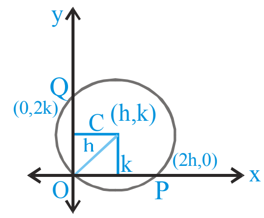
(iv) When circle touches y-axis then equation of circle is $(x-h)^2 + (y-k)^2 = h^2$.



(v) When circle touches both the axes (x-axis and y-axis) then equation of circle $(x-h)^2 + (y-h)^2 = h^2$.



(vi) When circle passes through the origin and centre of the circle is (h,k) then radius $\sqrt{h^2+k^2} = r$ and intercept cut on x-axis OP =2h, and intercept cut on y-axis is OQ = 2k and equation of circle is
$(x-h)^2 + (y-k)^2 = h^2 + k^2$ or $x^2 + y^2 - 2hx - 2ky = 0$



(b) General Equation of Circle

$x^2 + y^2 + 2gx + 2fy + c = 0$. where g,f,c are constants and centre is (–g,–f)

i.e. $\left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$ and radius $r = \sqrt{g^2+f^2-c}$

(i) If $(g^2 + f^2 - c) > 0$, then r is real and positive and the circle is a real circle.

(ii) If $(g^2 + f^2 - c) = 0$, then radius r = 0 and circle is a point circle.

(iii) If $(g^2 + f^2 - c) < 0$, then r is imaginary then circle is also an imaginary circle with real centre.

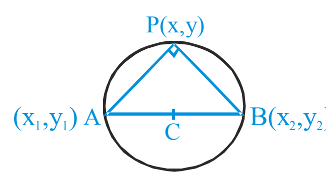
(iv) $x^2 + y^2 + 2gx + 2fy + c = 0$, has three constants and to get the equation of the circle at least three conditions should be known $\Rightarrow$ A unique circle passes through three non collinear points.

(v) The general second degree in x and y, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if :

- coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$
- coefficient of xy = 0 or h = 0
- $(g^2 + f^2 - c) \geq 0$ (for a real circle)

(c) Equation of Circle in Diameter Form :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are the end points of the diameter of the circle and P(x,y) is the point other then A and B on the circle then from geometry we know that $\angle APB = 90°$.



$\Rightarrow$ (Slope of PA) × (Slope of PB) = –1

$\Rightarrow$ $\therefore \left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$

$\Rightarrow$ $(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$

This will be the circle of least radius passing through (x_1, y_1) and (x_2, y_2).

(d) Equation of Circle in Parametric Forms :

(i) The parametric equation of the circle $x^2+y^2 = r^2$ are $x = r\cos\theta, y = r\sin\theta ; \theta \in [0, 2\pi)$ and $(r\cos\theta, r\sin\theta)$ are called the parametric co-ordinates.

(ii) The parametric equation of the circle $(x-h)^2 + (y-k)^2 = r^2$ is $x = h + r\cos\theta, y = k + r\sin\theta$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are $x = -g + \sqrt{g^2+f^2-c}\,\cos\theta$, $y = -f + \sqrt{g^2+f^2-c}\,\sin\theta$ where θ is parameter.

Equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$$x \cos\frac{\alpha+\beta}{2} + y \sin\frac{\alpha+\beta}{2} = a \cos\frac{\alpha-\beta}{2}.$$

Ex. Find the equation of the circle whose centre is (1, –2) and radius is 4.

Sol. The equation of the circle is $(x - 1)^2 + (y - (-2))^2 = 4^2$

$\Rightarrow \quad (x - 1)^2 + (y + 2)^2 = 16$

$\Rightarrow \quad x^2 + y^2 - 2x + 4y - 11 = 0$

Ex. Find the equation of the circle, the coordinates of the end points of whose diameter are (–1, 2) and (4, –3)

Sol. We know that the equation of the circle described on the line segment joining (x_1, y_1) and (x_2, y_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Here, $x_1 = -1, x_2 = 4, y_1 = 2$ and $y_2 = -3$.

So, the equation of the required circle is

$(x + 1)(x - 4) + (y - 2)(y + 3) = 0 \quad \Rightarrow \quad x^2 + y^2 - 3x + y - 10 = 0.$

Ex. If $y = 2x + m$ is a diameter to the circle $x^2 + y^2 + 3x + 4y - 1 = 0$, then find m

Sol. Centre of circle = (–3/2 , –2). This lies on diameter $y = 2x + m$

$\Rightarrow -2 = (-3/2) \times 2 + m \quad \Rightarrow \quad m = 1$

Ex. A circle has radius equal to 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through (7, 3).

Sol. Let the centre of the circle be (α, β). It lies on the line $y = x - 1$

$\Rightarrow \quad \beta = \alpha - 1$. Hence the centre is $(\alpha, \alpha - 1)$.

$\Rightarrow$ The equation of the circle is $(x - \alpha)^2 + (y - \alpha + 1)^2 = 9$

It passes through (7, 3) $\Rightarrow \quad (7 - \alpha)^2 + (4 - \alpha)^2 = 9$

$\Rightarrow \quad 2\alpha^2 - 22\alpha + 56 = 0 \quad \Rightarrow \quad \alpha^2 - 11\alpha + 28 = 0$

$\Rightarrow \quad (\alpha - 4)(\alpha - 7) = 0 \quad \Rightarrow \quad \alpha = 4, 7$

Hence the required equations are

$x^2 + y^2 - 8x - 6y + 16 = 0$ and $x^2 + y^2 - 14x - 12y + 76 = 0$.

Ex. Find the centre & radius of the circle whose equation is $x^2 + y^2 - 4x + 6y + 12 = 0$

Sol. Comparing it with the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$2g = -4 \quad \Rightarrow \quad g = -2$

$2f = 6 \quad \Rightarrow \quad f = 3$

& $\quad c = 12$

$\therefore$ centre is $(-g, -f)$ i.e. (2, –3)

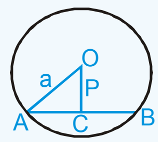
and radius = $\sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (3)^2 - 12} = 1$

INTERCEPTS CUT BY THE CIRCLE ON AXES

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on :

(i) x-axis $= 2\sqrt{g^2 - c}$ **(ii)** y-axis $= 2\sqrt{f^2 - c}$

(i) If the circle cuts the x-axis at two distinct point, then $g^2 - c > 0$
(ii) If the cirlce cuts the y-axis at two distinct point, then $f^2 - c > 0$
(iii) If circle touches x-axis then $g^2 = c$.
(iv) If circle touches y-axis then $f^2 = c$.
(v) Circle lies completely above or below the x-axis then $g^2 < c$.
(vi) Circle lies completely to the right or left to the y-axis, then $f^2 < c$.
(vii) Intercept cut by a line on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ or length of chord of the circle $= 2\sqrt{a^2 - P^2}$ where a is the radius and P is the length of perpendicular from the centre to the chord.



Ex. Find the equation to the circle touching the negative y-axis at a distance 3 from the origin and intercepting a length 8 on the x-axis.

Sol. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Since it touches y-axis at (0, –3) and (0, – 3) lies on the circle.

$\therefore$ $c = f^2$ ...... **(i)**

$9 - 6f + c = 0$ ...... **(ii)**

From **(i)** and **(ii)**, we get $9 - 6f + f^2 = 0$

$\Rightarrow$ $(f - 3)^2 = 0$ $\Rightarrow$ $f = 3$.

Putting f = 3 in **(i)** we obtain c = 9.

It is given that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intercepts length 8 on x-axis

$\therefore$ $2\sqrt{g^2 - c} = 8$ $\Rightarrow$ $2\sqrt{g^2 - 9} = 8$ $\Rightarrow$ $g^2 - 9 = 16$ $\Rightarrow$ $g = \pm 5$

Hence, the required circle is $x^2 + y^2 \pm 10x + 6y + 9 = 0$.

BASIC THEOREMS & RESULTS OF CIRCLES

(a) **Concentric circles :** Circles having same centre. **(b)** **Congruent circles :** Iff their radii are equal.
(c) **Congruent arcs :** Iff they have same degree measure at the centre.

Theorem 1

(i) If two arcs of a circle (or of congruent circles) are congruent, the corresponding chords are equal.

Converse : If two chords of a circle are equal then their corresponding arcs are congruent.

(ii) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

Converse : If the angle subtended by two chords of a circle (or of congruent circles) at the centre are equal, the chords are equal.

Theorem 2

(i) The perpendicular from the centre of a circle to a chord bisects the chord.
Converse : The line joining the mid point of a chord to the centre of a circle is perpendicular to the chord.
(ii) Perpendicular bisectors of two chords of a circle intersect at its centre.

Theorem 3

(i) There is one and only one circle passing through three non collinear points.
(ii) If two circles intersects in two points, then the line joining the centres is perpendicular bisector of common chords.

Theorem 4

(i) Equal chords of a circle (or of congruent circles) are equidistant from the centre.

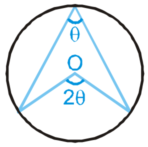
Converse : Chords of a circle (or of congruent circles) which are equidistant from the centre are equal in length.

(ii) If two equal chords are drawn from a point on the circle, then the centre of circle will lie on angle bisector of these two chords.

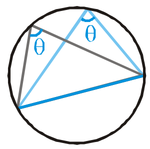
(iii) Of any two chords of a circle larger will be near to centre.

Theorem 5

(i) The degree measure of an arc or angle subtended by an arc at the centre is double the angle subtended by it at any point of alternate segment.

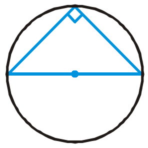


(ii) Angle in the same segment of a circle are equal.



(iii) The angle in a semi circle is right angle.

Converse : The arc of a circle subtending a right angle in alternate segment is semi circle.



Theorem 6

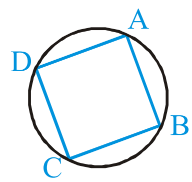
Any angle subtended by a minor arc in the alternate segment is acute and any angle subtended by a major arc in the alternate segment is obtuse.

Theorem 7

If a line segment joining two points subtends equal angles at two other points lying on the same side of the line segment, the four points are concyclic, i.e. lie on the same circle.

(d) Cyclic Quadrilaterals

A quadrilateral is called a cyclic quadrilateral if its all vertices lie on a circle.



Theorem 1

The sum of either pair of opposite angles of a cyclic quadrilateral is 180°

OR

The opposite angles of a cyclic quadrilateral are supplementary.

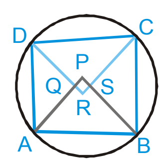
Converse : If the sum of any pair of opposite angle of a quadrilateral is 180°, then the quadrilateral is cyclic.

Theorem 2

If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle.

Theorem 3

The internal angle bisectors of a cyclic quadrilateral form a quadrilateral which is also cyclic.



Theorem 4

If two sides of a cyclic quadrilateral are parallel then the remaining two sides are equal and the diagonals are also equal.

OR

A cyclic trapezium is isosceles and its diagonals are equal.

Converse : If two non-parallel sides of a trapezium are equal, then it is cyclic.

OR

An isosceles trapezium is always cyclic.

Theorem 5

When the opposite sides of cyclic quadrilateral (provided that they are not parallel) are produced, then the exterior angle bisectors intersect at right angle.

POSITION OF A POINT WITH RESPECT TO A CIRCLE

The point (x_1, y_1) is inside, on or outside the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$.
according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c <, =$ or > 0.

The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $|AC - r|$ respectively.

Ex. Discuss the position of the points (1, 2) and (6, 0) with respect to the circle $x^2 + y^2 - 4x + 2y - 11 = 0$

Sol. We have $x^2 + y^2 - 4x + 2y - 11 = 0$ or $S = 0$, where $S = x^2 + y^2 - 4x + 2y - 11$.

For the point (1, 2), we have $S_1 = 1^2 + 2^2 - 4 \times 1 + 2 \times 2 - 11 < 0$

For the point (6, 0), we have $S_2 = 6^2 + 0^2 - 4 \times 6 + 2 \times 0 - 11 > 0$

Hence, the point (1, 2) lies inside the circle and the point (6, 0) lies outside the circle.

LINE AND A CIRCLE

Let $L = 0$ be a line & $S = 0$ be a circle. If r is the radius of the circle & p is the length of the perpendicular from the centre on the line, then:

(i) $p > r$ $\Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.

(ii) $p = r$ $\Leftrightarrow$ the line touches the circle. (It is tangent to the circle)

(iii) $p < r$ $\Leftrightarrow$ the line is a secant of the circle.

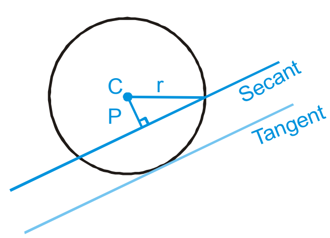
(iv) $p = 0$ $\Rightarrow$ the line is a diameter of the circle.

Also, if $y = mx + c$ is line and $x^2 + y^2 = a^2$ is circle then

(i) $c^2 < a^2 (1 + m^2)$ $\Leftrightarrow$ the line is a secant of the circle.

(ii) $c^2 = a^2 (1 + m^2)$ $\Leftrightarrow$ the line touches the circle. (It is tangent to the circle)

(iii) $c^2 > a^2 (1 + m^2)$ $\Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.



These conditions can also be obtained by solving $y = mx + c$ with $x^2 + y^2 = a^2$ and making the discriminant of the quadratic greater than zero for secant, equal to zero for tangent and less the zero for the last case.

Ex. For what value of c will the line y = 2x + c be a tangent to the circle $x^2 + y^2 = 5$?

Sol. We have $y = 2x + c$ or $2x - y + c = 0$ ......**(i)**

and $x^2 + y^2 = 5$ ....**(ii)**

If the line **(i)** touches the circle **(ii)**, then

length of the ⊥ from the centre (0, 0) = radius of circle **(ii)**

$$\Rightarrow \left|\frac{2\times 0 - 0 + c}{\sqrt{2^2+(-1)^2}}\right| = \sqrt{5} \quad \Rightarrow \quad \left|\frac{c}{\sqrt{5}}\right| = \sqrt{5}$$

$$\Rightarrow \frac{c}{\sqrt{5}} = \pm\sqrt{5} \quad \Rightarrow \quad c = \pm 5$$

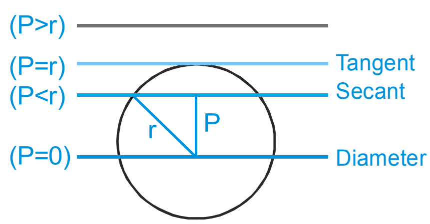
Hence, the line **(i)** touches the circle **(ii)** for c = ± 5

TANGENT LINE OF CIRCLE

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(a) Condition of Tangency :

The line L = 0 touches the circle S = 0 if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. P = r.



THEOREMS REGARDING TANGENT AND SECANT TO A CIRCLE

Theorem 1

A tangent to a circle is perpendicular to the radius through the point of contact.

Converse : A line drawn through the end point of a radius and perpendicular to it is a tangent to the circle.

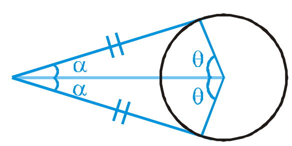
Theorem 2

If two tangents are drawn to a circle from an external point, then :

(i) they are equal.

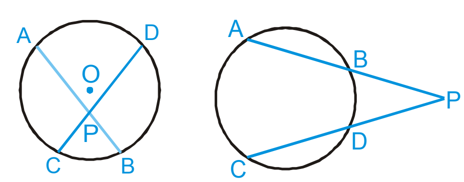
(ii) they subtend equal angles at the centre,

(iii) they are equally inclined to the segment, joining the centre to that point.



Theorem 3

If two chords of a circle intersect inside or outside the circle when produced, the rectangle formed by the two segments of one chord is equal in area to the rectangle formed by the two segments of the other chord.



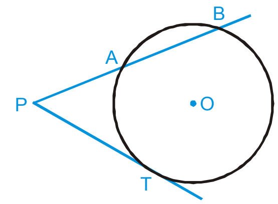
$PA \times PB = PC \times PD$

Theorem 4

If PAB is a secant to a circle intersecting the circle at A and B and PT is tangent segment, then $PA \times PB = PT^2$

OR

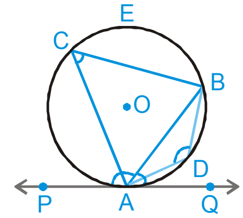
Area of the rectangle formed by the two segments of a chord is equal to the area of the square of side equal to the length of the tangent from the point on the circle.



Theorem 5

If a chord is drawn through the point of contact of a tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.

$\angle BAQ = \angle ACB$ and $\angle BAP = \angle ADB$



Converse :

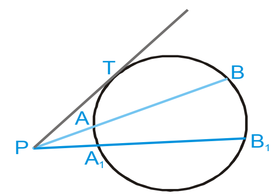
If a line is drawn through an end point of a chord of a circle so that the angle formed with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

POWER OF A POINT W.R.T. CIRCLE

Theorem : The power of point $P(x_1, y_1)$ w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is S_1

where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Note : If P outside, inside or on the circle then power of point is positive, negative or zero respectively.



If from a point $P(x_1, y_1)$, inside or outside the circle, a secant be drawn intersecting the circle in two points A & B, then PA . PB = constant. The product PA . PB is called power of point $P(x_1, y_1)$ w.r.t. the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, i.e. for number of secants $PA.PB = PA_1 . PB_1 = PA_2 . PB_2 = ... = PT^2 = S_1$

EQUATION OF THE TANGENT (T = 0)

Slope Form of Tangent

$y = mx + c$ is always a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$. Hence, equation of tangent is

$y = mx \pm a\sqrt{1+m^2}$ and the point of contact is $\left(-\frac{a^2 m}{c}, \frac{a^2}{c}\right)$.

The equation of tangent with slope m of the circle $(x - h)^2 + (y - k)^2 = a^2$ is

$$\mathbf{(y - k) = m(x - h) \pm a\sqrt{1 + m^2}}$$

Point Form of Tangent

(i) The equation of the tangent to the circle $x^2 + y^2 = a^2$ at its point (x_1, y_1) is, $x x_1 + y y_1 = a^2$.

(ii) The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1, y_1) is : $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$.

In general the equation of tangent to any second degree curve at point (x_1, y_1) on it can be obtained by replacing x^2 by $x x_1$, y^2 by yy_1, x by $\frac{x + x_1}{2}$, y by $\frac{y + y_1}{2}$, xy by $\frac{x_1 y + x y_1}{2}$ and c remains as c.

Parametric Form of Tangent

The equation of a tangent to circle $x^2 + y^2 = a^2$ at $(a\cos\alpha, a\sin\alpha)$ is $x\cos\alpha + y\sin\alpha = a$.

❖ The point of intersection of the tangents at the points $P(\alpha)$ & $Q(\beta)$ is $\left(\dfrac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \dfrac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Ex. Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0$ at $(4, -1)$.

Sol. Equation of tangent is

$$4x + (-y) - 30\left(\frac{x+4}{2}\right) + 6\left(\frac{y+(-1)}{2}\right) + 109 = 0$$

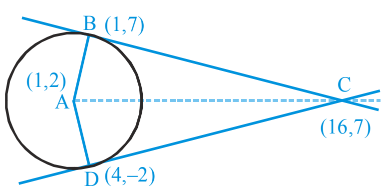
or $4x - y - 15x - 60 + 3y - 3 + 109 = 0$ or $-11x + 2y + 46 = 0$

or $11x - 2y - 46 = 0$

Hence, the required equation of the tangent is $11x - 2y - 46 = 0$

Ex. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ and B(1, 7) and D(4, –2) are points on the circle then, if tangents be drawn at B and D, which meet at C, then find area of quadrilateral ABCD.

Sol.



Here centre A(1, 2) and Tangent at (1, 7) is

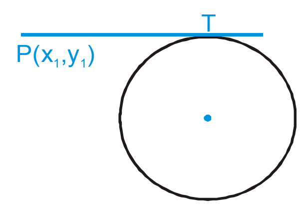
$x.1 + y.7 - 1(x + 1) - 2(y + 7) - 20 = 0$ or $y = 7$ ...... **(i)**

Tangent at D(4, –2) is $3x - 4y - 20 = 0$ ...... **(ii)**

Solving **(i)** and **(ii)**, C is (16, 7)

ABCD = AB × BC = 5 × 15 = 75 units.

LENGTH OF TANGENT $\left(\sqrt{S}\right)$



The length of tangent drawn from point (x_1,y_1) out side the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,

$$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

❖ When we use this formula the coefficient of x^2 and y^2 must be 1.

Ex. Find the length of the tangent drawn from the point (5, 1) to the circle $x^2 + y^2 + 6x - 4y - 3 = 0$

Sol. Given circle is $x^2 + y^2 + 6x - 4y - 3 = 0$ ...... **(i)**

Given point is (5, 1). Let P = (5, 1)

Now length of the tangent from P(5, 1) to circle **(i)** = $\sqrt{5^2 + 1^2 + 6.5 - 4.1 - 3} = 7$

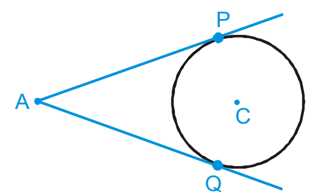
EQUATION OF PAIR OF TANGENTS ($SS_1 = T^2$)

The equation of a pair of tangents drawn from the point A (x_1, y_1) to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c.$



Ex. Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1)

Sol. Given circle is $S = x^2 + y^2 - 2x + 4y = 0$ ...... **(i)**

Let $P \equiv (0, 1)$

For point P, $S_1 = 0^2 + 1^2 - 2.0 + 4.1 = 5$

Clearly P lies outside the circle

and $T \equiv x . 0 + y . 1 - (x + 0) + 2 (y + 1)$

i.e. $T \equiv -x + 3y + 2.$

Now equation of pair of tangents from P(0, 1) to circle (1) is $SS_1 = T^2$

or $5 (x^2 + y^2 - 2x + 4y) = (- x + 3y + 2)^2$

or $5x^2 + 5y^2 - 10x + 20y = x^2 + 9y^2 + 4 - 6xy - 4x + 12y$

or $4x^2 - 4y^2 - 6x + 8y + 6xy - 4 = 0$

or $2x^2 - 2y^2 + 3xy - 3x + 4y - 2 = 0$ ...... **(ii)**

Separate equation of pair of tangents : From **(ii)**, $2x^2 + 3(y - 1) x - 2(2y^2 - 4y + 2) = 0$

$$\therefore \quad x = \frac{-3(y-1) \pm \sqrt{9(y-1)^2 + 8(2y^2 - 4y + 2)}}{4}$$

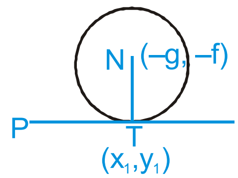
or $4x + 3y - 3 = \pm \sqrt{25y^2 - 50y + 25} = \pm 5(y - 1)$

$\therefore$ Separate equations of tangents are $2x - y + 1 = 0$ and $x + 2y - 2 = 0$

NORMAL OF CIRCLE

Normal at a point is the straight line which is perpendicular to the tangent at the point of contact.

If a line is normal / orthogonal to a circle, then it must pass through the centre of the circle. Using this fact normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is; $y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$.



The equation of normal on any point (x_1,y_1) of circle $x^2 + y^2 = a^2$ is $\frac{y}{x} = \frac{y_1}{x_1}$ -

If $x^2 + y^2 = a^2$ is the equation of the circle then at any point 't' of this circle (acost, asint), the equation of normal is **xsint – ycost = 0.**

Ex. Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

Sol. Since normal to the circle always passes through the centre so equation of the normal will be the line passing through (5, 6) & $\left(\frac{5}{2}, -1\right)$

i.e. $y + 1 = \frac{7}{5/2}\left(x - \frac{5}{2}\right) \Rightarrow 5y + 5 = 14x - 35$

$\Rightarrow$ $14x - 5y - 40 = 0$

DIRECTOR CIRCLE

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let P(h,k) is the point of intersection of two tangents drawn on the circle $x^2 + y^2 = a^2$. Then the equation of the pair of tangents is $SS_1 = T^2$

i.e. $(x^2 + y^2 - a^2)(h^2 + k^2 - a^2) = (hx + ky - a^2)^2$

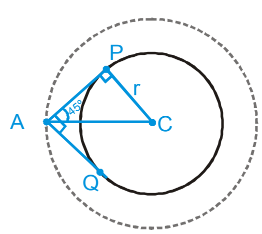
As lines are perpendicular to each other then, coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow$ $[(h^2 + k^2 - a^2) - h^2] + [(h^2 + k^2 - a^2) - k^2] = 0$

$\Rightarrow$ $h^2 + k^2 = 2a^2$

$\therefore$ locus of (h, k) is $x^2 + y^2 = 2a^2$ which is the equation of the director circle.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.



$AC = r \operatorname{cosec} 45° = r\sqrt{2}$

❖ The director circle of $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$

Ex. Find the equation of director circle of the circle $(x-2)^2+(y+1)^2=2$.

Sol. Centre & radius of given circle are $(2, -1)$ & $\sqrt{2}$ respectively.

Centre and radius of the director circle will be $(2, -1)$ & $\sqrt{2} \times \sqrt{2} = 2$ respectively.

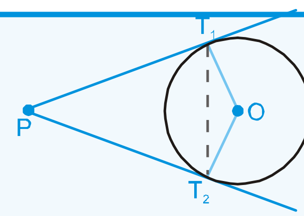
$\therefore$ equation of director circle is $(x-2)^2+(y+1)^2=4$

$\Rightarrow$ $x^2+y^2-4x+2y+1=0$.

CHORD OF CONTACT (T = 0)

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2+y^2+2gx+2fy+c=0$, then the equation of the chord of contact T_1T_2 is:

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$. (i.e., T = 0 same as equa. of tangent)



Here R = radius; L = length of tangent.

(a) Chord of contact exists only if the point 'P' is not inside.

(b) Length of chord of contact $T_1T_2 = \dfrac{2LR}{\sqrt{R^2+L^2}}$.

(c) Area of the triangle formed by the pair of the tangents & its chord of contact $= \dfrac{RL^3}{R^2+L^2}$

(d) Tangent of the angle between the pair of tangents from $(x_1, y_1) = \left(\dfrac{2RL}{L^2-R^2}\right)$

(e) Equation of the circle circumscribing the triangle PT_1T_2 is:

$(x-x_1)(x+g)+(y-y_1)(y+f)=0$.

Ex. Find the equation of the chord of contact of the tangents drawn from (1, 2) to the circle $x^2+y^2-2x+4y+7=0$

Sol. Given circle is $x^2+y^2-2x+4y+7=0$ ...... **(i)**

Let P = (1, 2)

For point P (1, 2), $x^2+y^2-2x+4y+7=1+4-2+8+7=18>0$

Hence point P lies outside the circle

For point P (1, 2), $T = x.1+y.2-(x+1)+2(y+2)+7$

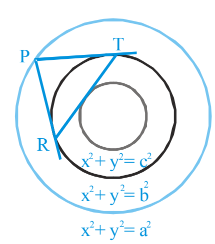
i.e. $T = 4y+10$

Now equation of the chord of contact of point P(1, 2) w.r.t. circle **(i)** will be

$4y+10=0$ or $2y+5=0$

Ex. The chord of contact of tangents drawn from a point on the circle $x^2+y^2=a^2$ to the circle $x^2+y^2=b^2$ touches the circle $x^2+y^2=c^2$. Show that a, b, c are in GP.

Sol. Let P(acosθ, asinθ) be a point on the circle $x^2+y^2=a^2$.



Then equation of chord of contact of tangents drawn from

P(acosθ, asinθ) to the circle $x^2+y^2=b^2$ is $ax\cos\theta+ay\sin\theta=b^2$ ..... **(i)**

This touches the circle $x^2+y^2=c^2$ ..... **(ii)**

$\therefore$ Length of perpendicular from (0, 0) to **(i)** = radius of **(ii)**

$$\therefore \quad \frac{|0+0-b^2|}{\sqrt{(a^2\cos^2\theta+a^2\sin^2\theta)}}=c$$

or $b^2=ac$ $\Rightarrow$ a, b, c are in GP.

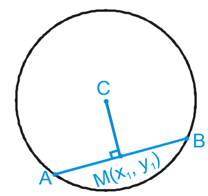
EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$)

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is

$\mathbf{y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)}$. This on simplification can be put in the form

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

which is designated by $\mathbf{T = S_1}$.



(i) The shortest chord of a circle passing through a point 'M' inside the circle is one chord whose middle point is M.

(ii) The chord passing through a point 'M' inside the circle and which is at a maximum distance from the centre is a chord with middle point M.

Ex. Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y - 11 = 0$, whose middle point is $(1, -1)$

Sol. Equation of given circle is $S \equiv x^2 + y^2 + 6x + 8y - 11 = 0$

Let $L \equiv (1, -1)$

For point $L(1, -1)$, $S_1 = 1^2 + (-1)^2 + 6.1 + 8(-1) - 11 = -11$

and $T \equiv x.1 + y(-1) + 3(x + 1) + 4(y - 1) - 11$

i.e. $T \equiv 4x + 3y - 12$

Now equation of the chord of circle **(i)** whose middle point is $L(1, -1)$ is

$T = S_1$ or $4x + 3y - 12 = -11$ or $4x + 3y - 1 = 0$

Second Method : Let C be the centre of the given circle, then $C \equiv (-3, -4)$. $L \equiv (1, -1)$ slope of $CL = \frac{-4+1}{-3-1} = \frac{3}{4}$

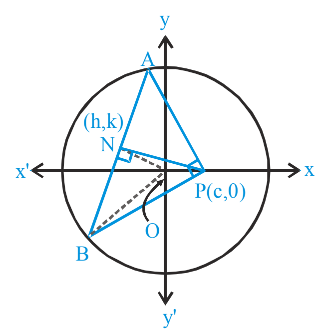
$\therefore$ Equation of chord of circle whose middle point is L, is $y + 1 = -\frac{4}{3}(x - 1)$

($\because$ chord is perpendicular to CL) or $4x + 3y - 1 = 0$

Ex. Find the locus of middle points of chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point $(c, 0)$.

Sol. Let N(h, k) be the middle point of any chord AB,

which subtend a right angle at P(c, 0).



Since $\angle APB = 90°$

$\therefore$ $NA = NB = NP$

(since distance of the vertices from middle point of the hypotenuse are equal)

or $(NA)^2 = (NB)^2 = (h - c)^2 + (k - 0)^2$ ..... **(i)**

But also $\angle BNO = 90°$

$\therefore$ $(OB)^2 = (ON)^2 + (NB)^2$

$\Rightarrow$ $-(NB)^2 = (ON)^2 - (OB)^2$ $\Rightarrow$ $-[(h - c)^2 + (k - 0)^2] = (h^2 + k^2) - a^2$

or $2(h^2 + k^2) - 2ch + c^2 - a^2 = 0$

$\therefore$ Locus of N(h, k) is $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$

EQUATION OF THE CHORD JOINING TWO POINTS OF CIRCLE

The equation of chord PQ to the circle $x^2 + y^2 = a^2$ joining two points $P(\alpha)$ and $Q(\beta)$ on it is given by the equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

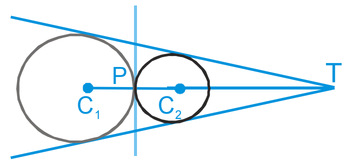
$$x \cos\frac{\alpha+\beta}{2} + y \sin\frac{\alpha+\beta}{2} = a \cos\frac{\alpha-\beta}{2}.$$

DIRECT AND TRANSVERSE COMMON TANGENTS

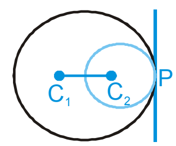
Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

(a) Both Circles will Touch

(i) Externally : if $C_1C_2 = r_1 + r_2$ i.e. the distance between their centres is equal to sum of their radii and point P & T divides C_1C_2 in the ratio $r_1 : r_2$ (internally & externally respectively). In this case there are **three common tangents.**

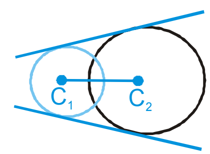


(ii) Internally : if $C_1C_2 = |r_1 – r_2|$ i.e. the distance between their centres is equal to difference between their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be only **one common tangent.**



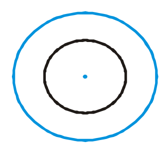
(b) The Circles will Intersect

when $|r_1 – r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents.**

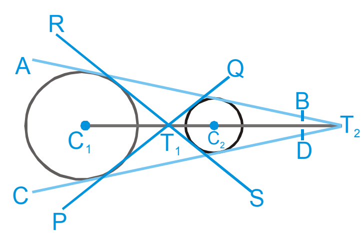


(c) The Circles will not Intersect

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 – r_2|$ In this case there will be no common tangent.



(ii) When circle are apart from each other then $C_1C_2 > r_1 + r_2$ and in this case there will be **four common tangents.** Lines PQ and RS are called **transverse** or **indirect** or **internal** common tangents and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called **direct** or **external** common tangents. These lines meet C_1C_2 produced on T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.



(i) The direct common tangents meet at a point which divides the line joining centre of circles externally in the ratio of their radii.

Transverse common tangents meet at a point which divides the line joining centre of circles internally in the ratio of their radii.

(ii) Length of an external (or direct) common tangent & internal (or transverse) common tangent to the two circles are given by: $L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$ & $L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$,

where d = distance between the centres of the two circles and r_1, r_2 are the radii of the two circles. Note that length of internal common tangent is always less than the length of the external or direct common tangent.

Ex. Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally.

Sol. Given circles are $x^2 + y^2 - 2x - 4y = 0$ ...... **(i)**

and $x^2 + y^2 - 8y - 4 = 0$ ...... **(ii)**

Let A and B be the centres and r_1 and r_2 the radii of circles **(i)** and **(ii)** respectively, then

$A \equiv (1, 2), B \equiv (0, 4), r_1 = \sqrt{5}, r_2 = 2\sqrt{5}$

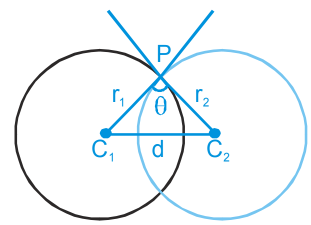
Now $AB = \sqrt{(1-0)^2 + (2-4)^2} = \sqrt{5}$ and $r_1 + r_2 = 3\sqrt{5}$, $|r_1 - r_2| = \sqrt{5}$

Thus $AB = |r_1 - r_2|$, hence the two circles touch each other internally.

THE ANGLE OF INTERSECTION OF TWO CIRCLES

Definition : The angle between the tangents of two circles at the point of intersection of the two circles is called angle of intersection of two circles. If two circles are $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and θ is the acute angle between them



then $\cos\theta = \left|\dfrac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\ \sqrt{g_2^2 + f_2^2 - c_2}}\right|$ or $\cos\theta = \left|\left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)\right|$

Here r_1 and r_2 are the radii of the circles and d is the distance between their centres.

If the angle of intersection of the two circles is a right angle then such circles are called **"Orthogonal circles"** and conditions for the circles to be orthogonal is -

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

(a) The centre of a variable circle orthogonal to two fixed circles lies on the radical axis of two circles.

(b) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent in a circle which is orthogonal to all the three circles.

(c) The centre of a circle which is orthogonal to three given circles is the radical centre provided the radical centre lies outside all the three circles.

Ex. Obtain the equation of the circle orthogonal to both the circles $x^2 + y^2 + 3x - 5y + 6 = 0$ and $4x^2 + 4y^2 - 28x + 29 = 0$ and whose centre lies on the line $3x + 4y + 1 = 0$.

Sol. Given circles are $x^2 + y^2 + 3x - 5y + 6 = 0$ ...... **(i)**

and $4x^2 + 4y^2 - 28x + 29 = 0$

or $x^2 + y^2 - 7x + \dfrac{29}{4} = 0.$ ...... **(ii)**

Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ...... **(iii)**

Since circle **(iii)** cuts circles **(i)** and **(ii)** orthogonally

$\therefore$ $2g\left(\dfrac{3}{2}\right) + 2f\left(-\dfrac{5}{2}\right) = c + 6$ or $3g - 5f = c + 6$ ...... **(iv)**

and $2g\left(-\dfrac{7}{2}\right) + 2f.0 = c + \dfrac{29}{4}$ or $-7g = c + \dfrac{29}{4}$ ...... **(v)**

From **(iv)** & **(v)**, we get $10g - 5f = -\frac{5}{4}$

or $\quad 40g - 20f = -5.$ ...... **(vi)**

Given line is $3x + 4y = -1$ ...... **(vii)**

Since centre $(-g, -f)$ of circle **(iii)** lies on line (vii),

$\therefore \quad -3g - 4f = -1$ ...... **(viii)**

Solving **(vi)** & (viii), we get $g = 0$, $f = \frac{1}{4}$

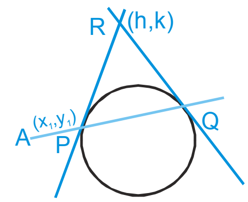
$\therefore$ from (5), $c = -\frac{29}{4}$

$\therefore$ from **(iii)**, required circle is

$$x^2 + y^2 + \frac{1}{2}y - \frac{29}{4} = 0 \quad \text{or} \quad 4(x^2 + y^2) + 2y - 29 = 0$$

POLE AND POLAR

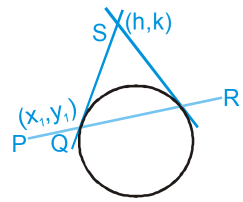
Let any straight line through the given point $A(x_1,y_1)$ intersect the given circle $S = 0$ in two points P and Q and if the tangent of the circle at P and Q meet at the point R then locus of point R is called polar of the point A and point A is called the pole, with respect to the given circle.



(a) The Equation of the Polar of Point (x_1,y_1) w.r.t. Circle $x^2 + y^2 = a^2$ (T = 0).

Let PQR is a chord which passes through the point $P(x_1,y_1)$ which intersects the circle at points Q and R and the tangents are drawn at points Q and R meet at point S(h,k) then equation of QR the chord of contact is $x_1h + y_1k = a^2$

$\therefore$ locus of point S(h,k) is $\mathbf{xx_1 + yy_1 = a^2}$ which is the equation of the polar.



(i) The equation of the polar is the T=0, so the polar of point (x_1,y_1) w.r.t circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(ii) If point is outside the circle then equation of polar and chord of contact is same. So the chord of contact is polar.

(iii) If point is inside the circle then chord of contact does not exist but polar exists.

(iv) If point lies on the circle then polar , chord of contact and tangent on that point are same.

(v) If the polar of P w.r.t. a circle passes through the point Q, then the polar of point Q will pass through P and hence P Q are conjugate points of each other w.r.t. the given circle.

(vi) If pole of a line w.r.t. a circle lies on second line. Then pole of second line lies on first line and hence both lines are conjugate lines of each other w.r.t. the given circle.

(vii) If O be the centre of a circle and P be any point, then OP is perpendicular to the polar of P.

(viii) If O be the centre of a circle and P any point, then if OP (produce, if necessary) meet the polar of P in Q, then OP. OQ = $(\text{radius})^2$

(b) Pole of a Given Line with Respect to a Circle

To find the pole of a line we assume the coordinates of the pole then from these coordinates we find the polar. This polar and given line represent the same line. Then by comparing the coefficients of similar terms we can get the coordinates of the pole. The pole of $\ell x + my + n = 0$

w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\frac{-\ell a^2}{n}, \frac{-ma^2}{n}\right)$

Ex. Find the equation of the polar of the point (2, –1) with respect to the circle $x^2 + y^2 - 3x + 4y - 8 = 0$

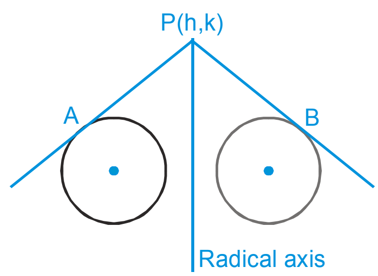
Sol. Given circle is $x^2 + y^2 - 3x + 4y - 8 = 0$ ...... **(i)**

Given point is (2, –1) let P = (2, –1). Now equation of the polar of point P with respect to circle **(i)**

$$x.2 + y(-1) - 3\left(\frac{x+2}{2}\right) + 4\left(\frac{y-1}{2}\right) - 8 = 0$$

or $4x - 2y - 3x - 6 + 4y - 4 - 16 = 0$ or $x + 2y - 26 = 0$

RADICAL AXIS OF THE TWO CIRCLES ($S_1 - S_2 = 0$)



Definition The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal and is called the radical axis. If two circles are -

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

Let P(h,k) is a point and PA,PB are length of two tangents on the circles from point P, Then from definition -

$$\sqrt{h^2 + k^2 + 2g_1h + 2f_1k + c_1} = \sqrt{h^2 + k^2 + 2g_2h + 2f_2k + c_2} \quad \text{or} \quad 2(g_1-g_2)h + 2(f_1-f_2)k + c_1 - c_2 = 0$$

$\therefore$ locus of (h,k)

$2x(g_1-g_2) + 2y(f_1-f_2)k + c_1 - c_2 = 0$

$\mathbf{S_1 - S_2 = 0}$

which is the equation of radical axis.

(i) To get the equation of the radical axis first of all make the coefficient of x^2 and $y^2 = 1$

(ii) If circles touch each other then radical axis is the common tangent to both the circles.

(iii) When the two circles intersect on real points then common chord is the radical axis of the two circles.

(iv) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.

(v) Radical axis (if exist) bisects common tangent to two circles.

(vi) The radical axes of three circles (taking two at a time) meet at a point.

(vii) If circles are concentric then the radical axis does not always exist but if circles are not concentric then radical axis always exists.

(viii) If two circles are orthogonal to the third circle then radical axis of both circle passes through the centre of the third circle.

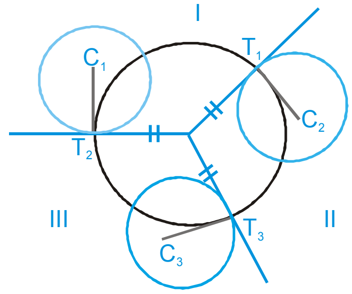
(ix) A system of circle, every pair of which have the same radical axis, is called a **coaxial system of circles.**

RADICAL CENTRE

The radical centre of three circles is the point from which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles.

To get the radical axis of three circles $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ we have to solve any two

$S_1 - S_2 = 0$, $S_2 - S_3 = 0$, $S_3 - S_1 = 0$



(i) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.

(ii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.

(iii) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.

(iv) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent is a circle which is orthogonal to all the three circles.

Ex. Find the co-ordinates of the point from which the lengths of the tangents to the following three circles be equal.

$$3x^2 + 3y^2 + 4x - 6y - 1 = 0$$
$$2x^2 + 2y^2 - 3x - 2y - 4 = 0$$
$$2x^2 + 2y^2 - x + y - 1 = 0$$

Sol. Here we have to find the radical centre of the three circles. First reduce them to standard form in which coefficients of x^2 and y^2 be each unity. Subtracting in pairs the three radical axes are

$$\frac{17}{6}x - y + \frac{5}{3} = 0\,; \qquad -x - \frac{3}{2}y - \frac{3}{2} = 0$$

$$-\frac{11}{6}x + \frac{5}{2}y - \frac{1}{6} = 0.$$

solving any two, we get the point $\left(-\frac{16}{21} - \frac{31}{63}\right)$ which satisfies the third also. This point is called the radical centre and by definition the length of the tangents from it to the three circles are equal.

Ex. A and B are two fixed points and P moves such that PA = nPB where $n \neq 1$. Show that locus of P is a circle and for different values of n all the circles have a common radical axis.

Sol. Let $A \equiv (a, 0)$, $B \equiv (-a, 0)$ and P(h, k)

So PA = nPB

$\Rightarrow \quad (h - a)^2 + k^2 = n^2[(h + a)^2 + k^2]$

$\Rightarrow \quad (1-n^2)h^2 + (1-n^2)k^2 - 2ah(1+n^2) + (1-n^2)a^2 = 0$

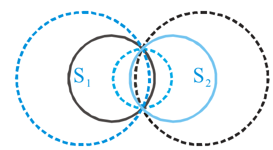
$\Rightarrow \quad h^2 + k^2 - 2ah\left(\dfrac{1+n^2}{1-n^2}\right) + a^2 = 0$

Hence locus of P is

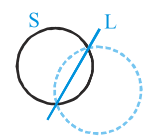
$x^2 + y^2 - 2ax\left(\dfrac{1+n^2}{1-n^2}\right) + a^2 = 0$, which is a circle of different values of n.

Let n_1 and n_2 are two different values of n so their radical axis is x = 0 i.e. y-axis. Hence for different values of n the circles have a common radical axis.

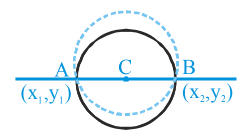
FAMILY OF CIRCLES



(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K S_2 = 0 \quad (K \neq -1)$.

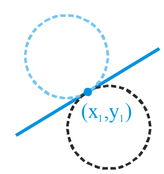


(b) The equation of the family of circles passing through the point of intersection of a circle S = 0 & a line L = 0 is given by S + KL = 0.

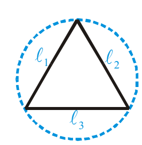


(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

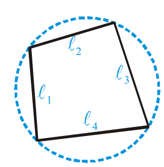
$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0 \text{ where K is a parameter.}$$



(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x-x_1)^2 + (y-y_1)^2 + K[y - y_1 - m(x-x_1)] = 0$, where K is a parameter.



(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of xy = 0 & coefficient of x^2 = coefficient of y^2.



(f) Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0, L_2 = 0, L_3 = 0$ & $L_4 = 0$ is $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0.

Ex. Find the equations of the circles passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$ and whose radius is 4.

Sol. Any circle through the intersection of given circles is $S_1 + \lambda S_2 = 0$

or $\quad (x^2 + y^2 - 2x - 4y - 4) + \lambda(x^2 + y^2 - 10x - 12y + 40) = 0$

or $\quad (x^2 + y^2) - 2\dfrac{(1+5\lambda)}{1+\lambda}x - 2\dfrac{(2+6\lambda)}{1+\lambda}y + \dfrac{40\lambda - 4}{1+\lambda} = 0 \qquad$ ....**(i)**

$r = \sqrt{g^2 + f^2 - c} = 4$, given

$$\therefore \quad 16 = \frac{(1+5\lambda)^2}{(1+\lambda)^2} + \frac{(2+6\lambda)^2}{(1+\lambda)^2} - \frac{40\lambda - 4}{1+\lambda}$$

$$16(1 + 2\lambda + \lambda^2) = 1 + 10\lambda + 25\lambda^2 + 4 + 24\lambda + 36\lambda^2 - 40\lambda^2 - 40\lambda + 4 + 4\lambda$$

or $\quad 16 + 32\lambda + 16\lambda^2 = 21\lambda^2 - 2\lambda + 9$

or $\quad 5\lambda^2 - 34\lambda - 7 = 0$

$\therefore \quad (\lambda - 7)(5\lambda + 1) = 0$

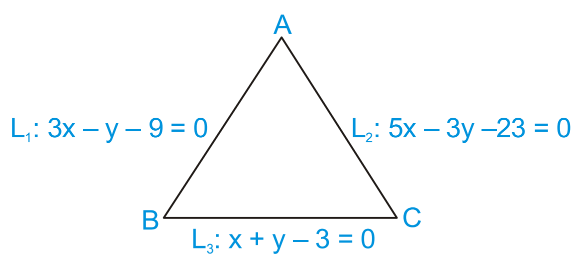
$\therefore \quad \lambda = 7, -1/5$

Putting the values of λ in **(i)** the required circles are $2x^2 + 2y^2 - 18x - 22y + 69 = 0$

and $x^2 + y^2 - 2y - 15 = 0$

Ex. Find the equation of circle circumscribing the triangle whose sides are $3x - y - 9 = 0$, $5x - 3y - 23 = 0$ & $x + y - 3 = 0$.

Sol.



$$L_1L_2 + \lambda L_2L_3 + \mu L_1L_3 = 0$$

$(3x - y - 9)(5x - 3y - 23) + \lambda(5x - 3y - 23)(x + y - 3) + \mu(3x - y - 9)(x + y - 3) = 0$

$(15x^2 + 3y^2 - 14xy - 114x + 50y + 207) + \lambda(5x^2 - 3y^2 + 2xy - 38x - 14y + 69)$

$+ \mu(3x^2 - y^2 + 2xy - 18x - 6y + 27) = 0$

$(5\lambda + 3\mu + 15)x^2 + (3 - 3\lambda - \mu)y^2 + xy(2\lambda + 2\mu - 14) - x(114 + 38\lambda + 18\mu) + y(50 - 14\lambda - 6\mu)$

$+ (207 + 69\lambda + 27\mu) = 0$ ...... **(i)**

coefficient of x^2 = coefficient of y^2

$\Rightarrow \quad 5\lambda + 3\mu + 15 = 3 - 3\lambda - \mu$

$2\lambda + \mu + 3 = 0$ ...... **(ii)**

coefficient of $xy = 0$

$\Rightarrow \quad \lambda + \mu - 7 = 0$ ...... **(iii)**

Solving **(ii)** and **(iii)**, we have

$\lambda = -10, \mu = 17$

Putting these values of λ & μ in equation **(i)**, we get $2x^2 + 2y^2 - 5x + 11y - 3 = 0$

TIPS & FORMULAS

1. Definition

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point (in the same given plane) remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

2. Standard Equations of the Circle

(a) Central Form

If (h, k) is the centre and r is the radius of the circle then its equation is

$(x-h)^2 + (y-k)^2 = r^2$

(b) General Equation of Circle

$x^2 + y^2 + 2gx + 2fy + c = 0$. where g,f,c are constants and centre is $(-g,-f)$

i.e. $\left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$ and radius $r = \sqrt{g^2 + f^2 - c}$

Note

The general quadratic equation in x and y, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if :

(i) coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$.

(ii) coefficient of $xy = 0$ or $h = 0$

(iii) $(g^2 + f^2 - c) \geq 0$ (for a real circle)

(c) Intercepts Cut by the Circle on Axes

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on :

(i) x-axis $= 2\sqrt{g^2 - c}$ (ii) y-axis $= 2\sqrt{f^2 - c}$

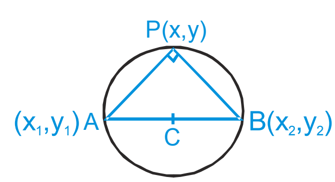
Note

Intercept cut by a line on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ or length of chord of the circle $= 2\sqrt{a^2 - P^2}$ where a is the radius and P is the length of perpendicular from the centre to the chord.

(d) Equation of Circle in Diameter Form

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are the end points of the diameter of the circle then the equation of circle is given by

$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$



(e) Equation of Circle in Parametric Forms

(i) The parametric equation of the circle $x^2+y^2 = r^2$ are $x = r\cos\theta, y = r\sin\theta$; $\theta \in [0, 2\pi)$ and $(r\cos\theta, r\sin\theta)$ are called the parametric co-ordinates.

(ii) The parametric equation of the circle $(x-h)^2 + (y-k)^2 = r^2$ is $x = h + r\cos\theta, y = k + r\sin\theta$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are $x = -g + \sqrt{g^2 + f^2 - c}\,\cos\theta$,

$y = -f + \sqrt{g^2 + f^2 - c}\,\sin\theta$ where θ is parameter.

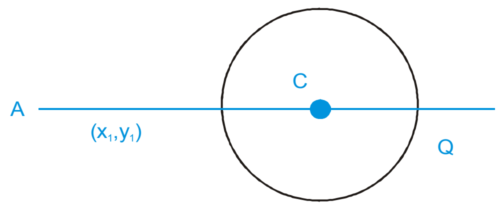
Note Equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}.$$

3. Position of a Point W.R.T Circle

(a) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1,y_1) then -

Point (x_1,y_1) lies out side the circle or on the circle or inside the circle according as



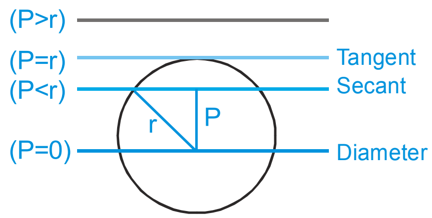
$\Rightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$ or $S_1 >, =, < 0$

(b) The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $|AC - r|$ respectively.

4. Tangent Line of Circle

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

(a) Condition of Tangency



The line L = 0 touches the circle S = 0 if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. P = r.

(b) Equation of the Tangent (T = 0)

(i) Tangent at the point (x_1,y_1) on the circle $x^2 + y^2 = a^2$ is $\mathbf{xx_1 + yy_1 = a^2}$.

(ii) **(1)** The tangent at the point (acost, asint) on the circle $x^2 + y^2 = a^2$ is **xcost + ysint = a**

(2) The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is $\left(\dfrac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \dfrac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

(iii) The equation of tangent at the point (x_1,y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0}$$

(iv) If line $y = mx + c$ is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and contact **points are**

$\left(\mp\dfrac{\mathbf{am}}{\sqrt{1+m^2}}, \pm\dfrac{\mathbf{a}}{\sqrt{1+m^2}}\right)$ **or** $\left(\mp\dfrac{a^2m}{c}, \pm\dfrac{a^2}{c}\right)$ and equation of tangent is $\mathbf{y = mx \pm a\sqrt{1+m^2}}$

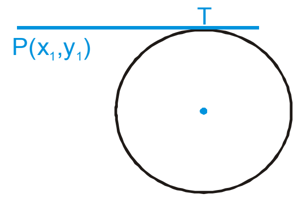
(v) The equation of tangent with slope m of the circle $(x-h)^2 + (y-k)^2 = a^2$ is $\mathbf{(y-k) = m(x-h) \pm a\sqrt{1+m^2}}$

Note To get the equation of tangent at the point $(x_1\ y_1)$ on any second degree curve we replace xx_1 in place of x^2, yy_1 in place of y^2, $\dfrac{x+x_1}{2}$ in place of x, $\dfrac{y+y_1}{2}$ in place of y, $\dfrac{xy_1+yx_1}{2}$ in place of xy and c in place of c.

(c) Length of Tangent ($\sqrt{S_1}$) :

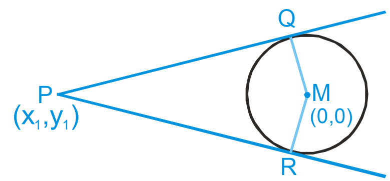
The length of tangent drawn from point (x_1,y_1) out side the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,

$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$



Note When we use this formula the coefficient of x^2 and y^2 must be 1.

(d) Equation of Pair of Tangents ($SS_1 = T^2$) :



Let the equation of circle $S \equiv x^2 + y^2 = a^2$ and $P(x_1,y_1)$ is any point outside the circle. From the point we can draw two real and distinct tangent PQ & PR and combine equation of pair of tangents is -

$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$ **or** $SS_1 = T^2$

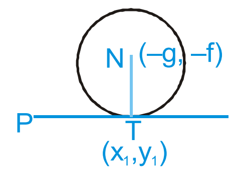
5. Normal of Circle

Normal at a point is the straight line which is perpendicular to the tangent at the point of contact.

Note : Normal at point of the circle passes through the centre of the circle.

(a) Equation of normal at point (x_1,y_1) of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

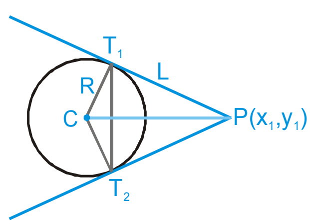
$$y - y_1 = \left(\frac{y_1 + f}{x_1 + g}\right)(x - x_1)$$



(b) The equation of normal on any point (x_1,y_1) of circle $x^2 + y^2 = a^2$ is $\frac{y}{x} = \frac{y_1}{x_1}$ -

(c) If $x^2 + y^2 = a^2$ is the equation of the circle then at any point 't' of this circle (acost, asint), the equation of normal is **$x\sin t - y\cos t = 0$.**

6. Chord of Contact



A line joining the two points of contacts of two tangents drawn from a point out side the circle, is called chord of contact of that point.

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. $T = 0$ same as equation of tangent).

7. Equation of The Chord with a Given Middle Point ($T = S_1$)

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is

$y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$. This on simplification can be put in the form

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by **$T = S_1$.**

8. Director Circle

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let P(h,k) is the point of intersection of two tangents drawn on the circle $x^2 + y^2 = a^2$. Then the equation of the pair of tangents is $SS_1 = T^2$

i.e. $(x^2 + y^2 - a^2)(h^2 + k^2 - a^2) = (hx + ky - a^2)^2$

As lines are perpendicular to each other then, coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow$ $[(h^2 + k^2 - a^2) - h^2] + [(h^2 + k^2 - a^2) - k^2] = 0$

$\Rightarrow$ $\mathbf{h^2 + k^2 = 2a^2}$

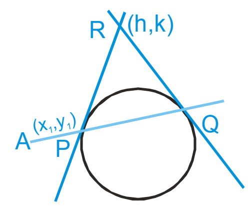
$\therefore$ locus of (h, k) is $x^2 + y^2 = 2a^2$ which is the equation of the director circle.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

Note The director circle of $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$

9. Pole and Polar

Let any straight line through the given point $A(x_1,y_1)$ intersect the given circle S = 0 in two points P and Q and if the tangent of the circle at P and Q meet at the point R then locus of point R is called polar of the point A and point A is called the pole, with respect to the given circle.



The equation of the polar is the T = 0, so the polar of point (x_1, y_1) w.r.t circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Pole of a Given Line with Respect to a Circle

To find the pole of a line we assume the coordinates of the pole then from these coordinates we find the polar. This polar and given line represent the same line. Then by comparing the coefficients of similar terms we can get the coordinates of the pole. The pole of $\ell x + my + n = 0$

w.r.t. circle $x^2 + y^2 = a^2$ will be $\left(\dfrac{-\ell a^2}{n}, \dfrac{-ma^2}{n}\right)$

10. Family of Circles

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K S_2 = 0$ $(K \neq -1)$.

(b) The equation of the family of circles passing through the point of intersection of a circle S = 0 & a line L = 0 is given by S + KL = 0.

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \text{ where K is a parameter.}$$

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

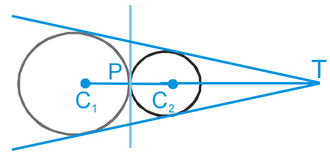
(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of xy = 0 & coefficient of x^2 = coefficient of y^2.

(f) Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ is $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0.

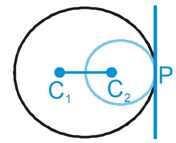
11. Direct and Transverse Common Tangents

Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

(a) Both Circles will Touch :



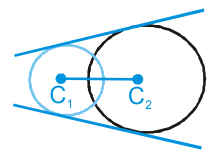
(i) Externally if $C_1C_2 = r_1 + r_2$ i.e. the distance between their centres is equal to sum of their radii and point P & T divides C_1C_2 in the ratio $r_1 : r_2$ (internally & externally respectively). In this case there are **three common tangents.**



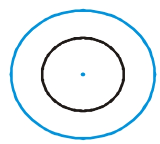
(ii) Internally if $C_1C_2 = |r_1 - r_2|$ i.e. the distance between their centres is equal to difference between their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ **externally** and in this case there will be only **one common tangent.**

(b) The Circles will Intersect :

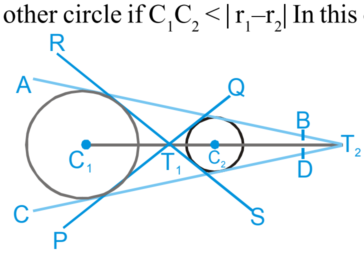
when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are **two common tangents.**



(c) The Circles will not Intersect :



(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$ In this case there will be no common tangent.



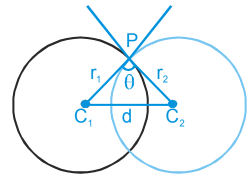
(ii) When circle are apart from each other then $C_1C_2 > r_1 + r_2$ and in this case there will be **four common tangents.** Lines PQ and RS are called **transverse** or **indirect** or **internal** common tangents and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called **direct** or **external** common tangents. These lines meet C_1C_2 produced on T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.

Note Length of direct common tangent $= \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent $= \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

12. The Angle of Intersection of Two Circles

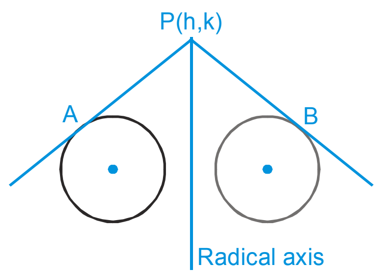
Definition : The angle between the tangents of two circles at the point of intersection of the two circles is called angle of intersection of two circles. If two circles are $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and θ is the acute angle between them then $\cos\theta = \left|\dfrac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\ \sqrt{g_2^2 + f_2^2 - c_2}}\right|$ or $\cos\theta = \left|\left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)\right|$



Here r_1 and r_2 are the radii of the circles and d is the distance between their centres.
If the angle of intersection of the two circles is a right angle then such circles are called **"Orthogonal circles"** and conditions for the circles to be orthogonal is -$\mathbf{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$

13. Radical axis of the Two Circles ($S_1 - S_2 = 0$)

Definition : The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal and is called the radical axis. If two circles are -



$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$
$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

Let P(h,k) is a point and PA,PB are length of two tangents on the circles from point P, Then from definition -

$$\sqrt{h^2 + k^2 + 2g_1h + 2f_1k + c_1} = \sqrt{h^2 + k^2 + 2g_2h + 2f_2k + c_2} \text{ or } 2(g_1 - g_2)h + 2(f_1 - f_2)k + c_1 - c_2 = 0$$

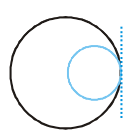
$\therefore$ locus of (h,k)

$2x(g_1 - g_2) + 2y(f_1 - f_2)k + c_1 - c_2 = 0$

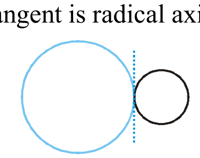
$\mathbf{S_1 - S_2 = 0}$

Note

(i) If two circles touches each other then common tangent is radical axis.

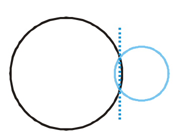


$S_1 - S_2 = 0$



$S_1 - S_2 = 0$

(ii) If two circle cuts each other then common chord is radical axis.



$S_1 - S_2 = 0$

(iii) If two circles cuts third circle orthogonally then radical axis of first two is locus of centre of third circle.