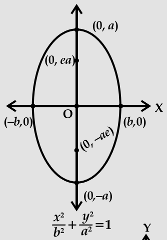
CHAPTER : 12 CONIC SECTIONS (PART-I)

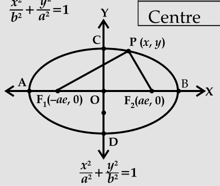
Conic Sections Part-I

Ellipse

- An ellipse is the set of all points in a plane, that the sum of their distances from two fixed points in the plane is constant.
- The two fixed points are called the 'foci' of the ellipse.
- The midpoint of line segment joining foci is called the 'centre' of the ellipse.
- The line segment through the foci of the ellipse is called' major axis'.
- The line segment through centre & perpendicular to major axis is called minor axis.



Forms of ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
	$a>b$	$a>b$
Equation of major axis	$y=0$	$x=0$
Length of major axis	$2a$	$2a$
Equation of Minor axis	$x=0$	$y=0$
Lenght of Minor axis	$2b$	$2b$
Directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$
Equation of latus rectum	$x=\pm ae$	$y=\pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Centre	(0, 0)	(0, 0)



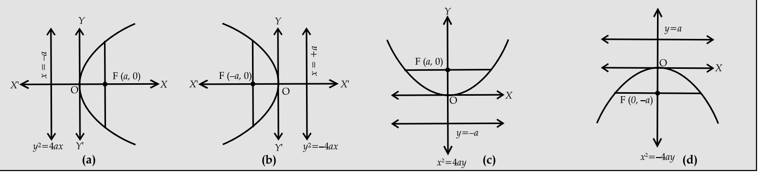
Here, $a>b$ and $b^2 = a^2(1–e^2)$, $e<1$

Parabola

- A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point in the plane. Fixed line is called 'directrix' of parabola. Fixed point F is called the 'focus'. A line through focus & perpendicular to directrix is called 'axis'. Point of intersection of parabola with axis is called 'vertex'.

Main facts about the parabola

Forms of Parabolas	$y^2=4ax$	$y^2=-4ax$	$x^2=4ay$	$x^2=-4ay$
Axis	$y=0$	$y=0$	$x=0$	$x=0$
Directix	$x=-a$	$x=a$	$y=-a$	$y=a$
Vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Lenght of latus rectum	$4a$	$4a$	$4a$	$4a$
Equations of latus rectum	$x=a$	$x=-a$	$y=a$	$y=-a$



Definition

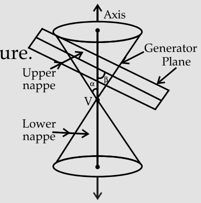
Circles, ellipses, parabolas and hyperbolas are known as conic sections because they can be obtained as intersections of plane with a double napped right circular cone α. From the given figure.

(i) Section will represent circle, if β =90°

(ii) Section will represent an Ellipse, if $\alpha<\beta<\pi/2$

(iii) Section will represent a parabola if if $\alpha=\beta$

(iv) Section will represent a hyperbola if $0 \le \beta<\alpha$



Circle

A circle is a set of all points in a plane that are equidistant from a fixed point in the plane. The fixed point is called the 'centre' of the circle and the distance from the centre to a point on the circle is called the 'radius' of the circle.

The equation of a circle with centre (h, k) and the radius r is $(x–h)^2+(y–k)^2=r^2$

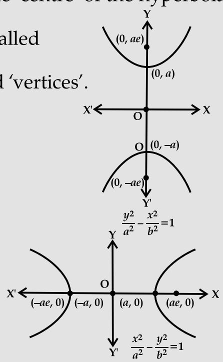
The general equation of circle is $x^2+y^2+2gx+2fy+c=0$

its centre is $(–g, –f)$ and radius $r=\sqrt{g^2+f^2-c}$

Hyperbola

- A hyperbola is the set of all points in a plane, that the difference of whose distances from two fixed points in the plane is a constant.
- The two fixed points are called the 'foci' of the hyperbola.
- The mid-point of the line segment joining the foci is called the 'centre' of the hyperbola.
- The line through the foci is called 'transverse axis'.
- Line through centre and perpendicular to transverse axis is called 'conjugate axis'.
- Points at which hyperbola intersects transverse axis are called 'vertices'.

Forms of the hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Equation of transverse axis	$y=0$	$x=0$
Equation of conjugate axis	$x=0$	$y=0$
Length of transverse axis	$2a$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm ae)$
Equation of latus rectum	$x=\pm ae$	$y=\pm ae$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Centre	(0, 0)	(0, 0)



Here, $b^2 = a^2(e^2–1)$, $e>1$

CHAPTER : 12 CONIC SECTIONS (PART-II)

Conic Sections Important Facts Part-II

Ellipse

- The standard euation of ellipse is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $b^2=a^2(1-e^2)$
- Eccentricity of ellipse $e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{(minor\ axis)^2}{(major\ axis)^2}}$
- The equation of tangent at a point (x_1, y_1) on the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$
- The point of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$
- Equations of normal on the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at a point (x_1, y_1)

  In point form $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2-b^2$

  In parametric form $(a\sec\theta)x-(b\operatorname{cosec}\theta)y=a^2-b^2$

  In slope form $y=mx\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$

Hyperbola

- Eccentricity of Hyperbola $e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{(conjugate\ axis)^2}{(transverse\ axis)^2}}$
- Length of latusrectum $= 2a(e^2-1)$
- The Conjugate hyperbola of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
- A line $y = mx+c$ will be the tangent on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if $c=\pm\sqrt{a^2m^2-b^2}$
- The equation of tangent at a point (x_1, y_1) is given by $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$
- The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, at $(a\sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$
- Equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point (x_1, y_1)
- In point form $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$
- In parametric form $ax\cos\theta + by\cot\theta = a^2+b^2$
- In slope form $y = mx\mp\frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$

Parabola

- The equation of the tangent slope m to the parabola $y^2=4ax$ at point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, $y=mx+\frac{a}{m}$
- The equation of the tangent slope m to the parabola $(y-k)^2=4a(x-h)$ is : $(y-k)=m(x-h)+\frac{a}{m}$
- The coordinate of the point of contact are $\left(h+\frac{a}{m^2}, k+\frac{2a}{m}\right)$
- The equation of normal to the parabola $y^2=4ax$ at a point (x_1, y_1) is : $y-y_1=\frac{-y_1}{2a}(x-x_1)$
- The equation of normal of slope m to the parabola $y^2=4ax$ at a point $(am^2, -2am)$ is : $y=mx-2am-am^3$
- The equation of normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$ is given by $y+tx=2at+at^3$

Circle

- Standard equation of a circle $(x-h)^2+(y-k)^2=r^2$
- When the coordinates of the end points of diameter are (x_1y_1) and (x_2,y_2), then the equation of the circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$
- General equation of the circle is $x^2+y^2+2gx+2fy+c=0$
- The above equation is called general equation of the circle, whose centre is $(-g, -f)$ and radius is $\sqrt{g^2+f^2-c}$.

CHAPTER

12 CONIC SECTIONS

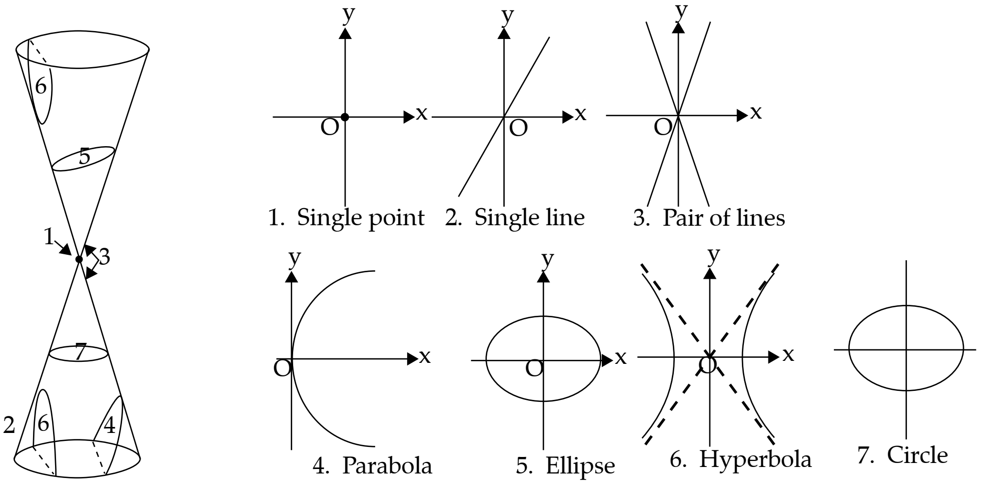
Chapter Objectives

Sections of a cone : circles, ellipse, parabola, hyperbola, a point a straight line and a pair of intersecting lines as a degenerated case of a conic section. Standard equations and simple properties of parabola, ellipse and hyperbola, Standard equation of a circle.

STUDY MATERIAL

I. Concept Clarified

1. Section of a Cone



Point -1 is a single point which is intersection of lines called vertex.

The fixed vertical lines are called axes.

The fixed angle is called vertex angle. and rotating line is called generator.

The vertex divides the cone in two parts (nappes)

When we cut an object with plane in many slices we get many parallel cross sections. Similarly if we cut a right circular cone in slices we will get so many cross sections. The possible cross sections from cone are circle, parabola, ellipse and hyperbola.

In the above diagram section **7** is called circle, section **4** is called parabola. section **5** is called ellipse and section **6** is called hyperbola.

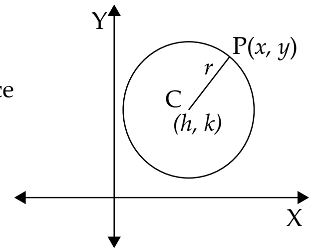
2. Circle

A circle is defined as locus of a point which moves in a plane such that the distance from a fixed point in that plane is always constant. The fixed point is called center of the circle and constant distance is called radius of the circle.

- **Standard equations of circle**

Let C(h, k,) be the centre of a circle and P(x, y) be any point on the circumference of the circle, then

$$(x-h)^2 + (y-k)^2 = r^2$$

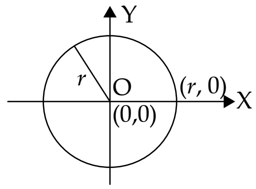


Cases :

1. When the centre of the circle coincide with origin then, $h = 0, k = 0$

So, equation of the circle is

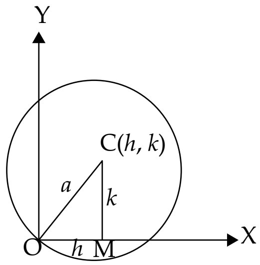
$$x^2 + y^2 = r^2$$



2. When the circle passes through the origin.

When the origin lies on the circumference of the circle then equation of circle is

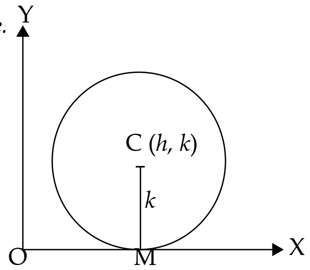
$$x^2 + y^2 - 2hx - 2ky = 0$$



3. When the circle touches x- axis

When the circle touches X-axis then radius of the circle r will be equal to k, *i.e.* $r = k$, in this case

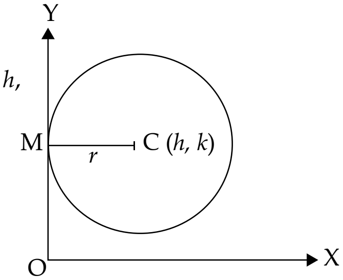
$$x^2 + y^2 - 2hx - 2ry + h^2 = 0$$



4. When the circle touches y-axis

When the circle touches Y-axis then radius of the circle r will be equal to h, *i.e.* $r = h$, in this case

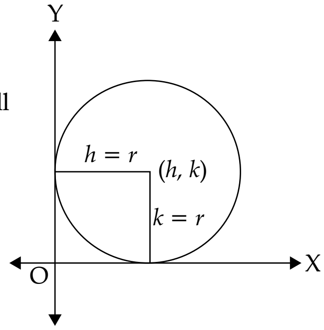
$$x^2 + y^2 - 2rx - 2ky + k^2 = 0$$



5. When the circle touches both the axes

When the circle touch X-axis as well as Y-axis then radius of the circle r will be equal to h and k *i.e.* $r = h = k$, in this case :

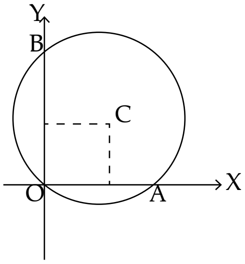
$$x^2 + y^2 - 2rx - 2ry + a^2 = 0$$



6. Equation of circle cutting intercept on X - axis and Y - axis as $(a, 0)$ and $(0, b)$ respectively

When the circle passes through origin and cuts the intercepts a and b on respectively then

$$x^2 + y^2 - ax - by = 0$$



7. When the coordinates of the end points of diameter are (x_1, y_1) and (x_2, y_2), then equation of circle is :

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

8. When the circle passes through three non-collinear points - (x_1, y_1), (x_2, y_2) and (x_3, y_3) then equation of circle is

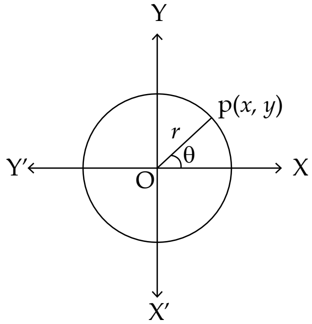
$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

9. **Parametric form of circle**

$x = r\cos\theta$ and $y = r\sin\theta$ for circle $x^2 + y^2 = r^2$

and, $x = h + r\cos\theta$ and $y = k + r\sin\theta$ for circle $(x - h)^2 + (y - k)^2 = r^2$,

where $0 \le \theta \le 2\pi$



General Equation of a circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

The above equation is called general equation of circle, whose centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

Cases :

- When $g^2 + f^2 - c > 0$, then the radius of circle is real and hence circle is real and possible to draw.
- When $g^2 + f^2 - c = 0$, then the radius of circle is zero and it is called as point circle.
- When $g^2 + f^2 - c < 0$, then the radius of circle is imaginary and hence the circle is not possible to draw.

Tangent to a circle at a given point

Trick : $x^2 = xx_1$, $y^2 = yy_1$, $x = \dfrac{x_1 + x}{2}$, $y = \dfrac{y_1 + y}{2}$ in the given conictamily of equation

The equation of tangent at the point $P(x_1, y_1)$ to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

The equation of tangent of slope form

The equation of tangent with slope 'm' on a circle $x^2 + y^2 = r^2$ is given by :

$$y = mx \pm a\sqrt{1 + m^2}$$

Where the coordinate of point of contact is $\left(\pm\dfrac{rm}{\sqrt{1+m^2}}, \mp\dfrac{r}{\sqrt{1+m^2}}\right)$

Normal to a circle at give point

The equation formula at point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ *is*

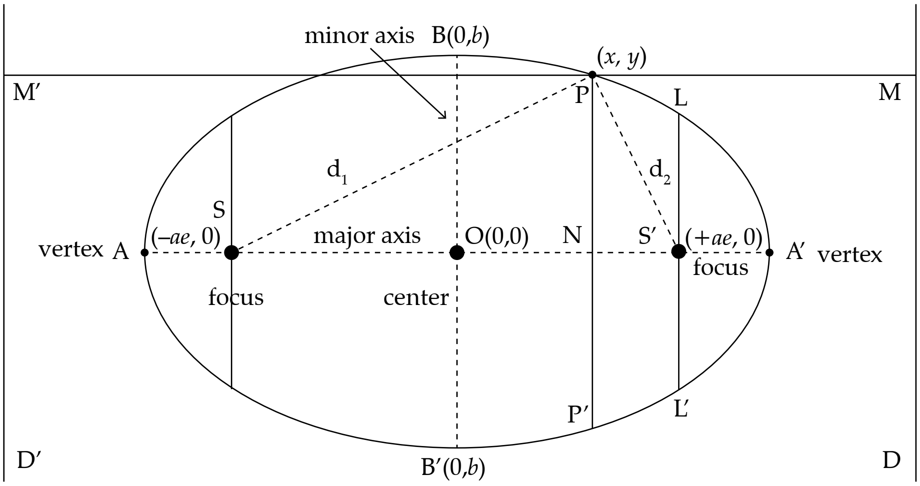
$(y_1 + f)x - (x_1 + g)y + (gy_1 - fx_1) = 0$

Properties of Circle

1. Circles having equal radii are congruent.
2. Circles having different radii are similar.
3. The central angle which intercepts an arc is the double of any inscribed angle that intercepts the same arc.
4. The radius perpendicular to a chord bisects the chord.
5. The chords equidistant from the centre are equal in length.
6. A tangent to a circle is at a right angle to the radius at the point of contact.
7. Two tangents drawn on a circle from a point outside are equal in length.
8. The angle subtended at the centre of a circle by its circumference is equal to four right angles.
9. The circumference of two different circles is proportional to their corresponding radii.
10. Arcs of the same circle are proportional to their corresponding angles.
11. Radii of the same circle or equal circles are equal.
12. Equal chords have equal circumferences.
13. The diameter of a circle is the longest chord.
14. Equal circles have equal circumferences.

3. Ellipse

Ellipse is defined as a locus of a point in a plane which moves in a plane in such a way that the ratio of its distance from a fixed point called focus in the same plane to its distance from a fixed line called directrix is always constant and less than unity.



The constant ratio is called eccentricity and denoted by letter 'e'.

$$\frac{SP}{PM} = e$$

Standard equation of Ellipse

The standard equation of ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(1 - e^2),\ e > 0$$

Some important definitions

1. **Vertices**

   The points A and A′ in the above figure of ellipse are called vertices of the ellipse where the curve meets the line joining the foci S and S′. The coordinates of A′ and A are $(a, 0)$ and $(-a, 0)$ respectively.

2. **Major and Minor axes**

   In the above figure the distances AA′ = $2a$ and BB′=2b are called the major and minor axes of the ellipse.

3. **Foci**

   In the above figure of ellipse S' and S are called the foci of the ellipse. The coordinate of foci of ellipse are $(ae, 0)$ and $(–ae, 0)$ respectively.

4. Directrix

In the above figure MD and M′D′ are called the directrix of the ellipse. The equation of directrix of the ellipse are $x = a/e$ and $x = -a/e$ respectively.

5. Centre

The centre of the ellipse is the point of intersection of major and minor axes. In the above diagram it is denoted by letter O and its coordinates are (0,0).

6. Eccentricity of the ellipse

The equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and $b^2 = a^2(1-e^2)$, $e<1$

$$e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{(\text{semi minor axis})^2}{(\text{semi major axis})^2}}$$

7. Ordinate and double ordinate

Suppose P be a point on the ellipse and PN be the perpendicular on the major axis SS′ in such a way that PN produced meet ellipse at point P′, then PN is called ordinate and PNP′ is called double ordinate of P.

8. Latus - recturm

It a double ordinate passing through the focus. In the above diagram LS′ L′ is called the Latus - recturm of the ellipse and LS′ is called the semi Latus-recturm.

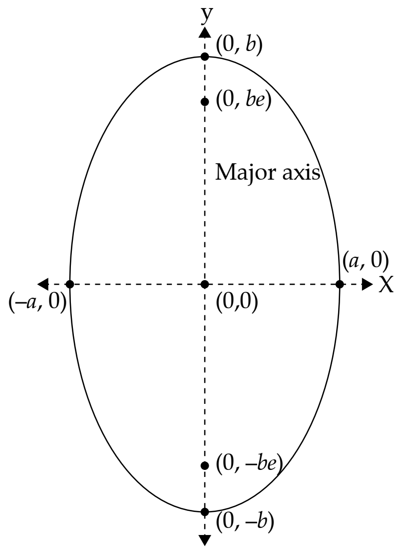
$$\text{Latus - recturm} = \frac{2b^2}{a} = 2a(1-e^2)$$

➢ Equation of ellipse in other forms

When $a < b$ then the equation of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{where } e=\sqrt{1-\frac{a^2}{b^2}}$$

1. Coordinate of the vertices are $(0, -b)$ and $(0, b)$
2. Coordinate of foci are $(0, be)$ and $(0, -be)$
3. Length of major axis is $2b$
4. Length of minor axis is $2a$
5. Equation of major axis is $x=0$,
6. Equation of minor axis is $y=0$,
7. Equation of directrices is $y=\pm\frac{b}{e}$
8. Length of latusrectum is $\frac{2a^2}{b}$



➢ Special form

If the centre of the ellipse is at point (h, k) and the directions of the axes are parallel to coordinate axes then

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

If one shifts the origin at (h, k) without rotating the coordinate axes then, $x = X + h$ and $y = Y + k$

So, $$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$$

➢ Parametric form of ellipse

When $x = a\cos\theta$ and $y = b\sin\theta$ taken together are called parametric equation of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{where } \theta \text{ is parameter.}$$

➢ Condition for a line to be tangent to ellipse

Let $y = mx + c$ be the tangent drawn on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $c=\pm\sqrt{a^2m^2+b^2}$

➢ Equation of tangent of ellipse

The equations of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $y= mx\pm\sqrt{a^2m^2+b^2}$ and the coordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}},\mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$.

The equation of tangent at a point (x_1, y_1) is given by $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

➢ Equation of normal on ellipse

1. **In point form**

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

2. **In parametric form** $(a \sec\theta)x - (b \operatorname{cosec}\theta)\, y = a^2 - b^2$
3. **In slope form**

$$y = mx \pm \frac{m(a^2 \quad b^2)}{\sqrt{a^2 \quad b^2m^2}}$$

Where m is slope of the normal to the ellipse.

➢ Equation of pair of tangent to Ellipse

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

➢ Properties of Ellipse

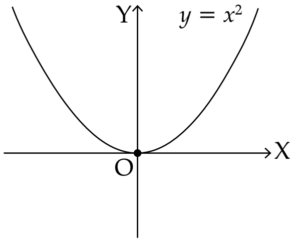
1. All ellipses have two focal points or foci.
2. The sum of the distances from every point on the ellipse to the two foci is a constant.
3. All ellipses have a centre and a major and minor axis.
4. All ellipses have eccentricity values greater than or equal to zero, and less than one.

4. Parabola

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point in the plane is always equal to its distance from a fixed straight line in the same plane.

If focus of a parabola is S (x, y) and equation of the directrix is $ax + by + c = 0$, then the equation of the parabola is $(a^2 + b^2)[(x - x_1)^2 + (y - y_1)^2] = (ax + by + c)^2$

The graph of the quadratic function $y = x^2$ is called a **parabola.** The basic shape can be seen by plotting a few points with $x = -3, -2, -1, 0, 1, 2$ and 3.

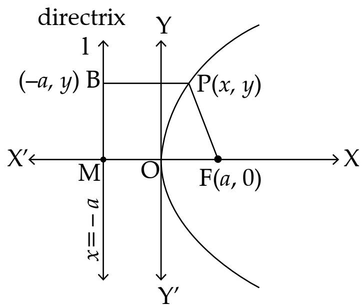


The above graph is the **basic parabola.** As we will see, all parabolas can be obtained from this one by translations, rotations, reflections and stretching.

The basic parabola has the following properties :

- It is symmetric about the y-axis, which is an **axis of symmetry.**
- The minimum value of y occurs at the origin, which is a minimum turning point. It is also known as the **vertex** of the parabola.
- The arms of the parabola continue indefinitely.

- **Definitions related to Parabola**



1. **Vertex**

   It is defined as the point of intersection of parabola with axis.

2. **Centre**

   It is defined as the point which bisects every chord of conic passing through it.

3. **Focal Chord**

   Any chord passing through focus is known as focal chord.

4. **Double ordinate**

   It is defined as any chord perpendicular to the axis of parabola is called double ordinate of the parabola.

5. **Latusrectum**

   It is defined as a double ordinate passing through the focus of the parabola and perpendicular to the axis.

6. **Focal distance**

   It is distance of a point P(x, y) from the focus F is called the focal distance of the point P.

- **Other forms of a Parabola**

If the vertex of a parabola is at a point A(h, k) and its latusrectum is of length $4a$ then its equation is

1. $(y - k)^2 = 4a\ (x - h)$, its axis is parallel to OX *i.e* parabola open rightward.
2. $(y - k)^2 = -4a\ (x - h)$, its axis is parallel to OX′ *i.e* parabola open leftward.
3. $(x - h)^2 = 4a\ (y - k)$, its axis parallel to OY *i.e* parabola open upward.
4. $(x - h)^2 = -4a\ (y - k)$, its axis parallel to OY′ *i.e* parabola open downward.
5. The general equation of a parabola whose axis is parallel to X-axis is $x = ay^2 + by + c$ and the general equation of parabola whose axis is parallel to Y-axis is $y = ax^2 + bx + c$.

- **Position of a Point**

The point(x_1, y_1) lies inside, outside or on the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 <,>,= 0$.

- **Equation of Tangent**

A line which touches the parabola at one point is called tangent of a parabola.

1. **Point form :**

   The equation of tangent to the parabola $y^2 = 4ax$ at *a* point (x_1, y_1) is given by $yy_1 = 2a(x + x_1)$

2. **Slope form :**

   - The equation of the tangent of slope m to the parabola $y^2 = 4ax$ at point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is $y = mx + \frac{a}{m}$

   - The equation of the tangent of slope m to the parabola $(y - k^2) = 4a(x - h)$ is given by $(y - k) = m(x - h) + \frac{a}{m}$

   - The coordinates of the point of contact are $\left(h + \frac{a}{m^2}, k + \frac{2a}{m}\right)$.

3. **Parametric form**

   The equation of the tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is given by

   $$yt = x + at^2$$

4. **Point of intersection of tangents at any two points**

   The coordinates of point of intersection of tangents at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $\{at_1t_2, a(t_1 + t_2)\}$

- **Equation of normal in Different form**

1. **Point Form**

   The equation of the normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is given by

   $$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

2. **Slope form**

The equation of the normal of the slope m to parabola $y^2 = 4ax$ at the point $(am^2, -2am)$ is given by

$$y = mx - 2am - am^3$$

3. **Parametric form**

The equation of normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is given by :

$$y + tx = 2at + at^3$$

➢ Properties of Parabola

1. The eccentricity of any parabola is 1.
2. The parabola is symmetric about its axis.
3. The axis is perpendicular to directrix.
4. The axis passes through the vertex and the focus.
5. The vertex is the midpoint of the focus and the point of intersection of directrix and axis.
6. Tangents drawn to any point on the directrix are perpendicular

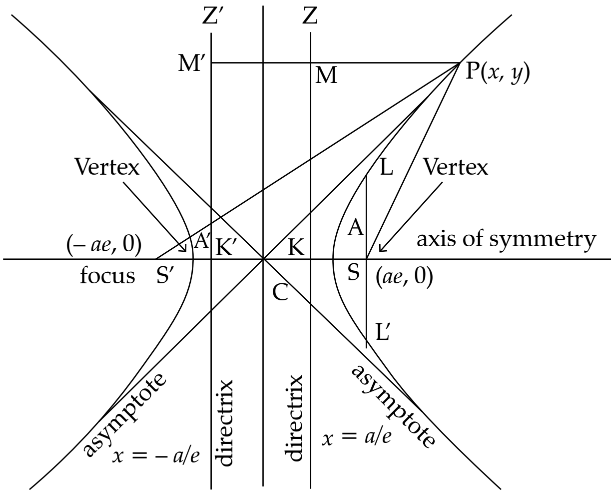
5. Hyperbola

A hyperbola is a locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed line (directrix) is always constant which is always greater than unity.

The equation of hyperbola is given as :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Here $b^2 = a^2(e^2 - 1), e > 0$



➢ Some important definitions

1. **Vertices**

The points A and A′ in the above figure of hyperbola are called vertices of hyperbola where the curve meets the line joining the foci S and S′. The coordinates of A and A′ are $(a, 0)$ and $(-a, 0)$ respectively.

2. **Foci**

In the above figure of hyperbola S and S′ are called the foci of the hyperbola. The coordinate of foci of hyperbola are $(ae, 0)$ and $(-ae, 0)$ respectively.

3. **Directrix**

In the above figure ZK and Z′K′ ae called the directrix of the hyperbola. The equation of directrix of the hyperbola are $x = a/e$ and $x = -a/e$ respectively.

4. **Centre**

The centre of hyperbola is the point of intersection of every chord of hyperbola. In the above diagram it is denoted by letter C and its coordinates are (0, 0).

5. **Eccentricity of hyperbola**

The equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and $b^2 = a^2(e^2 - 1)$

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}}$$

6. **Length of laturspectrum of hyperbola**
Length of laturspectrum = $2a(e^2 - 1)$

➢ Conjugate of Hyperbola

The hyperbola whose traverse and conjugate axis are respectively the conjugate and traverse axis of given hyperbola is called the conjugate hyperbola of given hyperbola.

The conjugate hyperbola of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

➢ Parametric Coordinates and parametric equations

Suppose equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If $x = a \sec \theta$ and $y = b \tan \theta$, then $\sec^2 \theta - \tan^2 \theta = 1$.

The coordinates $(a \sec \theta, b \tan \theta)$ are known as the parametric coordinates and $x = a \sec \theta$ and $y = b \tan \theta$ are called parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

➢ Condition for a line to be tangent to Hyperbola

Let $y = mx + c$ be the tangent drawn on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c = \pm\sqrt{a^2m^2 - b^2}$

This is required condition of tangency.

➢ Equation of tangent of Hyperbola

1. **In Slope form**

The equation of tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, are $y = mx \pm \sqrt{a^2m^2 - b^2}$ and the coordinates of the points of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$.

2. **In Point form**

The equation of tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, at a point (x_1, y_1) is given by $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

3. **In parametric form**

The equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec \theta, b \tan \theta)$ is given by $\left(\frac{\sec\theta}{a}\right)x - \left(\frac{\tan\theta}{b}\right)y = 1$

➢ Equation of normal on Hyperbola

1. **In point form**

The equation of normal in point form at point (x_1, y_1) is given by $\frac{a^2x}{x_1} + \frac{b^2y_1}{y} = a^2 + b^2$

2. **In parametric form**

The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} =$ at $(a \sec \theta, b \tan \theta)$ is given by $ax \cos \theta + by \cot \theta = a^2 + b^2$

3. **In slope form**

The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in slope form is given by $y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}$ and the coordinates of the point of contact are $\left(\pm\frac{a^2}{\sqrt{a^2 - b^2m^2}}, \mp\frac{b^2m}{\sqrt{a^2 - b^2m}}\right)$

➢ Equation of pair of tangent drawn to Hyperbola

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

➢ Asymptotes of Hyperbola

An asymptotes to a curve is a straight line, at a finite distance from the origin, to which the tangent to the curve tends as the point of contact goes to infinity.

The equation of asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$ or $\frac{x}{a}\pm\frac{y}{b}=0$

➢ Rectangular Hyperbola

A hyperbola whose asymptotes are perpendicular to each other is called rectangular hyperbola. Its equation is given by $xy = c^2$

In parametric form, it is represented as $x = ct$, $y=\frac{c}{t}$, where t ∈ R→{0}

➢ Properties of Hyperbola

1. If lengths of the traverse axis and the conjugate axis are equal then the hyperbola is said to be rectangular or equilateral hyperbola.
2. The eccentricity of rectangular hyperbola is $\sqrt{2}$ and the length of its latusrectum is same as its conjugate or traverse axis.
3. Through a given point four normal(real or imaginary) can be drawn to a hyperbola.
4. The tangent drawn at any point bisects the angle between the lines, joining the point to the foci, whereas the normal bisects the supplementary angle between the lines.
5. A rectangular hyperbola circumscribing a triangle passes through the orthocenter of this triangle.

II. Important Formulae

1. Standard equation of a circle $(x-h)^2 + (y-k)^2 = r^2$
2. When the coordinates of the end points of diameter are (x_1, y_1) and (x_2, y_2), then the equation of the circle is $(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$
3. General equation of the circle is

   $$x^2 + y^2 + 2gx + 2fy + c = 0$$

   The above equation is called general equation of the circle, whose centre is $(-g, -f)$ and radius is $\sqrt{g^2+f^2-c}$.
4. The standard equation of ellipse is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $b^2 = a^2(1-e^2)$, $e > 0$
5. Eccentricity of ellipse $e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{(\text{semi minor axis})^2}{(\text{semi major axis})^2}}$
6. The equation of tangent at a point (x_1, y_1) on the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$
7. The point of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b}}\right)$, when the equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $y=mx\pm\sqrt{a^2m^2+b^2}$
8. Equations of normal on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
   - In point form $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$
   - In parametric form $(a \sec\theta)\, x - (b \operatorname{cosec}\theta)\, y = a^2 - b^2$
   - In slope form $y=mx\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b\ m}}$
9. The equation of the tangent of slope m to the parabola $y^2 = 4ax$ at point $\left(\frac{a}{m^2},\frac{2a}{m}\right)$ is $y=mx+\frac{a}{m}$
10. The equation of the tangent of slope m to the parabola $(y-k)^2 = 4a(x-h)$ is $(y-k)=m(x-h)+\frac{a}{m}$.

11. The coordinate of the point of contact are $\left(h+\frac{a}{m^2}, k+\frac{2a}{m}\right)$.

12. The equation of normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

13. The equation of normal of slope m to the parabola $y^2 = 4ax$ at the point $(am^2, -2am)$ is $y = mx - 2am - am^3$

14. The equation of normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is given by $y + tx = 2at + at^3$

15. Eccentricity of Hyperbola $e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}}$

16. Length of latusrectum of hyperbola $= 2a\,(e^2 - 1)$

17. The conjugate hyperbola of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

18. A line $y = mx+c$ will be the tangent on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if $c=\pm\sqrt{a^2m^2-b^2}$

19. The equation of tangent at a point (x_1, y_1) is given by $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

20. The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, at $(a \sec\theta, b \tan\theta)$ is $\left(\frac{\sec\theta}{a}\right)x-\left(\frac{\tan\theta}{b}\right)y=1$

21. Equations of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

- In Point form $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$
- In parametric form $ax \cos\theta + by \cot\theta = a^2 + b^2$
- In slope form $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$