JEE - 2025

MATHS - MODULES

THEORY

EXERCISE

SOLUTIONS

- Useful for JEE MAINS and ADVANCED Exams
- Each topic contains Detailed Theory with images
- Every topic contains Exercises and Detailed solutions

DETERMINANTS

INTRODUCTION

If the equations $a_1x + b_1 = 0$, $a_2x + b_2 = 0$ are satisfied by the same value of x, then $a_1b_2 - a_2b_1 = 0$. The expression $a_1b_2 - a_2b_1$ is called a determinant of the second order, and is denoted by :

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

A determinant of second order consists of two rows and two columns.

Next consider the system of equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$

If these equations are satisfied by the same values of x and y, then on eliminating x and y we get.

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

The expression on the left is called a determinant of the third order, and is denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

A determinant of third order consists of three rows and three columns.

VALUE OF A DETERMINANT

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

❖ Sarrus diagram to get the value of determinant of order three :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{array}{c} -ve\ -ve\ -ve \\ \left|\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} \\ +ve\ +ve\ +ve \end{array} = (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)$$

Note that the product of the terms in first bracket (i.e. $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$) is same as the product of the terms in second bracket.

Ex. Evaluate **(i)** $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$ **(ii)** $\begin{vmatrix} 1 & \log_b a \\ \log_a b & 1 \end{vmatrix}$

Sol. **(i)** $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = \sin^2\theta - (-\cos^2\theta) = \sin^2\theta + \cos^2\theta = 1.$

(ii) $\begin{vmatrix} 1 & \log_b a \\ \log_a b & 1 \end{vmatrix} = 1 - \log_b a \times \log_a b = 1 - 1 = 0$ $\left[\because \log_b a = \dfrac{1}{\log_a b}\right]$

Ex. Eliminate ℓ, m, n from the equations $a\ell + cm + bn = 0$, $c\ell + bm + an = 0$, $b\ell + am + cn = 0$ and express the result in the simplest form.

Sol. The given set of equations can also be written as (if $n \neq 0$) :

$$a\left(\frac{\ell}{n}\right) + c\left(\frac{m}{n}\right) + b = 0\ ;\quad c\left(\frac{\ell}{n}\right) + b\left(\frac{m}{n}\right) + a = 0\ ;\quad b\left(\frac{\ell}{n}\right) + a\left(\frac{m}{n}\right) + c = 0$$

Then, let $\frac{\ell}{n} = x\ ;\ \frac{m}{n} = y$

$\Rightarrow$ System of equations :

$ax + cy + b = 0$ .....**(i)**

$cx + by + a = 0$ .....**(ii)**

$bx + ay + c = 0$ .....**(iii)**

We have to eliminate x & y from these simultaneous linear equations.

Since these equations are satisfied by the same values of x and y, then eliminating x and y we get,

$$\begin{vmatrix} a & c & b \\ c & b & a \\ b & a & c \end{vmatrix} = 0$$

Ex. The value of $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ is -

Sol. $$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = 1\begin{vmatrix} 3 & 6 \\ -7 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 3 \\ 2 & -7 \end{vmatrix}$$

$$= (27 + 42) - 2\,(-36 - 12) + 3\,(28 - 6) = 231$$

Alternative : By sarrus diagram

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}\begin{matrix} 1 & 2 \\ -4 & 3 \\ 2 & -7 \end{matrix}$$

$$= (27 + 24 + 84) - (18 - 42 - 72) = 135 - (18 - 114) = 231$$

MINORS & COFACTORS

Let Δ be a determinant. Then minor of element a_{ij}, denoted by M_{ij}, is defined as the determinant of the submatrix obtained by deleting i^{th} row & j^{th} column of Δ. Cofactor of element a_{ij}, denoted by C_{ij}, is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

e.g. 1 $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$M_{11} = d = C_{11}$

$M_{12} = c,\ C_{12} = -c$

$M_{21} = b,\ C_{21} = -b$

$M_{22} = a = C_{22}$

e.g. 2 $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

$M_{11} = \begin{vmatrix} q & r \\ y & z \end{vmatrix} = qz - yr = C_{11}.$

$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx,\ C_{23} = -(ay - bx) = bx - ay$ etc.

Ex. Find the minors and cofactors of elements of the matrix $A = [a_{ij}] = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{bmatrix}$.

Sol. Let M_{ij} and C_{ij} denote respectively the minor and cofactor of element a_{ij} in A. Then,

$M_{11} = \begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} = -10 - 30 = -40 \quad \Rightarrow \quad C_{11} = M_{11} = -40$

$M_{12} = \begin{vmatrix} 4 & 6 \\ 3 & 2 \end{vmatrix} = 8 - 18 = -10 \quad \Rightarrow \quad C_{12} = -M_{12} = 10$

$M_{13} = \begin{vmatrix} 4 & -5 \\ 3 & 5 \end{vmatrix} = 20 + 15 = 35 \quad \Rightarrow \quad C_{13} = M_{13} = 35$

$M_{21} = \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} = 6 + 10 = 16 \quad \Rightarrow \quad C_{21} = -M_{21} = -16$

$M_{22} = \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 2 + 6 = 8 \quad \Rightarrow \quad C_{22} = M_{22} = 8$

$M_{23} = \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = 5 - 9 = -4 \quad \Rightarrow \quad C_{23} = -M_{23} = 4$

Ex. Find the minors and cofactors of elements '–3', '5', '–1' & '7' in the determinant $\begin{vmatrix} 2 & -3 & 1 \\ 4 & 0 & 5 \\ -1 & 6 & 7 \end{vmatrix}$

Sol. Minor of $-3 = \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix} = 33$; Cofactor of $-3 = -33$

Minor of $5 = \begin{vmatrix} 2 & -3 \\ -1 & 6 \end{vmatrix} = 9$; Cofactor of $5 = -9$

Minor of $-1 = \begin{vmatrix} -3 & 1 \\ 0 & 5 \end{vmatrix} = -15$; Cofactor of $-1 = -15$

Minor of $7 = \begin{vmatrix} 2 & -3 \\ 4 & 0 \end{vmatrix} = 12$; Cofactor of $7 = 12$

Transpose of a Determinant

The transpose of a determinant is the determinant of transpose of the corresponding matrix.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

PROPERTIES OF DETERMINANTS

(a) The value of a determinant remains unaltered, if the rows & columns are inter-changed,

e.g. if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D_1 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D_1 = -D$.

(c) If all the elements of a row (or column) are zero, then the value of the determinant is zero.

(d) If all the elements of any row (or column) are multiplied by the same number, then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D_1 = KD$

(e) If all the elements of a row (or column) are proportional (or identical) to the element of any other row, then the determinant vanishes, i.e. its value is zero.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D = 0$; If $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D_1 = 0$

(f) If each element of any row (or column) is expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

e.g. $\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note that : If $D_r = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$

where $r \in N$ and a,b,c, a_1, b_1,c_1 are constants, then

$$\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

(g) **Row - column operation :** The value of a determinant remains unaltered under a column (C_i) operation of the form $C_i \to C_i + \alpha C_j + \beta C_k$ $(j, k \neq i)$ or row (R_i) operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$ $(j, k \neq i)$. In other words, the value of a determinant is not altered by adding the elements of any row (or column) to the same multiples of the corresponding elements of any other row (or column)

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$D = \begin{vmatrix} a_1 + \alpha a_2 & b_1 + \alpha b_2 & c_1 + \alpha c_2 \\ a_2 & b_2 & c_2 \\ a_3 + \beta a_1 & b_3 + \beta b_1 & c_3 + \beta c_1 \end{vmatrix} \quad (R_1 \to R_1 + \alpha R_2; R_3 \to R_3 + \beta R_2)$$

(i) By using the operation $R_i \to xR_i + yR_j + zR_k$ $(j, k \neq i)$, the value of the determinant becomes x times the original one.

(ii) While applying this property ATLEAST ONE ROW (OR COLUMN) must remain unchanged.

(h) **Factor theorem :** If the elements of a determinant D are rational integral functions of x and two rows (or columns) become identical when x = a then (x – a) is a factor of D.

Note that if r rows become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of D.

Ex. Simplify $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Sol. Given detereminant is equal to

$$= \frac{1}{abc} \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} = \frac{abc}{abc} \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $\quad C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a^2 - b^2 & b^2 - c^2 & c^2 \\ a^3 - b^3 & b^3 - c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (a – b) (b – c) [ab^2 + abc + ac^2 + b^3 + b^2C + bc^2 – a^2b – a^2c – ab^2 – abc – b^3 – b^2c]$

$= (a – b) (b – c) [c(ab + bc + ca) – a(ab + bc + ca)]$

$= (a – b) (b – c) (c – a) (ab + bc + ca)$

Ex. Find the value of the determinant $\begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix}$

Sol. $D = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = a\begin{vmatrix} a & b & c \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix} = abc\begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0$

Since all rows are same, hence value of the determinant is zero.

Ex. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Sol. Let $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying $R_1 \to R_1 - R_2 - R_3$ to Δ, we get

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Expanding along R_1, we obtain

$$\Delta = 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$= 2 c (a b + b^2 – bc) – 2 b (b c – c^2 – ac)$

$= 2 a b c + 2 cb^2 – 2 bc^2 – 2 b^2c + 2 bc^2 + 2 abc$

$= 4 abc$

Ex. Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Sol. $D = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$D = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$ $(R_1 \to R_1 + R_2 + R_3)$

$D = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$D = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$ $(C_3 \to C_3 - C_1 ; C_2 \to C_2 - C_1)$

$D = (a+b+c)^3$

Ex. Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

Sol. We have, $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$ (by property 5)

$= 0 + 0 = 0$ (using property 3 and property 4)

Ex. If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, then the value of k is-

Sol. Applying $(C_3 \to C_3 - C_1)$

$D = \begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 9 & 11 & 1 \end{vmatrix} = 0$ $(R_3 \to R_3 - R_2; R_2 \to R_2 - R_1)$

$\Rightarrow k - 1 = 0 \Rightarrow k = 1$

Ex. Prove that $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = m(x-a)(x-b)$

Sol. Using factor theorem,

Put $x = a$

$$D = \begin{vmatrix} a & a & a \\ m & m & m \\ b & a & b \end{vmatrix} = 0$$

Since R_1 and R_2 are proportional which makes $D = 0$, therefore $(x - a)$ is a factor of D.

Similarly, by putting $x = b$, D becomes zero, therefore $(x - b)$ is a factor of D.

$$D = \begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = \lambda(x-a)(x-b) \qquad \text{......(i)}$$

To get the value of λ, put $x = 0$ in equation **(i)**

$$\begin{vmatrix} a & a & 0 \\ m & m & m \\ b & 0 & b \end{vmatrix} = \lambda ab$$

$amb = \lambda ab \quad \Rightarrow \quad \lambda = m$

$\therefore \quad D = m(x-a)(x-b)$

MULTIPLICATION OF TWO DETERMINANTS

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D_1 is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D_1 = D^{n-1}$

Ex. Let α & β be the roots of equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ for $n \geq 1$. Evaluate the value of the determinant

$$\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}.$$

Sol. $$D=\begin{vmatrix}3 & 1+S_1 & 1+S_2\\ 1+S_1 & 1+S_2 & 1+S_3\\ 1+S_2 & 1+S_3 & 1+S_4\end{vmatrix}=\begin{vmatrix}1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2\\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3\\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \beta & \beta^2\end{vmatrix}\times\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & \beta\\ 1 & \alpha^2 & \beta^2\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \beta & \beta^2\end{vmatrix}^2=[(1-\alpha)(1-\beta)(\alpha-\beta)]^2$$

$$D=(\alpha-\beta)^2(\alpha+\beta-\alpha\beta-1)^2$$

$\because$ α & β are roots of the equation $ax^2+bx+c=0$

$\Rightarrow$ $\alpha+\beta=\dfrac{-b}{a}$ & $\alpha\beta=\dfrac{c}{a}$ $\Rightarrow$ $|\alpha-\beta|=\dfrac{\sqrt{b^2-4ac}}{|a|}$

$$D=\frac{(b^2-4ac)}{a^2}\left(\frac{a+b+c}{a}\right)^2=\frac{(b^2-4ac)(a+b+c)^2}{a^4}$$

Important Determinants

(i) $$\begin{vmatrix}0 & b & -c\\ -b & 0 & a\\ c & -a & 0\end{vmatrix}=0$$

(ii) $$\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ bc & ac & ab\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=(a-b)(b-c)(c-a)$$

(iii) $$\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$$

(iv) $$\begin{vmatrix}1 & 1 & 1\\ a^2 & b^2 & c^2\\ a^3 & b^3 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca)$$

(v) $$\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^4 & b^4 & c^4\end{vmatrix}=(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$$

APPLICATION OF DETERMINANTS IN GEOMETRY

(a) The lines : $a_1x + b_1y + c_1 = 0$ ......**(i)**

$a_2x + b_2y + c_2 = 0$ ......**(ii)**

$a_3x + b_3y + c_3 = 0$ ......**(iii)**

are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

This is the condition for consistency of three simultaneous linear equations in 2 variables.

(b) Equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(c) Area of a triangle whose vertices are (x_r, y_r) ; $r = 1, 2, 3$ is $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

If $D = 0$ then the three points are collinear.

(d) Equation of a straight line passing through points (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Ex. Find the area of the triangle whose vertices are (3, 8), (– 4, 2) and (5, 1).

Sol. The area of triangle is given by $\Delta = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$

$$= \frac{1}{2}\left[3(2-1) - 8(-4-5) + 1(-4-10)\right]$$

$$= \frac{1}{2}(3 + 72 - 14) = \frac{61}{2}$$

Singular & non singular matrix

A square matrix A is said to be singular or non-singular according as |A| is zero or non-zero respectively.

Cofactor matrix & adjoint matrix

Let $A = [a_{ij}]_n$ be a square matrix. The matrix obtained by replacing each element of A by corresponding cofactor is called as cofactor matrix of A, denoted as cofactor A. The transpose of cofactor matrix of A is called as adjoint of A, denoted as adj A.

i.e. if $A = [a_{ij}]_n$

then cofactor $A = [c_{ij}]_n$ when c_{ij} is the cofactor of a_{ij} $\forall$ i & j.

Adj $A = [d_{ij}]_n$ where $d_{ij} = c_{ji}$ $\forall$ i & j.

Properties of cofactor A and adj A

(a) $A \,.\, \text{adj } A = |A|\, I_n = (\text{adj } A)\, A$ where $A = [a_{ij}]_n$.

(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A.

In particular, for 3×3 matrix, $|\text{adj } A| = |A|^2$

(c) If A is a symmetric matrix, then adj A are also symmetric matrices.

(d) If A is singular, then adj A is also singular.

Ex. If $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 2 \\ 0 & -3 & 1 \end{bmatrix}$ and $B = [-1 \;\; 2 \;\; -1]$ find $B \cdot (\text{adj } A)$

Sol. The cofactors are

$c_{11} = +(4+6) = 10$ $\quad c_{12} = -(-1-0) = 1$

$c_{13} = +(3-0) = 3$ $\quad c_{21} = -(-1+9) = -8$

$c_{22} = +(2-0) = 2$ $\quad c_{23} = -(-6+0) = 6$

$c_{31} = +(-2-12) = -14$ $\quad c_{32} = -(4+3) = -7$

$c_{33} = +(8-1) = 7$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 10 & 1 & 3 \\ -8 & 2 & 6 \\ -14 & -7 & 7 \end{bmatrix}^T = \begin{bmatrix} 10 & -8 & -14 \\ 1 & 2 & -7 \\ 3 & 6 & 7 \end{bmatrix}$$

Then $\quad B \,.\, (\text{adj } A) = [-1 \;\; 2 \; -1] \begin{bmatrix} 10 & -8 & -14 \\ 1 & 2 & -7 \\ 3 & 6 & 7 \end{bmatrix}$

$= [-10 + 2 - 3 \quad 8 + 4 - 6 \quad 14 - 14 - 7]$

$= [-11 \quad 6 \quad -7]$

INVERSE OF A MATRIX (RECIPROCAL MATRIX)

Let A be a non-singular matrix. Then the matrix $\frac{1}{|A|}$ adj A is the multiplicative inverse of A (we call it inverse of A) and is denoted by A^{-1}.

We have $A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$

$$\Rightarrow \quad A\left(\frac{1}{|A|}\text{adj} A\right) = I_n = \left(\frac{1}{|A|}\text{adj} A\right) A, \text{ for A is non-singular}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} \text{ adj } A.$$

(i) The necessary and sufficient condition for existence of inverse of A is that A is non-singular.

(ii) A^{-1} is always non-singular.

(iii) If $A = \text{dia}\,(a_{11}, a_{22}, \ldots.., a_{nn})$ where $a_{ii} \neq 0\ \forall\ i$, then $A^{-1} = \text{diag}\,(a_{11}^{-1}, a_{22}^{-1}, \ldots., a_{nn}^{-1})$.

(iv) $(A^{-1})' = (A')^{-1}$ for any non-singular matrix A. Also adj $(A') = (\text{adj } A)'$.

(v) $(A^{-1})^{-1} = A$ if A is non-singular.

(vi) Let k be a non-zero scalar & A be a non-singular matrix. Then $(kA)^{-1} = \frac{1}{k} A^{-1}$.

(vii) $|A^{-1}| = \frac{1}{|A|}$ for $|A| \neq 0$.

(viii) Let A be a non-singular matrix. Then $AB = AC \Rightarrow B = C$ & $BA = CA \Rightarrow B = C$.

(ix) A is non-singular and symmetric $\Rightarrow A^{-1}$ is symmetric.

(x) $(AB)^{-1} = B^{-1} A^{-1}$ if A and B are non- singular.

(xi) In general $AB = 0$ does not imply $A = 0$ or $B = 0$. But if A is non-singular and $AB = 0$, then $B = 0$. Similarly B is non-singular and $AB = 0 \Rightarrow A = 0$. Therefore, $AB = 0 \Rightarrow$ either both are singular or one of them is 0.

Ex. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that A adj A = | A | I. Also find A^{-1}

Sol. We have $|A| = 1\,(16 - 9) - 3\,(4 - 3) + 3\,(3 - 4) = 1 \neq 0$

Now $A_{11} = 7, A_{12} = -1, A_{13} = -1, A_{21} = -3, A_{22} = 1, A_{23} = 0, A_{31} = -3, A_{32} = 0, A_{33} = 1$

Therefore adj $A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now
$$A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|.\, I$$

Also
$$A^{-1}\ \frac{1}{|A|}\ \text{adj}\ \ A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Ex. Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Using the equation, find A^{-1} .

Sol. We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now $A^2 - 4A + I = 0$

Therefore $A\,A - 4A = -I$

or $AA(A^{-1}) - 4\,A\,A^{-1} = -\,I\,A^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A\,(A\,A^{-1}) - 4I = -\,A^{-1}$

or $AI - 4I = -\,A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Ex. Find the inverse of the matrix, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$.

Sol. We have, $|A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = (14 - 12) = 12 \neq 0.$

So, A^{-1} exists.

The cofactors of the elements of $|A|$ are given by

$A_{11} = 7,$ $\qquad A_{12} = (-4) = 4.$

$A_{21} = -(-3) = 3,$ $\qquad A_{22} = 2.$

$\therefore$ $(\text{adj } A) = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

Hence, $A^{-1} = \dfrac{1}{|A|}\,.\,(\text{adj } A)$

$= \dfrac{1}{2}.\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & \frac{3}{2} \\ 2 & 1 \end{bmatrix}.$

Finding inverse using Elementry operations

(i) Using row transformations :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations,

Step I : Write A = IA and

Step II : Apply a sequence of row operation on A = IA till we get, I = BA.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary row operations on the matrix equation X = AB, we will apply these row operations simultaneously on X and on the first matrix A of the product AB on RHS.

Ex. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ by using elementary row transformations.

Sol. We know that A = I A

or, $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ (Applying $R_2 \to R_2 + R_1$)

$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} A$ (Applying $R_2 \to R_2 + 2R_3$)

$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -4 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$ (Applying $R_1 \to R_1 + (-2) R_2, R_3 \to R_3 + 2R_2$)

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$ (Applying $R_1 \to R_1 + 2R_3$)

Hence, $A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

(ii) Using column transformations :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary column operations,

Step I : Write A = AI and

Step II : Apply a sequence of column operations on A = AI till we get, I = AB.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary column operations on the matrix equation $X = AB$, we will apply these row operatdions simultaneously on X and on the second matrix B of the product AB on RHS.

CRAMER'S RULE (SYSTEM OF LINEAR EQUATIONS)

- Simultaneous linear equations
  - Consistent (at least one solution)
    - Exactly one solution or Unique solution
      - Trivial solution
        - All variable zero is the only solution
      - Non trivial solution
        - At least one non zero variable satisfies the system
    - Infinite solutions
  - Inconsistent (no solution)

(a) Equations involving two variables

(i)	Consistent Equations	:	Definite & unique solution (Intersecting lines)
(ii)	Inconsistent Equations	:	No solution (Parallel lines)
(iii)	Dependent Equations	:	Infinite solutions (Identical lines)

Let, $a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$ then :

(1) $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ $\Rightarrow$ Given equations are consistent with unique solution

(2) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ $\Rightarrow$ Given equations are inconsistent

(3) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ $\Rightarrow$ Given equations are consistent with infinite solutions

(b) Equations Involving Three variables

Let $a_1x + b_1y + c_1z = d_1$ ....**(i)**

$a_2x + b_2y + c_2z = d_2$ ....**(ii)**

$a_3x + b_3y + c_3z = d_3$ ....**(iii)**

Then, $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$, $z = \dfrac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

(i) If $D \neq 0$ and atleast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations is consistent and has unique non trivial solution.

(ii) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent and has trivial solution only.

(iii) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent and has infinite solutions.

Note that In case $\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_1x + b_1y + c_1z &= d_2 \\ a_1x + b_1y + c_1z &= d_3 \end{aligned}\right\}$ (Atleast two of d_1 , d_2 & d_3 are not equal)

$D = D_1 = D_2 = D_3 = 0$. But these three equations represent three parallel planes. Hence the system is inconsistent.

(iv) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

(c) Homogeneous system of linear equations

Let $a_1x + b_1y + c_1z = 0$ ....**(i)**

$a_2x + b_2y + c_2z = 0$ ....**(ii)**

$a_3x + b_3y + c_3z = 0$ ....**(iii)**

$\Rightarrow$ $D_1 = D_2 = D_3 = 0$

$\therefore$ The system always possesses atleast one solution $x = 0, y = 0, z = 0$, which is called Trivial solution, i.e. this system is always consistent.

Check value of D

$D \neq 0$ — **Unique Trivial** solution

$D = 0$ — **Trivial & Non-Trivial** solutions (infinite solutions)

Note that if a given system of linear equations has Only Zero solutions for all its variables then the given equations are said to have TRIVIAL SOLUTION.

Also, note that if the system of equations $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$; $a_3x + b_3y + c_3 = 0$

is always consistent then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ but converse is not true.

Ex. Find the nature of solution for the given system of equations :

$x + 2y + 3z = 1$; $2x + 3y + 4z = 3$; $3x + 4y + 5z = 0$

Sol. $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

Now, $D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 0 & 4 & 5 \end{vmatrix} = 5$

$\because$ $D = 0$ but $D_1 \neq 0$ Hence no solution.

Ex. Find the value of λ, if the following equations are consistent :

$$x + y - 3 = 0; (1+\lambda)x + (2+\lambda)y - 8 = 0; x - (1+\lambda)y + (2+\lambda) = 0$$

Sol. The given equations in two unknowns are consistent, then $\Delta = 0$

i.e. $\begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$

$\therefore$ $\begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & 3\lambda - 5 \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$

$\Rightarrow$ $(5+\lambda) - (3\lambda - 5)(-2-\lambda) = 0$ $\Rightarrow$ $3\lambda^2 + 2\lambda - 5 = 0$

$\therefore$ $\lambda = 1, -5/3$

Ex. If x, y, z are not all simultaneously equal to zero, satisfying the system of equations

$\sin(3\theta)\,x - y + z = 0;\ \cos(2\theta)x + 4y + 3z = 0;\ 2x + 7y + 7z = 0$, then find the values of $\theta (0 \le \theta \le 2\pi)$.

Sol. Given system of equations is a system of homogeneous linear equations which posses non-zero solution set, therefore $D = 0$.

$\Rightarrow$ $D = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix}$ $\Rightarrow$ $D = \begin{vmatrix} \sin 3\theta & -1 & 0 \\ \cos 2\theta & 4 & 7 \\ 2 & 7 & 14 \end{vmatrix}$ $(C_3 \to C_3 + C_2)$

$D = \begin{vmatrix} \sin 3\theta & -1 & 0 \\ \cos 2\theta - 1 & 0.5 & 0 \\ 2 & 7 & 14 \end{vmatrix}$ $(R_2 \to R_2 - \frac{R_3}{2})$

$D = 14\left(\frac{\sin 3\theta}{2} + \cos 2\theta - 1\right)$

$\because$ $D = 0$

$\therefore$ $\sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow$ $3\sin\theta - 4\sin^3\theta = 4\sin^2\theta$ $\Rightarrow$ $(\sin\theta)(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow$ $(\sin\theta)(2\sin\theta - 1)(2\sin\theta + 3) = 0$ $\Rightarrow$ $\sin\theta = 0$; $\sin\theta = \frac{1}{2}$; $\sin\theta = -\frac{3}{2}$

$\sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi$; $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$; $\sin\theta = -\frac{3}{2} \Rightarrow$ no solution.

$\Rightarrow$ $\theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$

TIPS & FORMULAS

1. Minors

The minor of any element of determinant is the determinant of the elements which remain after deleting the row & the column in which the given element stands.

For example, the minor of a_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ & the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order threee will have "9 minors".

2. Cofactors

If M_{ij} represents the minor of the element belonging to i^{th} row and j^{th} column then the cofactor of that element :
$C_{ij} = (-1)^{i+j} . M_{ij}$

Important Notev :

Consider $\quad \Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Let A_1 be cofactor of a_1, B_2 be cofactor of b_2 and so on, then,

(i) $a_1A_1 + b_1B_1n + c_1C_1 = a_1A_1 + a_2A_2 + a_3A_3 = \Delta$

(ii) $a_2A_1 + b_2B_1 + c_2C_1 = b_1A_1 + b_2A_2 + b_3A_3 = O$

3. Properties of Determinants

(a) The value of a determinants remains unaltered, if the rows & columns are interchanged.

(b) If any two rows (or columns) of a determinant be interchanged the value of determinant is changed in sign only. e.g.

Let $\quad D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \& \quad D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$

(c) If a determinant has any two rows (or columns) identical, then its value is zero.

(d) If all the elements of any row (or columns) be multiplied by the same number, then the determinant is multiplied by that number.

(e) $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(f) The value of a determinant is not altered by adding to the elements of any row (or column) the multiples of the corresponding elements of any other row (or column) e.g.

Let $\quad D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$D' = \begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 & c_1 + mc_2 \\ a_2 & b_2 & c_2 \\ a_3 + na_1 & b_3 + nb_2 & c_3 + nc_1 \end{vmatrix}. \qquad \text{Then } D' = D$$

Note : While applying this property atleast one row (or column) must remain unchanged.

(g) If the elements of a determinant Δ are rational function of x and two rows (or columns) become identical when $x = a$, then $x - a$ is a factor of Δ.

Again, if r rows become identical when a is substituted for x, then $(x-a)^{r-1}$ is a factor of Δ.

(h) If $D(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$, where f_r, g_r, h_r ; $r = 1, 2, 3$ are three differential function.

then $$\frac{d}{dx}D(x) = \begin{vmatrix} f'_1 & f'_2 & f'_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g'_1 & g'_2 & g'_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h'_1 & h'_2 & h'_3 \end{vmatrix}$$

4. Multiplication of two Determinants

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1l_1 + b_2l_2 & a_1m_1 + b_1m_2 \\ a_2l_1 + b_2l_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D' is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D' = D^{n-1}$.

5. Special Determinants

(a) Symmetric Determinant

Elements of a determinant are such that $a_{ij} = a_{ji}$.

e.g. $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

(b) Skew Symmetric Determinant

If $a_{ij} = -a_{ji}$ then the determinant is said to be a skew symmetric determinant. here all the principal diagonal elements are zero. The value of a skew symmetric determinant of odd order is zero and of even order is perfect square.

e.g. $$\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$$

(c) Other Important Determinants

(i) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(ii) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

(iii) $$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

(iv) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$$

6. System of Equation

(a) System of equation involving two variable

$a_1x+b_1y+c_1=0$
$a_2x+b_2y+c_2=0$

- Consistent (System of equation has solution)
  - unique solution: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ or $a_1b_2-a_2b_1 \neq 0$ (Equations represents intersecting lines)
  - Infinite solution: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (Equations represents coincident lines)
- Inconsistent (System of equation has no solution): $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ (Equations represents parallel disjoint lines)

If $\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, then $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}$

or $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}; y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$

(b) System of Equations Involving three Variables

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

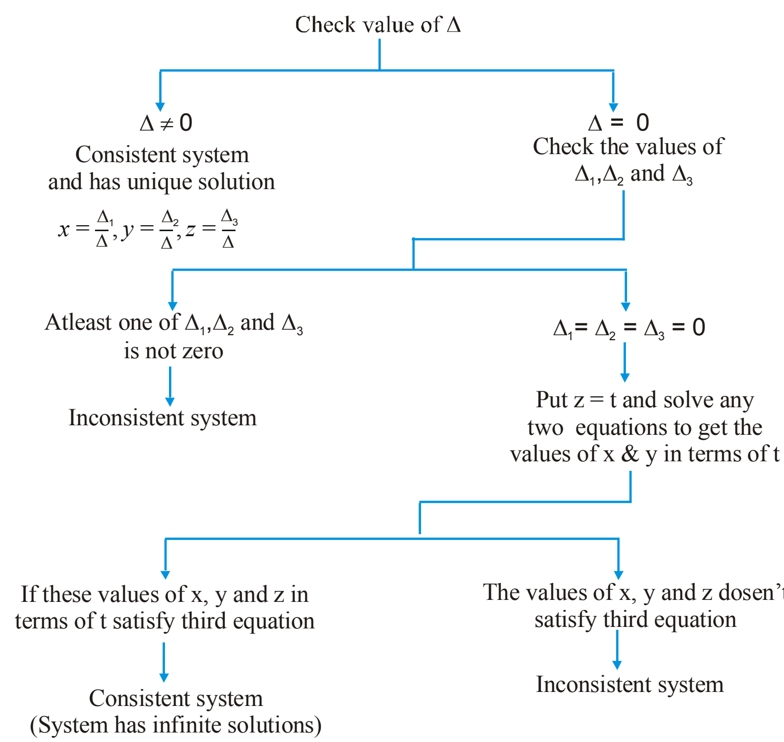
$a_3x + b_3y + c_3z = d_3$

To solve this system we first define following determinants

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \qquad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Now following algorithm is used to solve the system.



Note

(i) **Trival Solution :** In the solution set of system of equation if all the variables assumes zero, then such a solution set is called Trival solution otherwise the solution is called non-trival solution.

(ii) If $d_1 = d_2 = d_3 = 0$ then system of linear equation is known as system of Homogeneous linear equation which always posses atleast one solution (0, 0, 0).

(iii) If system of homogeneous linear equation posses non-zero/non-trival solution then $\Delta = 0$. In such case given system has infinite solutions.