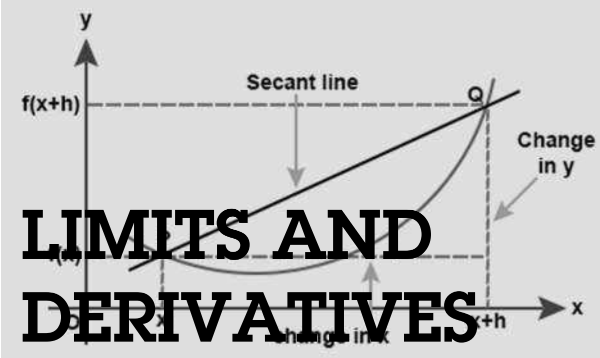
CHAPTER

13 LIMITS AND DERIVATIVES



Chapter Objectives

> *Definition of Limits; Limits of polynomials and rational functions, trigonometric, exponential and logarithmic functions; Algebra of limits; Sandwich theorem; some standard limits; Derivative introduced as rate of change both as that of distance function and geometrically; Derivatives of sum, difference, product and quotient of function; Derivatives of polynomials and trigonometric functions.*

TOPIC-1
Limit And Its Fundamentals

Quick Review

- **Definition of Limit**

  Let $y = f(x)$ be a function of x. If at $x = a$, $f(x)$ takes indeterminate form, then we consider the value of the function which is very near to a. If these values tend to a definite unique number as x tends to a then the unique number, so obtained is called the limit of $f(x)$ at $x = a$ and is written as $\lim_{x \to a} f(x)$.

  OR

  If $f(x)$ approaches to a real number l, when x approaches to a *i.e.*, if $f(x) \longrightarrow l$ when $x \longrightarrow a$, then l is called the limit of the function $f(x)$. In symbolic form, it on be written as—

$$\lim_{x \to a} f(x) = l$$

- **Left hand and right hand limit.**

  A real number l, is the left hand limit of function $f(x)$ at $x = a$, if the value of $f(x)$ can be made as close as l, at point closed to a and on the left of a. Symbolically,

$$\text{L.H.L} = \lim_{x \to a^+} f(x) = l_1$$

  A real number l_2 is the right hand limit of function $f(x)$ at $x = a$, if the values of $f(x)$ can be made as close as l_2 at points closed to a and on the right of a symbolically,

$$\text{R.H.L.} = \lim_{x \to a^+} f(x) = l_2$$

TIPS...

- *The students must not think that $\frac{dy}{dx}$ is a fraction i.e. dy divides by dx, for dy and dx have no meaning according to ever definition $\frac{dy}{dx}$ is merely a symbol for the derivative of y, which denotes the limiting value of fraction $\frac{\delta y}{\delta x}$.*
- *When the derivative of a function is evaluated directly by using the definition, then this method is called 'differentiation from the first principle'. In evaluating derivatives by definition, standard formulae of derivatives are not be used.*
- *Always remember for $0 < |x| < \frac{\pi}{2}$; $\cos x < \frac{\sin x}{x} < 1$*
- *Learn all the standard derivatives, to make the calculation fast.*
- *Limit is used when we have to find value of a function near to some value.*

- **Method to find left hand and right hand limit**

Step I For left hand limit, write the given function as $\lim_{x \to a-} f(x)$ and for right hand limit, write the given function as $\lim_{x \to a^+} f(x)$.

Step II For left hand limit, put $x = a - h$ and change the limit $x \to a^-$ by $h \to 0$. Then limit obtained in step I in $\lim_{h\to 0} f(a-h)$.

For right hand limit, put $x = a + h$ and change the limit $x \to a^+$ by $h \to 0$. Then, Limit obtained in step I is $\lim_{h\to 0} f(a+h)$

Step III Simplify the result obtained in step II *i.e.*, $\lim_{h\to 0} f(a-h)$ or $\lim_{h\to 0} f(a+h)$.

➢ **Existence of limit**

If the right hand limit and left hand limit coincide, then we say that limit exists and their common value is called the limit of $f(x)$ at $x = a$ and is denoted by $\lim_{x\to a} f(x)$.

➢ **Algebra of limits**

Let 'f' and 'g' be two real function with common domain D, such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists, then,

(i) Limit of sum of two function is sum of the limits of the function *i.e.*,

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

(ii) Limit of difference of two functions is difference of the limits of the function *i.e.*,

$$\lim_{x\to a}(f-g)(x) = \lim_{x\to 0} f(x) - \lim_{x\to a} g(x)$$

(iii) Limit of product of two functions is product of the limits of the function *i.e.*,

$$\lim_{x\to 0}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

(iv) Limit of quotient of two functions is quotient of the limits of the function *i.e.*,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \text{ where } \lim_{x\to a} g(x) \neq 0 \cdot$$

TRICKS...

- *Remember some standard limits of following forms.*

  (i) $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ *(ii)* $\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1$

  (iii) $\lim_{x\to 0}\frac{\sin^{-1}x}{x} = 1$ *(iv)* $\lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1$

- $\lim_{x\to a}\frac{x^n - a^n}{x - a} = na^{n-1}$ *is true, if n is any rational number and a > 0.*

- *To find the differential coefficient of x^n, decrease the index of x by 1 and multiply the result by the original index.*

- *We can generalise the formula of sum or difference for more that two functions, i.e.*

  $$\frac{d}{dx}[f_1(x) \pm f_2(x) \pm f_3(x) \pm .... \pm f_n(x)]$$

  $$= \frac{d}{dx}\{f_1(x)\} \pm \frac{d}{dx}\{f_2(x)\} \pm \frac{d}{dx}\{f_3(x)\} \pm .... \pm \frac{d}{dx}\{f_n(x)\}$$

- *To find the derivative by first principle, revise the following trigonometric identities.*

  $$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

  $$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

  $$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

  $$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

(v) Limit of product of a constant and on function is the product of that constant and limit of the function *i.e.*,

$$\lim_{x\to a}\{c\cdot f(x)\} = c \lim_{x\to 0} f(x), \text{ where 'c' is a constant.}$$

➢ **Limit of polynomial function**

Let $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx_n$ be a polynomial function.

then,

$$\lim_{x\to a} f(x) = \lim_{x\to a}[a_0 + a_1x + a_2x^2 + ... + a_nx^n]$$

$$= a_0 + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + ... + a_n\lim_{x\to a} x^n$$

$$= a_0 + a_1a + a_2a_2 + ... + a_na^n = f(a).$$

➢ **Limit of Rational Function**

A function f is said to be a rational functional if $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomial functions such that $h(x) \neq 0$. $x \to a$

(a) $$\lim_{x\to a} f(x) = \lim_{x\to a}\frac{g(x)}{h(x)} = \frac{g(a)}{h(a)}$$

If $g(a) = 0$ and $h(a) = 0$ *i.e.*, this is of the form $\frac{0}{0}$, then factor $(x - a)$ of $g(x)$ and $h(x)$ are determined and then $(x - a)$ cancelled out.

Let,
$$g(x) = (x - a)\, p(x)$$
$$h(x) = (x - a)\, q(x)$$

Then,

$$\lim_{x\to a} f(x) = \lim_{x\to a}\frac{g(x)}{h(x)} = \lim_{x\to a}\frac{(x-a)p(x)}{(x-a)q(x)} \qquad (x \neq a)$$

$$= \lim_{x\to a}\frac{p(x)}{q(x)}$$

$$= \frac{p(a)}{q(a)}$$

(b) For any positive integer n,

$$\lim_{x\to a}\frac{x^n - a^n}{x - a} = n\, a^{n-1}$$

- **Limits of Trigonometric, exponential and logarithmic Functions.**

  To find the limits of trigonometric functions, we use the following theorems—

  (i) Let f and g be two real valued functions with the same domain, such that $f(x) \leq g(x)$ for all x in domain in definition. For some a, if both limit exist, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

  (ii) Sandwich Theorem—Let f, g and h be real functions, such that $f(x) \leq g(x) \leq h(x)$ for all x in the common domain in definition. For some real number a, if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = 1$, then $\lim_{x\to a} g(x) = 1$.

- **Some Standard Limits**

  (i) $\lim_{x\to 0}\frac{x^n - a^n}{x - a} = na^{n-1}$

  (ii) $\lim_{x\to 0}\frac{\sin x}{x} = 1$

  (iii) $\lim_{x\to 0}\frac{\tan x}{x} = 1$

  (iv) $\lim_{x\to a}\frac{\sin(x-a)}{x-a} = 1$

  (v) $\lim_{x\to a}\frac{\tan(x-a)}{x-a} = 1$

  (vi) $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

  (vii) $\lim_{x\to 0}\frac{a^x - 1}{x} = \log_e a \neq 0, a > 1$

  (viii) $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$

(ix) $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

➢ **Note :**

- $x \to a^-$ is read as x tends to 'a' from left and it means that x is very close to 'a' but it is always less than a.
- $x \to a^+$ is read as x tends to 'a' from right and it means that x is very close to 'a' but it is always greater than 'a'.
- $x \to a$ is read as x tends to 'a' and it means that x is very close to a but it is not equal to 'a'.
- Left hand limit and right hand limit of a constant function is the constant itself. *e.g.*,
  $\lim_{x\to 1^-} 3=3,\ \lim_{x\to 3^+} 4=4.$
- Some factorization formulae which we use in finding limit of a function are—
  (i) If $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.
  (ii) $a^2 - b^2 = (a - b)(a + b)$
  (iii) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
  (iv) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
  (v) $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a+b)(a - b)$.
- The result $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$ is also true for any rational number 'n' and positive 'a'.
- The domain of exponential function $f(x) = e^x$ is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- The domain of logarithmic function $f(x) = \log_e x$ is $(0, \infty)$ and its range is $(-\infty, \infty)$.

Multiple Choice Questions **(1 mark each)**

[A] Objective Type Questions

Q. 1. $\lim_{x\to\pi}\frac{\sin x}{x-\pi}$ **is equal to :**

(a) 1 **(b)** 2

(c) –1 **(d)** –2

[NCERT Exemplar Q. 54, Page 242]

Ans. Correct option : (C)

Given, $\lim_{x\to\pi}\frac{\sin x}{\pi-x}=\frac{\sin(\pi-x)}{-(\pi-x)}=-1$

$\left[\because \lim_{x\to 0}\frac{\sin x}{\pi-x}=1 \text{ and } \pi-x\to 0 \Rightarrow x\to\pi\right]$

Q. 2. $\lim_{x\to 0}\frac{x^2\cos x}{1-\cos x}$ **is equal to :**

(a) 2 **(b)** $\frac{2}{3}$

(c) $\frac{-3}{2}$ **(d)** 1

[NCERT Exemplar Q. 55, Page 242]

Ans. Correct option : (A)

Given, $\lim_{x\to 0}\frac{x^2\cos x}{1-\cos x}=\lim_{x\to 0}\frac{x^2\cos x}{2\sin^2 x\frac{x}{2}}$

$\left[\because 1-\cos x = 2\sin^2\frac{x}{2}\right]$

$$=\lim_{x\to 0}\frac{\frac{x^2}{4}\times 4\cos x}{2\sin^2\frac{x}{2}}$$

$$=\frac{\lim_{x\to 0\Rightarrow\frac{x}{2}\to 0}\left(\frac{x}{2}\right)^2 .2\cos x}{\sin^2\frac{x}{2}}$$

$= 2\cos 0° = 2 \times 1 = 2$ $\left[\because \lim_{x\to 0}\frac{x}{\sin x}=1\right]$

Q. 3. $\lim_{x\to 0}\frac{(1+x)^n-1}{x}$ **is equal to :**

(a) n **(b)** 1

(c) $-n$ **(d)** 0

[NCERT Exemplar Q. 56, Page 242]

Ans. Correct option : (A)

Given, $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=\lim_{x\to 0}\frac{(1+x)^n-(1)^n}{(1+x)-1}$

$$=\lim_{1+x\to 1}\frac{(1+x)^n-(1)^n}{(1+x)-(1)} = n(1)^{n-1} = n$$

$\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$

Q. 4. $\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1}$ **is equal to :**

(a) 1 **(b)** $\dfrac{m}{n}$

(c) $-\dfrac{m}{n}$ **(d)** $\dfrac{m^2}{n^2}$

[NCERT Exemplar Q. 57, Page 242]

Ans. Correct option : (B)

Given, $\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1} = \lim\limits_{x\to 1}\dfrac{\dfrac{x^m-1^m}{x-1}}{\dfrac{x^n-1^n}{x-1}} = \dfrac{m(1)^{m-1}}{n(1)^{n-1}}=\dfrac{m}{n}$

$\left[\because \lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}\right]$

Q. 5. $\lim\limits_{\theta\to 0}\dfrac{1-\cos 4\theta}{1-\cos 6\theta}$ **is equal to :**

(a) $\dfrac{4}{9}$ **(b)** $\dfrac{1}{2}$

(c) $\dfrac{-1}{2}$ **(d) –1**

[NCERT Exemplar Q. 58, Page 242]

Ans. Correct option : (A)

Given, $\lim\limits_{\theta\to 0}\dfrac{1-\cos 4\theta}{1-\cos 6\theta} = \lim\limits_{\theta\to 0}\dfrac{2\sin^2 2\theta}{2\sin^2 3\theta}$

$[\because 1-\cos\theta = 2\sin^2\dfrac{\theta}{2}]$

$= \lim\limits_{\theta\to 0}\dfrac{\sin^2 2\theta}{\sin^2 3\theta} = \lim\limits_{\theta\to 0}\left[\dfrac{\sin 2\theta}{\sin 3\theta}\right]^2$

$= \lim\limits_{\substack{\theta\to 0\\2\theta\to 0\\3\theta\to 0}}\left[\dfrac{\dfrac{\sin 2\theta}{2\theta}\times 2\theta}{\dfrac{\sin 3\theta}{3\theta}\times 3\theta}\right]^2 = \left[\dfrac{2\theta}{3\theta}\right]^2 = \dfrac{4}{9}$

Q. 6. $\lim\limits_{x\to 0}\dfrac{\operatorname{cosec} x-\cot x}{x}$ **is equal to :**

(a) $\dfrac{-1}{2}$ **(b) 1**

(c) $\dfrac{1}{2}$ **(d) 1**

[NCERT Exemplar Q. 59, Page 243]

Ans. Correct option : (C)

Given, $\lim\limits_{x\to 0}\dfrac{\operatorname{cosec}x-\cos+x}{x} = \lim\limits_{x\to 0}\dfrac{\dfrac{1}{\sin x}-\dfrac{\cos x}{\sin x}}{x}$

$= \lim\limits_{x\to 0}\dfrac{1-\cos x}{x\sin x}$

$= \lim\limits_{x\to 0}\dfrac{2\sin^2\dfrac{x}{2}}{x.2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}$

$= \lim\limits_{x\to 0}\dfrac{\sin\dfrac{x}{2}}{x\cos\dfrac{x}{2}} = \lim\limits_{x\to 0}\dfrac{\tan\dfrac{x}{2}}{x} = \lim\limits_{x\to 0}\dfrac{\tan\dfrac{x}{2}}{2\times\dfrac{x}{2}}$

$= \dfrac{1}{2}\times 1=\dfrac{1}{2}$ $\left[\because \lim\limits_{x\to 0}\dfrac{\tan x}{x}=1\right]$

Q. 7. $\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt{x+1}-\sqrt{1-x}}$ **is equal to :**

(a) 2 **(b) 0**

(c) 1 **(d) –1**

[NCERT Exemplar Q. 60, Page 243]

Ans. Correct option : (C)

Given, $\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt{x+1}-\sqrt{1-x}}$

$= \lim\limits_{x\to 0}\dfrac{\sin x\left[\sqrt{x+1}+\sqrt{1-x}\right]}{\left(\sqrt{x+1}-\sqrt{1-x}\right)\left(\sqrt{x+1}+\sqrt{1-x}\right)}$

$= \lim\limits_{x\to 0}\dfrac{\sin x\left[\sqrt{x+1}-\sqrt{1-x}\right]}{x+1-1+x}$

$= \lim\limits_{x\to 0}\dfrac{\sin x\left[\sqrt{x+1}+\sqrt{1-x}\right]}{2x}$

$= \dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{\sin x}{x}\left[\sqrt{x+1}+\sqrt{1-x}\right]$

Taking limit, we get $= \dfrac{1}{2}\times 1\times\left[\sqrt{0+1}-\sqrt{0-1}\right]$

$= \dfrac{1}{2}\times 1\times 2=1$

Q. 8. $\lim\limits_{x\to\pi/4}\dfrac{\sec^2 x-2}{\tan x-1}$ **is :**

(a) 3 **(b) 1**

(c) 0 **(d) 2**

[NCERT Exemplar Q. 61, Page 243]

Ans. Correct option : (D)

Given, $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sec^2 x-2}{\tan x-1} = \lim\limits_{x\to\frac{\pi}{4}}\dfrac{1+\tan^2 x-2}{\tan x-1}$

$= \lim\limits_{x\to\frac{\pi}{4}}\dfrac{\tan^2 x-1}{\tan x-1}$

$= \lim\limits_{x\to\frac{\pi}{4}}\dfrac{(\tan x-1)(\tan x+1)}{(\tan x-1)}$

$= \lim\limits_{x\to\frac{\pi}{4}}(\tan x+1) = \tan\dfrac{\pi}{4}+1 = 1 + 1 = 2.$

Q. 9. $\lim\limits_{x\to 1}\dfrac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$ **is equal to :**

(a) $\dfrac{1}{10}$ **(b)** $\dfrac{-1}{10}$

(c) 1 **(d) None of these**

[NCERT Exemplar Q. 62, Page 243]

Ans. Correct option : (B)

Given, $\lim_{x\to1}\frac{(\sqrt{x}-1)(2x-3)}{2x^2+3x-2x-3}$

$=\lim_{x\to1}\frac{(\sqrt{x}-1)(2x-3)}{x(2x+3)-1(2x+3)} = \lim_{x\to1}\frac{(\sqrt{x}-1)(2x-3)}{(x-1)(2x+3)}$

$=\lim_{x\to1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)(2x-3)}{(x-1)(\sqrt{x}+1)(2x+3)}$

$=\lim_{x\to1}\frac{(x-1)(2x-3)}{(x-1)(\sqrt{x}+1)(2x+3)} = \lim_{x\to1}\frac{2x-3}{(\sqrt{x}+1)(2x+3)}$

Taking limit, we have,

$=\frac{2(1)-3}{(\sqrt{1}+1)(2\times1+3)} = \frac{-1}{2\times5}=\frac{-1}{10} \qquad (x\neq1)$

Q. 10. If $f(x)\begin{cases}\frac{\sin[x]}{[x]}, & [x]\neq0\\ 0, & [x]=0\end{cases}$, where [,] denotes the greatest integer function, then $\lim_{x\to0}f(x)$ is equal to :

(a) 1 **(b) 0**

(c) –1 **(d) Does not exist**

[NCERT Exemplar Q. 63, Page 243]

Ans. Correct option : (D)

Given, $f(x)=\begin{cases}\frac{\sin[x]}{[x]}, & [x]\neq0\\ & [x]=0\end{cases}$

LHL $=\lim_{x\to0^-}\frac{\sin[x]}{[x]} = \lim_{h\to0}\frac{\sin[0-h]}{[0-h]}$

$=\lim_{h\to0}\frac{-\sin[-h]}{[-h]}=-1$

RHL $=\lim_{h\to0^+}\frac{\sin[x]}{[x]}=\lim_{h\to0}\frac{\sin[0+h]}{[0+h]}$

$=\lim_{h\to0}\frac{\sin[h]}{[h]}=1$

LHL ≠ RHL; So, limit does not exist.

Q. 11. If $f(x)=\begin{cases}x^2-1, & 0<x<2\\ 2x+3, & 2\le x<3\end{cases}$, then the quadratic equation whose roots are $\lim_{x\to2^-}f(x)$ and $\lim_{x\to2^+}f(x)$ is :

(a) $x^2 - 6x + 9 = 0$ **(b) $x^2 - 7x + 8 = 0$**

(c) $x^2 - 14x + 49 = 0$ **(d) $x^2 - 10x + 21 = 0$**

[NCERT Exemplar Q. 65, Page 243]

Ans. Correct option : (D)

Given, $f(x)=\begin{cases}x^2-1, 0<x<2\\ 2x+3, 2\le x<3\end{cases}$

$\therefore \lim_{x\to2^-} = f(x) \lim_{x\to2^-}(x^2-1) = \lim_{h\to0}[(2-h)^2-1]$

$=\lim_{h\to0}(4-h^2-4h-1) = \lim_{h\to0}[h^2-4h+3]=3$

and $\lim_{x\to2^+}f(x)=\lim_{x\to2^+}(2x+3)$

$=\lim_{h\to0}[2(2+h)+3]=7$

Therefore, the quadratic equation whose roots are 3 and 7 is $x^2-(3+7)x+3\times7=0$

i.e. $x^2-10x+21=0$

Q. 12. $\lim_{x\to0}\frac{\tan2x-x}{3x-\sin x}$ is equal to :

(a) 2 **(b) $\frac{1}{2}$**

(c) $\frac{-1}{2}$ **(d) $\frac{1}{4}$**

[NCERT Exemplar Q. 66, Page 244]

Ans. Correct option : (B)

Given, $f(x)=\lim_{x\to0}\frac{\tan2x-x}{3x-\sin x} = \lim_{x\to0}\frac{x\left[\frac{\tan2x}{x}-1\right]}{x\left[3-\frac{\sin x}{x}\right]}$

$=\lim_{\substack{x\to0\\ \therefore 2x\to0}}\frac{\left[\frac{\tan2x}{2x}\times2-1\right]}{\left[3-\frac{\sin x}{x}\right]} = \frac{1.2-1}{3-1}=\frac{2-1}{2}=\frac{1}{2}$

[B] Fill in the Blanks

Q. 1. If $f(x)=\frac{\tan x}{x-\pi}$, then $\lim_{x\to\pi}f(x)$ = ...

[NCERT Exemplar Q. 77, Page 245]

Q. 2. $\lim_{x\to0}\left(\sin mx\cot\frac{x}{\sqrt{3}}\right)$ = 2, then m = ...

[NCERT Exemplar Q. 78, Page 245]

Q. 3. $\lim_{h\to0}\frac{(3+h)}{3}$ = ...

[NCERT Exemplar Q. 80, Page 245]

Answers

1. $-\frac{\tan(\pi-x)}{-(\pi-x)}$

2. $\frac{2\sqrt{3}}{3}$

3. $1+6y+1=0.$

Very Short Answer Type Questions (1 mark each)

Evaluate the following limits (Q. 1 to 6)

Q. 1. Evaluate $\lim_{x\to 3}\frac{x^2-9}{x-3}$

[NCERT Exemplar, Q. 1, Page 239]

Sol.

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{x^2-(3)^2}{x-3}$$

$$= \lim_{x\to 3}\frac{(x+3)(x-3)}{(x-3)}$$

$$= \lim_{x\to 3}(x+3) \quad [x\neq 3]$$

$$= 3+3=6$$

Q. 2. Evaluate $\lim_{x\to 1/2}\frac{4x^2-1}{2x-1}$

[NCERT Exemplar, Q. 2, Page 239]

Sol.

$$\lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1} = \lim_{x\to\frac{1}{2}}\frac{(2x)^2-(1)^2}{2x-1}$$

$$= \lim_{x\to\frac{1}{2}}\frac{(2x+1)(2x-1)}{(2x-1)}$$

$$= \lim_{x\to\frac{1}{2}}(2x+1) \quad \left[x\neq\frac{1}{2}\right]$$

$$= 2\times\frac{1}{2}+1=1+1=2$$

Q. 3. Evaluate $\lim_{x\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

[NCERT Exemplar, Q. 3, Page 239]

Sol. $\lim_{x\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$

$$\lim_{x\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$\lim_{x\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$\lim_{x\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \quad [h\neq 0]$$

$$\lim_{x\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

Q. 4. Evaluate $\lim_{x\to 0}\frac{(x+2)^{1/3}-2^{1/3}}{x}$

[NCERT Exemplar, Q. 4, Page 239]

Sol. $\lim_{x\to 0}\frac{(x+2)^{1/3}-2^{1/3}}{x}$

Let $x+2=y$

$$\therefore \lim_{x\to 0}\frac{y^{1/3}-2^{1/3}}{y-2}$$

$$= \frac{1}{3}(2)^{\frac{1}{3}-1} \quad \left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$= \frac{1}{3}(2)^{-\frac{2}{3}}$$

$$= \frac{1}{3(2)^{2/3}}$$

Q. 5. Evaluate $\lim_{x\to 0}\frac{\sin^2 2x}{\sin^2 4x}$

[NCERT Exemplar, Q. 16, Page 240]

Sol. Given, $\lim_{x\to 0}\frac{\sin^2 2x}{\sin^2 4x} = \lim_{x\to 0}\frac{\sin^2 2x}{(2\sin 2x\cos 2x)^2}$

$$= \lim_{x\to 0}\frac{\sin^2 2x}{4\sin^2 2x\cos^2 2x}$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

$$= \lim_{x\to 0}\frac{1}{4\cos^2 2x} = \frac{1}{4}$$

$[\because \cos 0 = 1]$

Q. 6. Evaluate $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$

[NCERT Exemplar, Q. 17, Page 240]

Sol.

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2}$$

$(1-\cos 2x = 2\sin^2 x)$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2}$$

$$= 2\lim_{x\to 0}\frac{\sin^2 x}{x^2}$$

$$= 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$

$$\left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$= 2\times 1 = 2$$

Short Answer Type Questions (2 marks each)

Q. 1. Evaluate $\lim_{x\to 0}\frac{(1+x)^6-1}{(1+x)^2-1}$

[NCERT Exemplar, Q. 5, Page 239]

Sol. $\lim_{x\to 0}\frac{(1+x)^6-1}{(1+x)^2-1}$

Let $1 + x = y$

$\therefore \lim_{y\to 1}\frac{y^6-1}{y^2-1}$

$\Rightarrow = \lim_{y\to 1}\frac{(y^2-1)(y^4+y^2+1)}{(y^2-1)}$

$= \lim_{y\to 1}(y^4+y^2+1)$ $[y \neq 1]$

$= 1 + 1 + 1 = 3$

Q. 2. Evaluate $\lim_{x\to 0}\frac{(2+x)^{5/2}-(a+2)^{5/2}}{x-a}$

[NCERT Exemplar, Q. 6, Page 239]

Sol. $\lim_{x\to a}\frac{(2+x)^{5/2}-(a+2)^{5/2}}{x-a}$

Let $2 + x = y$

When $x \to a$ the $y \to (a + 2)$

$\therefore \lim_{y\to(a+2)}\frac{y^{5/2}-(a+2)^{5/2}}{y-(a+2)}$

$= \frac{5}{2}(a+2)^{\frac{5}{2}-1}$ $\left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$

$= \frac{5}{2}(a+2)^{3/2}$

Q. 3. Evaluate $\lim_{x\to 0}\frac{x^4-\sqrt{x}}{\sqrt{x}-1}$

[NCERT Exemplar, Q. 1, Page 240]

Ans. $\lim_{x\to 1}\frac{x^4-\sqrt{x}}{\sqrt{x}-1}$

$\lim_{x\to 1}\frac{\sqrt{x}(x^{7/2}-1)}{\sqrt{x}-1}$

Let $\sqrt{x} = y$

$\therefore \lim_{y\to 1}\frac{y(y^7-1)}{y-1}$ $\left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{x-1}\right]$

$\therefore \lim_{y\to 1}y \lim_{y\to 1}\frac{y^7-1}{y-1}$

$= 1 \times 7(1)^6$

$= 7$

Q. 4. Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}$

[NCERT Exemplar, Q. 11, Page 240]

Sol. $\lim_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}$

$= \lim_{x\to 0}\frac{\sqrt{1+x^3}-\sqrt{1-x^3}}{x^2}\cdot\frac{\sqrt{1+x^3}+\sqrt{1-x^3}}{\sqrt{1+x^3}+\sqrt{1-x^3}}$

$= \lim_{x\to 0}\frac{(1+x^3)-(1-x^3)}{x^2(\sqrt{1+x^3}+\sqrt{1-x^3})}$

$= \lim_{x\to 0}\frac{1+x^3-1+x^3}{x^2(\sqrt{1+x^3}+\sqrt{1-x^3})}$

$= \lim_{x\to 0}\frac{2x^3}{x^2(\sqrt{1+x^3}+\sqrt{1-x^3})}$

$= \lim_{x\to 0}\frac{2x}{(\sqrt{1+x^3}+\sqrt{1-x^3})}$

$= 0$

Q. 5. Evaluate $\lim_{x\to -3}\frac{x^3+27}{x^5+243}$ **[NCERT Exemplar, Q. 12, Page 240]**

Sol. Given, $\lim_{x\to -3}\frac{x^3+27}{x^5+243}$

$= \lim_{x\to -3}\frac{\frac{x^3+27}{x+3}}{\frac{x^5+243}{x+3}}$

$= \lim_{x\to -3}\frac{\frac{x^3-(-3)^3}{x-(-3)}}{\frac{x^5-(-3)^5}{x-(-3)}} = \frac{\lim_{x\to -3}\frac{x^3-(-3)^3}{x-(-3)}}{\lim_{x\to -3}\frac{x^5-(-3)^5}{x-(-3)}}$

$\left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$

$= \frac{3(-3)^2}{5(-3)^4} = \frac{1}{15}$

Q. 6. Find the value of n, if $\lim_{x\to 2}\frac{x^n-2^n}{x-2} = 80$**, $n \in$ N.**

[NCERT Exemplar, Q. 14, Page 240]

Sol. Given, $\lim_{x\to 2}\frac{x^n-2^n}{x-2} = 80$

$\Rightarrow n(2)^{n-1} = 80$

$\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$

$\Rightarrow n(2)^{n-1} = 5 \times 16$

$\Rightarrow n \times 2^{n-1} = 5 \times (2)^4$

$\Rightarrow n \times 2^{n-1} = 5 \times (2)^{5-1}$

$\therefore n = 5$

Q. 7. Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 7x}$

[NCERT Exemplar, Q. 15, Page 240]

Sol. Given, $\lim\limits_{x\to 0}\dfrac{\dfrac{\sin 3x}{3x}3x}{\dfrac{\sin 7x}{7x}.7x} = \dfrac{\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}}{\lim\limits_{x\to 0}\dfrac{\sin 7x}{7x}}\left(\dfrac{3x}{7x}\right)$

$$= \frac{3}{7}\cdot\frac{\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}}{\lim\limits_{x\to 0}\dfrac{\sin 7x}{7x}}$$

$\left[\because \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1\right]$

$$= \frac{3}{7}\times\frac{1}{1} = \frac{3}{7}$$

Q. 8. Evaluate $\lim\limits_{x\to 0}\dfrac{2\sin x-\sin 2x}{x^3}$ **[NCERT Exemplar, Q. 18, Page 240]**

Sol. Given, $\lim\limits_{x\to 0}\dfrac{2\sin x-\sin 2x}{x^3} = \lim\limits_{x\to 0}\dfrac{2\sin x-2\sin x\cos x}{x^3}$ $[\because \sin 2x = 2\sin x\cos x]$

$$= \lim_{x\to 0}\frac{2\sin x(1-\cos x)}{x^3}$$

$$= 2\lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)$$

$$= 2.1\lim_{x\to 0}\frac{1-\cos x}{x^2}$$ $\left[\because \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1\right]$

$$= 2\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = 4\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{4\times\frac{x^2}{4}}$$ $\left[(1-\cos x)=2\sin^2\dfrac{x}{2}\right]$

$$= \frac{4}{4}\lim_{x\to 0}\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2 = \lim_{x\to 2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = 1$$ Apply the limit

Q. 9. Evaluate $\lim\limits_{x\to\frac{\pi}{3}}\dfrac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$ **[NCERT Exemplar, Q. 20, Page 240]**

Sol. Given, $\lim\limits_{x\to\pi/3}\dfrac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim\limits_{x\to\pi/3}\dfrac{\sqrt{2}\sin 3x}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$

$$= \lim_{x\to\pi/3}\frac{\sin 3x}{\frac{\pi}{3}-x} = \lim_{x\to\pi/3}\frac{\sin(\pi-3x)}{\frac{\pi-3x}{3}}$$ $(y = \pi - 3x)$ $\left[\because \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1\right]$

$$= \lim_{y\to 0}\frac{3\sin y}{y} = 3\times 1$$

$$= 3$$

Q. 10. Evaluate $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin x-\cos x}{x-\frac{\pi}{4}}$ **[NCERT Exemplar, Q. 21, Page 240]**

Sol. $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}\left(\sin x\cdot\frac{1}{\sqrt{2}}-\cos x\cdot\frac{1}{\sqrt{2}}\right)}{\left(x-\frac{\pi}{4}\right)} = \lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}\left(\sin x\cos\frac{\pi}{4}-\cos x\cdot\sin\frac{\pi}{4}\right)}{\left(x-\frac{\pi}{4}\right)}$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\left\{\sin\left(x-\frac{\pi}{4}\right)\right\}}{\left(x-\frac{\pi}{4}\right)}$$

$$= \sqrt{2} \lim_{x \to \frac{\pi}{4}} \frac{\sin\left(x - \frac{\pi}{4}\right)}{\left(x - \frac{\pi}{4}\right)} \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$\therefore \qquad = \sqrt{2} \lim_{y \to 0} \frac{\sin y}{y} = \sqrt{2} \qquad \text{Let } x - \frac{\pi}{4} = y$$

Q. 11. Evaluate $\lim_{x \to \frac{\pi}{6}} \frac{\sqrt{3}\sin x - \cos x}{x - \frac{\pi}{6}}$ **[NCERT Exemplar, Q. 22, Page 240]**

Sol.
$$\lim_{x \to \frac{\pi}{6}} \frac{\sqrt{3}\sin x - \cos x}{x - \frac{\pi}{6}} = \lim_{x \to \frac{\pi}{6}} \frac{2\left(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right)}{\left(x - \frac{\pi}{6}\right)}$$

$$= \lim_{x \to \frac{\pi}{6}} \frac{2\left(\sin x \cos\frac{\pi}{6} - \cos x \sin\frac{\pi}{6}\right)}{\left(x - \frac{\pi}{6}\right)} = \frac{2\sin\left(x - \frac{\pi}{6}\right)}{\left(x - \frac{\pi}{6}\right)} \qquad \left[y = x - \frac{\pi}{6}\right]$$

$$= 2\lim_{y \to 0} \frac{\sin y}{y} \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right] \quad [\because \sin A \cos B - \cos A \sin B = \sin(A - B)]$$

$$= 2$$

Q. 12. Evaluate $\lim_{x \to 0} \frac{\sin 2x + 3x}{2x + \tan 3x}$ **[NCERT Exemplar, Q. 23, Page 240]**

Sol.
$$\lim_{x \to 0} \frac{\sin 2x + 3x}{2x + \tan 3x} = \lim_{x \to 0} \frac{\frac{\sin 2x + 3x}{2x} \cdot 2x}{\frac{2x + \tan 3x}{3x} \cdot 3x}$$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin 2x}{2x} + \frac{3x}{2x}\right)2x}{\left(\frac{2x}{3x} + \frac{\tan 3x}{3x}\right)3x} = \lim_{x \to 0} \frac{\frac{\sin 2x}{2x} + \frac{3}{2}}{\frac{2}{3} + \frac{\tan 3x}{3x}} \frac{2}{3}$$

$$= \frac{2}{3} \lim_{x \to 0} \frac{\frac{\sin 2x}{2x} + \frac{3}{2}}{\frac{2}{3} + \lim_{x \to 0} \frac{\tan 3x}{3x}}$$

$$= \frac{2}{3}\left(\frac{1 + \frac{3}{2}}{\frac{2}{3} + 1}\right) \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \text{ and } \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1\right]$$

$$= \frac{2}{3} \times \frac{\frac{5}{2}}{\frac{5}{3}} = \frac{2}{3} \times \frac{5}{2} \times \frac{3}{5} = 1$$

Q. 13. Evaluate $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$ **[NCERT Exemplar, Q. 25, Page 240]**

Sol.
$$\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = \lim_{x \to \frac{\pi}{6}} \frac{\operatorname{cosec}^2 x - 1 - 3}{\operatorname{cosec} x - 2} \qquad [\because \operatorname{cosec}^2 x = 1 + \cot^2 x]$$

$$= \lim_{x \to \frac{\pi}{6}} \frac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 2} = \lim_{x \to \frac{\pi}{6}} \frac{(\operatorname{cosec} x)^2 - (2)^2}{\operatorname{cosec} x - 2}$$

$$= \lim_{x\to\frac{\pi}{6}} \frac{(\text{cosec}\, x+2)(\text{cosec}\, x-2)}{(\text{cosec}\, x-2)} = \lim_{x\to\frac{\pi}{6}} (\text{cosec}\, x+2) \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$= 2 + 2 = 4 \qquad \text{(Apply to limit)}$$

Q. 14. Evaluate $\lim_{x\to 0}\frac{\sin x - 2\sin 3x + \sin 5x}{x}$ **[NCERT Exemplar, Q. 27, Page 240]**

Sol.

$$\lim_{x\to 0}\frac{\sin x-2\sin 3x+\sin 5x}{x} = \lim_{x\to 0}\frac{\sin 5x+\sin x-2\sin 3x}{x}$$

$$= \lim_{x\to 0}\frac{2\sin 3x\cos 2x-2\sin 3x}{x}$$

$$= \lim_{x\to 0}\frac{2\sin 3x(\cos 2x-1)}{x} = \lim_{x\to 0}\frac{2\sin 3x}{\frac{1}{3}\times 3x}(\cos 2x-1)$$

$$= 6\lim_{x\to 0}\frac{\sin 3x}{3x}(\cos 2x-1) = 6 \times 1 \times 0 = 0$$

Q. 15. Evaluate $\lim_{x\to 1}\frac{x^4-1}{x-1} = \lim_{x\to k}\frac{x^3-k^3}{x^2-k^2}$, **then find the value of** k. **[NCERT Exemplar, Q. 28, Page 240]**

Sol.

$$\lim_{x\to 1}\frac{x^4-1}{x-1} = \lim_{x\to k}\frac{x^3-k^3}{x^2-k^2}$$

$$4\,(1)^3 = \lim_{x\to k}\frac{\frac{x^3-k^3}{x-k}}{\frac{x^2-k^2}{x-k}}$$

$$\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$\Rightarrow \quad 4 = \frac{\lim_{x\to k}\frac{x^3-k^3}{x-k}}{\lim_{x\to k}\frac{x^2-k^2}{x-k}}$$

$$\Rightarrow \quad 4 = \frac{3}{2}(k)$$

$$\left[\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n} = \frac{m}{n}a^{m-n}\right]$$

$$\therefore \quad k = \frac{8}{3}$$

Q. 16. Show that $\lim_{x\to\pi/4}\frac{|x-4|}{x-4}$ **does not exist.** **[NCERT Exemplar, Q. 51, Page 241]**

Sol. Given, $\lim_{x\to\pi/4}\frac{|x-4|}{x-4}$

Let
$$f(x) = \begin{cases}-(x-4), & x<4\\ (x-4), & x>4\end{cases}$$

$$\text{LHL} = \lim_{x\to\frac{\pi^-}{4}}\frac{-(x-4)}{x-4}$$

$$= -1$$

$$\text{RHL} = \lim_{x\to\frac{\pi^+}{4}}\frac{(x-4)}{x-4} = 1$$

$$\therefore \quad \text{LHL} \neq \text{RHL}$$

So, limit does not exist.

Long Answer Type Questions-I (4 marks each)

Q. 1. Evaluate $\lim_{x\to 2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$ **[NCERT Exemplar, Q. 8, Page 240]**

Sol.

$$\lim_{x\to 2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}} = \lim_{x\to 2}\frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x+2})}{(\sqrt{3x-2})-(\sqrt{x+2})(\sqrt{3x-2}-\sqrt{x+2})}$$

$$= \lim_{x\to 2}\frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x+2})}{(\sqrt{3x-2})^2-(\sqrt{x+2})^2} \qquad [\because (a+b)(a-b) = a^2-b^2]$$

$$= \lim_{x\to 2}\frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x+2})}{(3x-2)-(\sqrt{x+2})}$$

$$= \lim_{x\to 2}\frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x+2})}{3x-2-x-2}$$

$$= \lim_{x\to 2}\frac{(x^2-4)(\sqrt{3x-2}+\sqrt{x+2})}{2x-4}$$

$$= \lim_{x\to 2}\frac{(x+2)(x-2)(\sqrt{3x-2}+\sqrt{x+2})}{2(x-2)}$$

$$= \lim_{x\to 2}\frac{(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2} \qquad [x\neq 2]$$

$$= \frac{(2+2)(\sqrt{6-2}+\sqrt{2+2})}{2}$$

$$= \frac{4(2+2)}{2} = 8$$

Q. 2. Evaluate $\lim_{x\to\sqrt{2}}\frac{x^4-4}{x^2+3\sqrt{2}x-8}$

[NCERT Exemplar, Q. 9, Page 240]

Sol. $\lim_{x\to\sqrt{2}}\frac{x^4-4}{x^2+3\sqrt{2}x-8}$

$$= \lim_{x\to\sqrt{2}}\frac{(x^2)^2-(2)^2}{x^2+3\sqrt{2}x-8}$$

$$= \lim_{x\to\sqrt{2}}\frac{(x^2)^2-(2)^2}{x^2+3\sqrt{2}x-8}$$

$$= \lim_{x\to\sqrt{2}}\frac{(x^2-2)(x^2+2)}{x^2+4\sqrt{2}x-\sqrt{2}x-8}$$

$$= \lim_{x\to\sqrt{2}}\frac{(x-\sqrt{2})(x+\sqrt{2})(x^2+2)}{(x-\sqrt{2})(x+4\sqrt{2})} \qquad [x\neq\sqrt{2}]$$

$$= \lim_{x\to\sqrt{2}}\frac{(x+\sqrt{2})(x^2+2)}{(x+4\sqrt{2})}$$

$$= \frac{(\sqrt{2}+\sqrt{2})\left[(\sqrt{2})^2+2\right]}{(\sqrt{2}+4\sqrt{2})}$$

$$= \frac{2\sqrt{2}(2+2)}{5\sqrt{2}} = \frac{8}{5}$$

Q. 3. Evaluate $\lim_{x\to 1}\frac{x^7-2x^5+1}{x^3-3x^2+2}$

[NCERT Exemplar, Q. 10, Page 240]

Sol. $\lim_{x\to 1}\frac{x^7-2x^5+1}{x^3-3x^2+2}$

$$= \lim_{x\to 1}\frac{x^7-x^5-x^5+1}{x^3-x^2-2x^2+2}$$

$$= \lim_{x\to 1}\frac{x^5(x^2-1)-1(x^5-1)}{x^2(x-1)-2(x^2-1)}$$

$$= \lim_{x\to 1}\frac{\frac{x^5(x^2-1)}{(x-1)}-\frac{1(x^5-1)}{(x-1)}}{\frac{x^2(x-1)}{(x-1)}-\frac{2(x^2-1)}{(x-1)}}$$

$$= \frac{\lim_{x\to 1}x^5(x+1)-\lim_{x\to 1}\left(\frac{x^5-1}{x-1}\right)}{\lim_{x\to 1}x^2-2\lim_{x\to 1}(x+1)}$$

$$= \frac{1\times 2-5\times(1)^4}{1-2\times 2} = \frac{2-5}{1-4}$$

$$= \frac{-3}{-3} = 1$$

Q. 4. Evaluate $\lim_{x\to\frac{1}{2}}\left(\frac{8x-3}{2x-1}-\frac{4x^2+1}{4x^2-1}\right)$

[NCERT Exemplar, Q. 13, Page 240]

Sol. $\lim_{x\to\frac{1}{2}}\left(\frac{8x-3}{2x-1}-\frac{4x^2+1}{4x^2-1}\right)$

$$= \lim_{x\to\frac{1}{2}}\left[\frac{(8x-3)(2x+1)-(4x^2+1)}{(4x^2-1)}\right]$$

$$= \lim_{x\to\frac{1}{2}}\left[\frac{16x^2+8x-6x-3-4x^2-1}{4x^2-1}\right]$$

$$= \lim_{x\to\frac{1}{2}}\left[\frac{12x^2+2x-4}{4x^2-1}\right]$$

$$= \lim_{x\to\frac{1}{2}}\frac{2\left(6x^2+x-2\right)}{4x^2-1}$$

$$= \lim_{x\to\frac{1}{2}}\frac{2\left(6x^2+4x-3x-2\right)}{4x^2-1}$$

$$= \lim_{x\to\frac{1}{2}}\frac{2\left[(3x+2)(2x-1)\right]}{(2x)^2-(1)^2}$$

$$= \lim_{x\to\frac{1}{2}}\frac{2(3x+2)(2x-1)}{(2x-1)(2x+1)} \qquad \left[x\neq\frac{1}{2}\right]$$

$$= \lim_{x\to\frac{1}{2}}\frac{2(3x+2)}{2x+1} = \frac{2\left(3\times\frac{1}{2}+2\right)}{2\times\frac{1}{2}+1}$$

$$= \frac{3}{2}+2 = \frac{7}{2}$$

Q. 5. Evaluate $\lim\limits_{x\to 0}\dfrac{1-\cos mx}{1-\cos nx}$ **[NCERT Exemplar, Q. 19, Page 240]**

Sol.

$$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx} = \lim_{x\to 0}\frac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{nx}{2}} = \lim_{x\to 0}\frac{\dfrac{\sin^2\frac{mx}{2}}{\left(\frac{mx}{2}\right)^2}\cdot\left(\frac{mx}{2}\right)^2}{\dfrac{\sin^2\frac{nx}{2}}{\left(\frac{nx}{2}\right)^2}\cdot\left(\frac{nx}{2}\right)^2} = \frac{\lim\limits_{x\to 0}\left(\dfrac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right)^2}{\lim\limits_{x\to 0}\left(\dfrac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right)^2}\cdot\frac{m^2\frac{x^2}{4}}{n^2\frac{x^2}{4}} \qquad [x\neq 0]$$

$$= \frac{m^2}{n^2}\cdot\frac{\lim\limits_{x\to 2}\left(\dfrac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right)}{\lim\limits_{x\to 2}\left(\dfrac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right)^2} = \frac{m^2}{n^2} \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

Q. 6. Evaluate $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}}$

[NCERT Exemplar, Q. 24, Page 240]

Sol. $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}}$

$$= \lim_{x\to a}\frac{\sin x-\sin a}{\sqrt{x}-\sqrt{a}}\times\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}$$

$$= \lim_{x\to a}\left(\sqrt{x}+\sqrt{a}\right)\lim_{x\to a}\frac{\sin x-\sin a}{x-a}$$

$$= \left(\sqrt{a}+\sqrt{a}\right)\lim_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a}$$

$$= 2\sqrt{a}\lim_{x\to a}2\cos\left(\frac{x+a}{2}\right)\lim_{x\to a}\frac{\sin\frac{x-a}{2}}{x-a}$$

$$= 2\sqrt{2}\cdot 2\cos a\lim_{x\to a}\frac{\sin\left(\frac{x-a}{2}\right)}{x-a}$$

Let $x-a = y$

$$\therefore \quad = 4\sqrt{2}\cos a\lim_{y\to 0}\frac{\sin\frac{y}{2}}{y}$$

$$= 4\sqrt{2}\cos a\lim_{y\to 0}\frac{\sin\frac{y}{2}}{2\frac{y}{2}} \qquad \left[\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$= 4\sqrt{2}\cos a\times\frac{1}{2}\times 1$$

$$= 4\sqrt{2}\cos a$$

Q. 7. Evaluate $\lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$ **[NCERT Exemplar, Q. 26, Page 240]**

Sol. Given,

$$\lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} = \lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+2\cos^2\frac{x}{2}-1}}{\sin x^2} \qquad \left[\because \cos x = 2\cos^2\frac{x}{2}-1\right]$$

$$= \lim_{x\to 0}\frac{\sqrt{2}-\sqrt{2\cos^2\frac{x}{2}}}{\sin^2 x} \qquad \left[\because \cos x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$$

$$= \lim_{x\to 0}\frac{\sqrt{2}\left(1-\cos\frac{x}{2}\right)}{\sin^2 x} = \lim_{x\to 0}\frac{\sqrt{2}\left(1-1+2\sin^2\frac{x}{4}\right)}{\sin^2 x}$$

$$= \lim_{x\to 0}\frac{\sqrt{2}\left(2\sin^2\frac{x}{4}\right)}{\sin^2 x} = \lim_{x\to 0}2\sqrt{2}\frac{\sin^2\frac{x}{4}}{\left(\frac{x}{4}\right)^2}\cdot\frac{\left(\frac{x}{4}\right)^2}{\sin^2 x}$$

$$= 2\sqrt{2}\lim_{x\to0}\left(\frac{\sin\frac{x}{4}}{\frac{x}{4}}\right)^2\cdot\lim_{x\to0}\left(\frac{x}{\sin x}\right)^2\cdot\frac{1}{16} \qquad \left(\lim_{\theta\to0}\frac{\theta}{\sin\theta}=1\right)$$

$$= 2\sqrt{2}\cdot1\cdot1\cdot\frac{1}{16} = \frac{1}{4\sqrt{2}}$$

Q. 8. Evaluate $\lim_{x\to\frac{\pi}{4}}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}$

[NCERT Exemplar, Q. 49, Page 241]

Sol. Given, $\lim_{x\to\frac{\pi}{4}}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}$ $\left[\frac{0}{0}\text{form}\right]$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\tan x(\tan^2 x-1)}{\cos\left(x+\frac{\pi}{4}\right)}$$

$$= \lim_{x\to\frac{\pi}{4}}\tan x\cdot\lim_{x\to\pi/4}\left(\frac{-(1-\tan^2 x)}{\cos\left(x+\frac{\pi}{4}\right)}\right)$$

$$= -1\times\lim_{x\to\frac{\pi}{4}}\frac{(1+\tan x)(1-\tan x)}{\cos\left(x+\frac{\pi}{4}\right)}$$

$[\because a^2-b^2=(a+b)(a-b)]$

$$= -\lim_{x\to\frac{\pi}{4}}(1+\tan x)\lim_{x\to\frac{\pi}{4}}\left[\frac{(\cos x-\sin x)}{\cos x\cdot\cos\left(x+\frac{\pi}{4}\right)}\right]$$

$$= -(1+1)\times\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\left[\frac{1}{\sqrt{2}}\cdot\cos x-\frac{1}{\sqrt{2}}\cdot\sin x\right]}{\cos x\cdot\cos\left(x+\frac{\pi}{4}\right)}$$

$$= -2\sqrt{2}\times\lim_{x\to\frac{\pi}{4}}\left[\frac{\cos\frac{\pi}{4}\cdot\cos x-\sin\frac{\pi}{4}\cdot\sin x}{\cos x\cdot\cos\left(x+\frac{\pi}{4}\right)}\right]$$

$[\because \cos A\cdot\cos B-\sin A\sin B = \cos(A+B)]$

$$= -2\sqrt{2}\lim_{x\to\frac{\pi}{4}}\frac{\cos\left(x+\frac{\pi}{4}\right)}{\cos x\cdot\cos\left(x+\frac{\pi}{4}\right)}$$

$$= -2\sqrt{2}\times\frac{1}{\frac{1}{\sqrt{2}}} = -2\sqrt{2}\times\sqrt{2} = -4$$

Q. 9. $\lim_{x\to\pi}\frac{1-\sin\frac{x}{2}}{\cos\frac{x}{2}\cos\frac{x}{4}-\sin\frac{x}{4}}$ **[NCERT Exemplar, Q. 50, Page 241]**

Sol. $$\lim_{x\to\pi}\frac{1-\sin\frac{x}{2}}{\cos\frac{x}{2}\left(\sin\frac{x}{4}-\sin\frac{x}{4}\right)\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)} = \lim_{x\to\pi}\frac{\cos^2\frac{x}{4}+\sin^2\frac{x}{4}-2\sin\frac{x}{4}\cos\frac{x}{4}}{\cos\frac{x}{4}\cdot(\cos\frac{x}{4}-\sin\frac{x}{4})}$$

$[\because \sin^2\theta+\cos^2\theta=1, \sin2\theta=2\sin\theta\cos\theta]$

$$= \lim_{x\to\pi}\frac{\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}{\left(\cos\frac{x}{4}+\sin\frac{x}{4}\right)\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)} \quad [\because \cos2\theta=\cos^2\theta-\sin^2\theta]$$

$$= \lim_{x\to\pi}\frac{\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}{\left(\cos\frac{x}{4}+\sin\frac{x}{4}\right)\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)} \quad [\because a^2-b^2=(a+b)(a-b)]$$

$$= \lim_{x\to\pi}\frac{1}{\cos\frac{x}{4}+\sin\frac{x}{4}} = \frac{1}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Q. 10. If $f(x) = \begin{cases} x+2, & x\le-1 \\ cx^2, & x\ge-1\end{cases}$**, then find** c **when** $\lim_{x\to-1}$ $f(x)$ **exists.**

[NCERT Exemplar, Q. 53, Page 242]

Sol. $f(x) = \begin{cases} x+2, & x\le-1 \\ cx^2, & x\ge-1\end{cases}$,

$$\text{LHL} = \lim_{x\to1^-}f(x) = \lim_{x\to-1^-}(x+2)$$

$= \lim_{h\to 0}(-1-h+2) = \lim_{h\to 0} c(-1+h)^2$

RHL $= \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} cx^2$

$= \lim_{x\to 0} c(-1+h)^2$

$\therefore \quad = c$

If $\lim_{x\to -1} f(x)$ exist, then LHL = RHL

$\therefore \quad c = 1$

Long Answer Type Questions-II (6 marks each)

Q. 1. $\lim_{y\to 0} \dfrac{(x+y)\sec(x+y) - x\sec x}{y}$

[NCERT Exemplar Q. 47, Page 241]

Sol. Given, $\lim_{y\to 0} \dfrac{(x+y)\sec(x+y) - x\sec x}{y}$

$$= \lim_{y\to 0} \frac{\dfrac{x+y}{\cos(x+y)} - \dfrac{x}{\cos x}}{y}$$

$$= \lim_{y\to 0} \frac{(x+y)\cos x - x\cos(x+y)}{y\cos x\cos(x+y)}$$

$$= \lim_{y\to 0}\left[\frac{x\cos x + y\cos x - x\cos(x+y)}{y\cos x\cos(x+y)}\right]$$

$$= \lim_{y\to 0}\left[\frac{x\cos x - x\cos(x+y) + y\cos x}{y\cos x\cos(x+y)}\right]$$

$$= \lim_{y\to 0} \frac{x(\cos x - \cos(x+y) + y\cos x}{y\cos x\cos(x+y)}$$

$$= \lim_{y\to 0} \frac{x\left[-2\sin\left(x+\dfrac{y}{2}\right)\sin\left(\dfrac{-y}{2}\right)\right] + y\cos x}{y\cos x\cos(x+y)}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}\right]$$

$$= \lim_{y\to 0}\left[\frac{x\left\{2\sin\left(x+\dfrac{y}{2}\right)\sin\dfrac{y}{2}\right\} + y\cos x}{y\cos x\cos(x+y)}\right]$$

$$= \lim_{y\to 0} \frac{2x\sin\left(x+\dfrac{y}{2}\right)}{\cos x\cos(x+y)}\cdot \lim_{y\to 0}\frac{\sin\dfrac{y}{2}}{\dfrac{y}{2}}\cdot\frac{1}{2} + \lim_{y\to 0}\sec(x+y)$$

$$= \lim_{x\to 0} \frac{2x\sin\left(x+\dfrac{y}{2}\right)}{\cos x\cos(x+y)}\cdot\frac{1}{2} + \lim_{y\to 0}\sec(x+y)$$

$$= \lim_{x\to 0} \frac{2x\sin\left(x+\dfrac{y}{2}\right)}{\cos x\cos(x+y)}\cdot\frac{1}{2} + \lim_{y\to 0}\sec(x+y)$$

$$= \frac{2x\sin x}{\cos x\cos x}\cdot\frac{1}{2} + \sec x$$

$= x \tan x \sec x + \sec x$

$= \sec x\,(x \tan x + 1)$

Q. 2. Evaluate :

$\lim_{x\to 0} \dfrac{\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{\cos 2\beta x - \cos 2\alpha x}\cdot x$

[NCERT Exemplar Q. 48, Page 241]

Sol. Given, $\lim_{x\to 0} \dfrac{\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{\cos 2\beta x - \cos 2\alpha x}\cdot x$

$$= \lim_{x\to 0} \frac{[2\sin\alpha x\cdot\cos\beta x + \sin 2\alpha x]}{\cos 2\beta x - \cos 2\alpha x}\cdot x \qquad \left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= \lim_{x\to 0} \frac{[2\sin\alpha x\cos\beta x + \sin 2\alpha x]x}{2\sin(\alpha+\beta)x\sin(\alpha-\beta)x} \qquad \left[\because \cos C - \cos D = 2\sin\frac{C+D}{2}\cdot\sin\frac{D-C}{2}\right]$$

$$= \lim_{x\to 0} \frac{[2\sin\alpha x\cos\beta x + \sin 2\alpha x]x}{2\sin(\alpha+\beta)x\sin(\alpha-\beta)x}$$

$$= \lim_{x\to 0} \frac{[2\sin\alpha x\cos\beta x + 2\sin\alpha x\cos\alpha x]x}{2\sin(\alpha+\beta)x\sin(\alpha-\beta)x}$$

$$= \lim_{x\to 0} \frac{2\sin\alpha x[\cos\beta x + \cos\alpha x]x}{2\sin(\alpha+\beta)x.\sin(\alpha-\beta)x}$$

$$\left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2} \text{ and } \sin 2\theta = 2\sin\theta\cos\theta\right]$$

$$= \lim_{x\to 0}\frac{\sin\alpha x.x}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)x\cos\left(\frac{\alpha-\beta}{2}\right)x}$$

$$= \lim_{x\to 0}\frac{\frac{\sin\alpha x}{\alpha x}.x.(\alpha x)}{\left[2\sin\frac{\left(\frac{\alpha+\beta}{2}\right)x}{\left(\frac{\alpha+\beta}{2}\right)x}.\sin\frac{\left(\frac{\alpha-\beta}{2}\right)x}{\left(\frac{\alpha-\beta}{2}\right)x}\right]\left(\frac{\alpha+\beta}{2}\right)x\cdot\left(\frac{\alpha-\beta}{2}\right)x}$$

$$= \frac{\frac{1}{2}\lim_{x\to 0}\frac{\sin\alpha x}{\alpha x}.\alpha x^2}{\lim_{x\to 0}\sin\frac{\left(\frac{\alpha+\beta}{2}\right)x}{\left(\frac{\alpha+\beta}{2}\right)x}\lim_{x\to 0}\frac{\left(\frac{\alpha-\beta}{2}\right)x}{\left(\frac{\alpha-\beta}{2}\right)x}\cdot\left(\frac{\alpha^2-\beta^2}{2}\right)x^2} \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$= \frac{1}{2}\cdot\frac{\alpha\cdot 4}{\alpha^2-\beta^2}\left[\frac{\lim_{x\to 0}\frac{\sin\alpha x}{\alpha x}}{\lim_{x\to 0}\sin\frac{\left(\frac{\alpha+\beta}{2}\right)x}{\left(\frac{\alpha+\beta}{2}\right)x}\lim_{x\to 0}\sin\frac{\left(\frac{\alpha-\beta}{2}\right)x}{\left(\frac{\alpha-\beta}{2}\right)x}}\right]$$

$$= \frac{1}{2}\cdot\frac{4\alpha}{\alpha^2-\beta^2} = \frac{2\alpha}{\alpha^2-\beta^2} = \frac{2\alpha}{\alpha^2-\beta^2}$$

TOPIC-2
Derivatives

Quick Review

- **Derivative at a point**

  Suppose f is a real valued function and 'a' is a point in its domain. Then, Derivative of f at a is defined by $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, provided this limit exists.

  The derivative of $f(x)$ at a is denoted by $f'(a)$.
- **First Principle of Derivative**

  Suppose f is a real valued function, the function defined by $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, wherever the limit exists and is defined to be the derivative of f and is denoted by $f'(x)$. This definition of derivative is called the first principle of derivative. Thus, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

  $f'(x)$ is also denoted by $\frac{d}{dx}[f(x)]$ or if $y = f(x)$, then it is denoted by $\frac{dy}{dx}$ and referred to as derivative of $f(x)$ or y with respect to x.
- **Algebra of Derivative of Functions**

  Let f and g be two function such that their derivatives are defined in a common domain. Then,

  (i) Derivative of sum of two functions is sum of the derivatives of the functions.

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

or $\qquad (u+v)' = u' + v'$

(ii) Derivative of difference of two function is difference of the derivative of the functions.

$$\frac{d}{dx}\left[f(x)-g(x)\right] = \frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

or $(u-v)' = u' - v'$

(iii) Derivative of product of two functions is given by the following product rule.

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \left[\frac{d}{dx}f(x)\right]g(x)+f(x)\left[\frac{d}{dx}g(x)\right]$$

or $(u.v)' = u' . v + v . u'$

(iv) Derivative of quotient of two functions is given by the following quotient rule.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{\left[g(x)\right]^2}, g(x)\neq 0$$

or $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$

- **Some Important derivatives**

- **(i)** $\frac{d}{dx}x^n = nx^{n-1} = nx^{n-1}$

  (ii) $\frac{d}{dx}\left[Cf(x)\right] = C\frac{d}{dx}f(x)$ C is a constant.

  (iii) $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

  (iv) If $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$,
  then $f'(x) = na_n x^{n-1} + (n-1)\, a_{n-1}x^{n-2} + (n-2)\, a_{n-2}x^{n-3} + ... + a_1$.

  (v) $\frac{d}{dx}(\sin x) = \cos x$

  (vi) $\frac{d}{dx}(\cos x) = -\sin x$

  (vii) $\frac{d}{dx}(\tan x) = \sec^2 x$

  (viii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

  (ix) $\frac{d}{dx}(\sec x) = \sec x \tan x$

  (x) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

  (xi) $\frac{d}{dx}\left(a^x\right) = a^x \log_e a$

  (xii) $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}$

  (xiii) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

  (xiv) $\frac{d}{dx}e^x = e^x$

  (xv) $\frac{d}{dx}[C] = 0$, where 'c' is a constant.

- **Geometrical meaning of Derivative at a point**

  Geometrically, derivative of a function at a point x = the slope of tangent to the curve $y = f(x)$ at the point $\{C\}$.

Slope of tangent at P = $\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}$

$= \left\{\frac{df(x)}{dx}\right\}_{x=c}$ or $f'(c)$

➢ **Note :**

- Derivative of f at $x = a$ is also given by substituting $x = a$ in $f'(x)$ and it is denoted by $\left.\frac{d}{dx}f(x)\right|_a$ or $\left.\frac{d}{dx}\right|_a$ or $\left(\frac{df}{dx}\right)_{x=a}$.

Multiple Choice Questions (1 mark each)

[A] Objective Type Questions

Q. 1. If $f(x) = x - [x]$, $\in$ R, then $f'\left(\frac{1}{2}\right)$ is equal to :

(a) $\frac{3}{2}$ (b) 1

(c) 0 (d) –1

[NCERT Exemplar Q. 67, Page 244]

Ans. Correct option : (a)

Given, $f(x) = x - [x]$; we to first check for differentiability of $f(x)$ at $x = \frac{1}{2}$

$\therefore Lf^1\left(\frac{1}{2}\right)$ = LHD = $\lim\limits_{h\to 0}\frac{f\left[\frac{1}{2}-h\right]-f\left[\frac{1}{2}\right]}{-h}$

$= \lim\limits_{h\to 0}\frac{\left(\frac{1}{2}-h\right)-\left(\frac{1}{2}-h\right)-\frac{1}{2}+\frac{1}{2}}{-h}$

$= \lim\limits_{h\to 0}\frac{\frac{1}{2}-h-0-\frac{1}{2}+0}{-h} = \frac{-h}{-h} = 1$

$Rf^1\left(\frac{1}{2}\right)$ = RHD = $\lim\limits_{x\to 0}\frac{f\left(\frac{1}{2}+h\right)-f\left(\frac{1}{2}\right)}{h}$

$= \lim\limits_{x\to 0}\frac{\frac{1}{2}+h-1-\frac{1}{2}+1}{h} = \frac{h}{h} = 1$

LHD = RHD

$\therefore\ f^1\left(\frac{1}{2}\right) = 1$

Q. 2. If $y = \sqrt{x}+\frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at $x = 1$ is equal to :

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) 0

[NCERT Exemplar Q. 68, Page 244]

Ans. Correct option : (d)

Given that $y = \sqrt{x}+\frac{1}{\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$

$\left(\frac{dy}{dx}\right)_{at\,x=1} = \frac{1}{2}-\frac{1}{2} = 0$

Q. 3. If $f(x) = \frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is equal to :

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$

(c) 1 (d) 0

[NCERT Exemplar Q. 69, Page 244]

Ans. Correct option : (a)

Given, that $f(x) = \frac{x-4}{2\sqrt{x}}$

$\therefore f^1(x) = \frac{1}{2}\left[\frac{\sqrt{x}.1-(x-4).\frac{1}{2\sqrt{x}}}{x}\right]$

$= \frac{1}{2}\left[\frac{2x-x+4}{2\sqrt{x}.x}\right] = \frac{1}{2}\left[\frac{x+4}{2(x)^{3/2}}\right]$

$\therefore [f^1(x)]_{\text{at } x=1} = \frac{1}{2}\left[\frac{1+4}{2\times 1}\right] = \frac{5}{4}$

Q. 4. If $y = \frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$, then $\frac{dy}{dx}$ is equal to :

(a) $\frac{-4x}{(x^2-1)^2}$ (b) $\frac{-4x}{x^2-1}$

(c) $\frac{1-x^2}{4x}$ (d) $\frac{4x}{x^2-1}$

[NCERT Exemplar Q. 70, Page 244]

Ans. Correct option : (a)

Given, $y = \dfrac{1+\dfrac{1}{x^2}}{1-\dfrac{1}{x^2}} \Rightarrow y = \dfrac{x^2+1}{x^2-1}$

$$\therefore \frac{dy}{dx} = \frac{(x^2-1).2x-(x^2+1).2x}{(x^2-1)^2}$$

$$= \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$= \frac{2x(-2)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

Q. 5. If $y = \dfrac{\sin x+\cos x}{\sin x-\cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is equal to :

(a) –2 **(b) 0**

(c) $\lim\limits_{x\to\pi} f(x)$ **(d) Does not exist**

[NCERT Exemplar Q. 71, Page 244]

Ans. Correct option : (a)

Given, $y = \dfrac{\sin x+\cos x}{\sin x-\cos x}$

$$\frac{dy}{dx} = \frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$= \frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$= \frac{-[\sin^2 x+\cos^2 x-2\sin x\cos x]+[\sin^2 x+\cos^2 x+2\sin x\cos x]}{(\sin x-\cos x)^2}$$

$$= \frac{-2}{(\sin x-\cos x)^2}$$

$$\therefore \left(\frac{dy}{dx}\right)_{at\,x=0} = \frac{-2}{(\sin 0-\cos 0)^2}$$

$$= \frac{-2}{(-1)^2} = -2$$

Q. 6. If $y = \dfrac{\sin(x+9)}{\cos x}$ then $\dfrac{dy}{dx}$ at $x = 0$ is equal to :

(a) cos 9 **(b) sin 9**

(c) 0 **(d) 1**

[NCERT Exemplar Q. 72, Page 245]

Ans. Correct option : (a)

Given, $y = \dfrac{\sin(x+9)}{\cos x}$

$$\frac{dy}{dx} = \frac{\cos x.\cos(x+9)-\sin(x+9)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos x\cos(x+9)+\sin x\sin(x+9)}{\cos^2 x}$$

$$= \frac{\cos(x+9-x)}{\cos^2 x} = \frac{\cos 9}{\cos^2 x}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } x=0} = \frac{\cos 9}{\cos^2 0} = \frac{\cos 9}{1} = \cos 9$$

Q. 7. If $f(x) = 1+x+\dfrac{x^2}{2}+...+\dfrac{x^{100}}{100}$, then $f'(1)$ is equal to :

(a) $\dfrac{1}{100}$ **(b) 100**

(c) 0 **(d) Does not exist**

[NCERT Exemplar Q. 73, Page 245]

Ans. Correct option : (b)

Given, $f(x) = 1 + x + \dfrac{x^2}{2}+.......+\dfrac{x^{100}}{100}$

$f^1(x) = 1 + \dfrac{2x}{2}+.......+\dfrac{100x^{99}}{100}$

$\therefore f^1(x) = 1 + 1 + 1$ .... + 1 (100 times)

= 100

Q. 8. If $f(x) = \dfrac{x^n - a^n}{x - a}$ for some constant a, then $f'(a)$ is equal to :

(a) 1 **(b) 0**

(c) $\dfrac{1}{2}$ **(d) Does not exist**

[NCERT Exemplar Q. 74, Page 244]

Ans. Correct option : (c)

Given, $f(x) = \dfrac{x^n-a^n}{x-a}$

$$f^1(x) = \frac{(x-a)(nx^{n-1})-(x^n-a^n).1}{(x-a)^2}$$

$$\therefore f^1(a) = \frac{(a-a)(na^{n-1})-(a^n-a^n).1}{(a-a)^2}$$

So, $f^1(a) = \dfrac{0}{0}$ = does not exist.

Q. 9. If $f(x) = x^{100} + x^{99} + ... + x + 1$, then $f'(1)$ is equal to :

(a) 5050 **(b) 5049**

(c) 5051 **(d) 50051**

[NCERT Exemplar Q. 75, Page 244]

Ans. Correct option : (a)

Given, $f(x) = x^{100} + x^{99} + ... + x + 1$

$\therefore f^1(x) = 100x^{99} + 99.x^{98} + ... + 1$

So, $f^1(1) = 100 + 99 + 98 + ... + 1$

$$= \frac{100}{2}[2\times 100+(100-1)(-1)]$$

$= 50\,[200 - 99] = 50 \times 101$

$= 5050$

Q. 10. If $f(x) = 1 - x + x^2 - x^3 + ... + x^{99} + x^{100}$, then $f'(1)$ is equal to :

(a) 150 **(b) –50**

(c) –150 **(d) 50**

[NCERT Exemplar Q. 76, Page 244]

Ans. Correct option : (d)

Given, $f(x) = 1 - x + x^2 - x^3 + ... - x^{99} + x^{100}$

$f^1(x) = -1 + 2x - 3x^2 + ... + 99x^{98} + 100x^{99}$

$\therefore f^1(1) = -1 + 2 - 3 .... - 99 + 100$

$= (-1 - 3 - 5 .... - 99) + (2 + 4 + 6 + ... + 100)$

$= \frac{50}{2}\left[2\times(-1)+(50-1)(-1)\right]+\frac{50}{2}\left[2\times2+(50-1)^2\right]$

$= 25\,[-2 - 98] + 25\,[4 + 98]$

$= 25 \times (-100) + 25\,(102) = 25[-100 + 102]$

$= 25 \times 2 = 50$

[B] Fill in the blanks of the following

Q. 1. If $y = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}$, then $\frac{dy}{dx}$ = ...

[NCERT Exemplar Q. 79, Page 245]

Sol. $y = e^x$

$$\frac{dy}{dx} = \frac{d}{dx}e^x = e^x$$

$$\frac{dy}{dx} = y$$

Short Answer Type Questions

(2 marks each)

Q. 1. Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + ... + a^{n-1}x + a^n$, for some real number a.

[NCERT Exemplar Q. 6, Page 313]

Sol. Let, $f(x) = x^n + ax^{n-1} + a^2x^{n-2} + ... + ax^{n-1} + a^n$

$$\Rightarrow \frac{d}{dx}\{f(x)\} = \frac{d}{dx}[x^n + ax^{n-1} + a^2x^{n-2} + ... + a^{n-1}x + a^n]$$

$$\Rightarrow f'(x) = nx^{n-1} + (n-1)ax^{n-2} + (n-2)ax^{n-3} + ... + a^{n-1}$$

$$\left[\because \frac{d}{dx}x^n = nx^{n-1} \text{ and } \frac{d}{dx}a^n = 0\right]$$

since a is a real no.

$$= \quad f'(x) = nx^{n-1} + (n-1)\,ax^{n-2} + (n-2)\,ax^{n-3} + ... + a^{n-1}.$$

Q. 2. For some constants a and b, find the derivative of

(i) $(x - a)(x - b)$

[NCERT Exemplar Q. 7, Page 313]

Sol. Let, $f(x) = (x - a)(x - b)$

$$\therefore \frac{d}{dx}f(x) = \frac{d}{dx}\{(x-a)(x-b)\}$$

$$\Rightarrow f'(x) = (x-a)\frac{d}{dx}(x-b)+(x-b)\frac{d}{dx}(x-a)$$

$$= (x-a)\left[\frac{d}{dx}x-\frac{d}{dx}b\right]+(x-b)\left[\frac{d}{dx}x-\frac{d}{dx}a\right]$$

$= (x - a)\,[1 - 0] + (x - b)\,[1 - 0]$

$= (x - a) + (x - b)$

$= 2x - (a + b)$

(ii) $(ax^2 + b)^2$

Sol. Let $f(x) = (ax^2 + b)^2$

$$\Rightarrow \frac{d}{dx}f(x) = \frac{d}{dx}(ax^2+b)^2$$

$$= \frac{d}{dx}(a^2x^4+2abx^2+b^2)$$

$$= a^2\frac{d}{dx}x^4+2ab\frac{d}{dx}x^2+\frac{d}{dx}b^2$$

$= a^2 \cdot 4x^3 + 2ab \cdot 2x + 0$

$\Rightarrow f'(x) = 4a^2x^3 + 4abx$

(iii) $\frac{x-a}{x-b}$

Sol. Let $f(x) = \frac{x-a}{x-b}$

$$\Rightarrow \frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

$$= \frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)\left[\frac{d}{dx}x-\frac{d}{dx}a\right]-(x-a)\left[\frac{d}{dx}x-\frac{d}{dx}b\right]}{(x-b)^2}$$

$$= \frac{(x-b)[1-0]-(x-a)[1-0]}{(x-b)^2}$$

$$= \frac{(x-b)-(x-a)}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

Q. 3. Find the derivatives of—

(i) $2x-\frac{3}{4}$ **[NCERT Exemplar Q. 8, Page 313]**

Sol. Let $f(x) = 2x - \frac{3}{4}$

$\therefore \quad f'(x) = \frac{d}{dx}\left(2x - \frac{3}{4}\right)$

$= 2\frac{d}{dx}x - \frac{d}{dx}\left(\frac{3}{4}\right)$

$= 2\cdot 1 - 0$

$= 2$

(ii) $(5x^3 + 3x - 1)(x - 1)$

Sol. Let, $f(x) = (5x^3 + 3x - 1)(x - 1)$

$\therefore \quad f'(x) = \frac{d}{dx}\{(5x^3 + 3x - 1)(x - 1)\}$

$= (5x^3 + 3x - 1)\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}(5x^3 + 3x - 1)$

$= (5x^3 + 3x - 1)\cdot 1 + (x-1)(15x^2 + 3)$

$= 20x^3 - 15x^2 + 6x - 4$

(iii) $x^{-3}(5 + 3x)$

Sol. Let, $f(x) = x^{-3}(5 + 3x) = 5x^{-3} + 3x^{-2}$

$\therefore \quad \frac{d}{dx}f(x) = \frac{d}{dx}(5x^{-3} + 3x^{-2})$

$= 5\frac{d}{dx}x^{-3} + 3\frac{d}{dx}x^{-2}$

$= 5(-3)x^{-3-1} + 3\cdot(-2)x^{-2-1}$

$\Rightarrow \quad f'(x) = \frac{-3(2x+5)}{x^4}$

(iv) $x^5(3 - 6x^{-9})$

Sol. Let, $f(x) = x^5(3 - 6x^{-9}) = 3x^5 - 6x^{-4}$

$\therefore \quad \frac{d}{dx}f(x) = \frac{d}{dx}(3x^5 - 6x^{-4})$

$= \frac{3d}{dx}(x^5) - 6\frac{d}{dx}(x^{-4})$

$= 3(5)x^{5-1} - 6(-4)x^{-4-1}$

$= 15x^4 + 24x^{-5}$

$\Rightarrow f^1(x) = 15x^4 + 24x^{-5}$

(v) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Sol. Let, $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$\therefore \quad \frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{2}{x+1} - \frac{x^2}{3x-1}\right)$

$= 2\frac{d}{dx}\left(\frac{1}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$

By quotient rule

$$f^1(x) = \left[\frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x) - x^2(3)}{(3x-1)^2}\right]$

$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right]$

$= \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x}{(3x-1)^2}\right]$

$= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$

Q. 4. Find the derivatives of the following functions.

(i) $\sin x \cos x$.

[NCERT Exemplar Q. 11, Page 313]

Sol. Let $f(x) = \sin x \cos x$

$\therefore \quad \frac{d}{dx}f(x) = \sin x \cos x$

$= \frac{d}{dx}\{\sin x \cos x\}$

$= \sin x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}\sin x.$

$= \sin x\cdot(-\sin x) + \cos x\cdot\cos x$

$\Rightarrow \quad f'(x) = \cos 2x$

Alternative method

$f(x) = \frac{2\sin\cos x}{2}$

$f(x) = \frac{1}{2}\sin 2x$

$f(x) = \frac{1}{2}\sin 2x$

$f'(x) = \frac{1}{2}\frac{d}{dx}\sin 2x$

$f'(x) = \cos 2x$

(ii) $\sec x$.

Sol. Let $f(x) = \sec x = \frac{1}{\cos x}$

$\therefore \quad \frac{d}{dx}f(x) = \frac{d}{dx}\cdot\frac{1}{\cos x}$

$$= \frac{\cos x \cdot \frac{d}{dx}1 - 1 \cdot \frac{d}{dx}\cos x}{(\cos x)^2}$$

$$= \frac{\cos x\, 0 - 1 \cdot (-\sin x)}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$\Rightarrow \quad f'(x) = \tan x \cdot \sec x$

(iii) 5 sec x + 4 cos x

Sol. Let $f(x) = 5 \sec x + 4 \cos x$

$$\therefore \quad \frac{d}{dx} f(x) = \frac{d}{dx} (5 \sec x + 4 \cos x)$$

$$= 5\frac{d}{dx}\sec x + 4\frac{d}{dx}\cos x$$

$$= 5 \cdot \sec x \tan x + 4 (-\sin x)$$

$\Rightarrow \quad f'(x) = 5 \sec x \tan x + 4 (-\sin x)$

(iv) cosec x

Sol. Let $\quad f(x) = \text{cosec } x = \frac{1}{\sin x}$

$$\therefore \quad \frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x \cdot \frac{d}{dx}1 - 1 \cdot \frac{d}{dx}\sin x}{(\sin x)^2}$$

$$= \frac{\sin x \cdot 0 - 1 \cdot \cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$\Rightarrow \quad f'(x) = -\cot x.\ \text{cosec } x$

(v) 3 cot x + 5 cosec x

Sol. Let, $\quad f(x) = 3 \cot x + \text{cosec } x$

$$\therefore \quad \frac{d}{dx} f(x) = \frac{d}{dx} (3 \cot x + 5 \text{ cosec } x)$$

$$= 3\frac{d}{dx}\cot x + 5\frac{d}{dx}\text{cosec } x$$

$$= 3 (-\text{cosec}^2 x) + 5 (-\cot x \text{ cosec } x)$$

$\Rightarrow \quad f'(x) = -\text{cosec } x\, (3 \text{ cosec } x + 5 \cot x)$

(vi) 5 sin x – 6 cos x + 7

Sol. Let $\quad f(x) = 5 \sin x - 6 \cos x + 7$

$$\therefore \quad \frac{d}{dx} f(x) = \frac{d}{dx} (5 \sin x - 6 \cos x + 7)$$

$$= 5\frac{d}{dx}\sin x - 6\frac{d}{dx}\cos x + \frac{d}{dx}7$$

$$= 5 \cdot \cos x - 6 \cdot (-\sin x) + 0$$

$$= 5 \cos x + 6 \sin x$$

(vii) 2 tan x – 7 sec x

Sol. Let $\quad f(x) = 2 \tan x - 7 \sec x$

$$\therefore \quad \frac{d}{dx} f(x) = \frac{d}{dx} (2 \tan x - 7 \sec x)$$

$$= 2\frac{d}{dx}\tan x - 7\frac{d}{dx}\sec x$$

$$= 2 \cdot \sec^2 x - 7 \sec x \cdot \tan x$$

$\Rightarrow \quad f'(x) = \sec x\, (2 \sec x - 7 \tan x)$

Long Answer Type Questions-I

(4 marks each)

Q. 1. Find the derivative of the following function : $\frac{\cos x}{1 + \sin x}$ [NCERT Exemplar Q. 16, Page 317]

Sol. Lef $f(x) = \frac{\cos x}{1 + \sin x}$

By quotient rule,

$$f'(x) = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{-1}{1+\sin x}$$

Q. 2. Find the derivative of the following function : $\dfrac{\sin x + \cos x}{\sin x - \cos x}$ **[NCERT Exemplar Q. 17, Page 318]**

Sol. Let $f(x) = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

By quotient rule,

$$f'(x) = \frac{(\sin x - \cos x)\frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x)\frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

$$= \frac{-[\sin^2 x + \cos^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x]}{(\sin x - \cos x)^2}$$

$$= \frac{-[1+1]}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

Q. 3. Find the derivative of the following function : $\dfrac{\sec x - 1}{\sec x + 1}$ **[NCERT Exemplar Q. 18, Page 419]**

Sol. Let $f(x) = \dfrac{\sec x - 1}{\sec x + 1}$

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By quotient rule,

$$f'(x) = \frac{(1+\cos x)\frac{d}{dx}(1 - \cos x) - (1 - \cos x)\frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2}$$

$$= \frac{(1 + \cos x)(\sin x) - (1 - \cos x)(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{\sin x + \cos x \sin x + \sin x - \sin x \cos x}{(1 + \cos x)^2}$$

$$= \frac{2\sin x}{(1 + \cos x)^2}$$

$$= \frac{2\sin x}{\left(1 + \frac{1}{\sec x}\right)^2} = \frac{2\sin x}{\frac{(\sec x + 1)^2}{\sec^2 x}}$$

$$= \frac{2\sin x \sec^2 x}{(\sec x + 1)^2}$$

$$= \frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x + 1)^2}$$

$$= \frac{2\sec x \tan x}{(\sec x + 1)^2}$$

Q. 4. Find the derivative of the following functions. $\dfrac{4x + 5\sin x}{3x + 7\cos x}$ **[NCERT Exemplar Q. 26, Page 318]**

Sol. Let $f(x) = \dfrac{4x + 5\sin x}{3x + 7\cos x}$

By quotient rule,

$$f'(x) = \frac{(3x + 7\cos x)\frac{d}{dx}(4x + 5\sin x) - (4x + 5\sin x)\frac{d}{dx}(3x + 7\cos x)}{(3x + 7\cos x)^2}$$

$$= \frac{(3x + 7\cos x)\left[4\frac{d}{dx}(x) + 5\frac{d}{dx}(\sin x)\right] - (4x + 5\sin x)\left[3\frac{d}{dx}(x) + 7\frac{d}{dx}(\cos x)\right]}{(3x + 7\cos x)^2}$$

$$= \frac{(3x + 7\cos x)[4 + 5\cos x] - (4x + 5\sin x)[3 - 7\sin x]}{(3x + 7\cos x)^2}$$

$$= \frac{12x + 15x\cos x + 28\cos x + 35\cos^2 x - 12x + 28x\sin x - 15\sin x + 35\sin^2 x}{(3x + 7\cos x)^2}$$

$$= \frac{15x\cos x + 28\cos x + 28x\sin x - 15\sin x + 35(\cos^2 x + \sin^2 x)}{(3x+7\cos x)^2}$$

$$= \frac{35 + 15x\cos x + 28\cos x + 28x\sin x - 15\sin x}{(3x+7\cos x)^2}$$

Differentiate each of the function with respect to *x* in following question using first principle.

Q. 5. cos (x^2 + 1) **[NCERT Exemplar Q. 43, Page 241]**

Sol. Let $f(x) = \cos(x^2+1)$ and $f(x+h) = \cos\{(x+h)^2+1\}$

$$\therefore \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\cos\{(x+h)^2+1\} - \cos(x^2+1)}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\left\{\frac{(x+h)^2+1+x^2+1}{2}\right\}\sin\left\{\frac{(x+h)^2+1-x^2-1}{2}\right\}}{h}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\sin\left\{\frac{(x+h)^2-x^2}{2}\right\}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\sin\left\{\frac{x^2+h^2+2xh-x^2}{2}\right\}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\sin\left\{\frac{h^2+2xh}{2}\right\}\right]$$

$$= -2\lim_{h\to 0}\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\lim_{h\to 0}\left\{\frac{\sin h\left(\frac{h+2x}{2}\right)}{h\left(\frac{h+2x}{2}\right)}\right\}\times\left(\frac{h+2x}{2}\right) \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right]$$

$$= -2x\sin(x^2+1)$$

Q. 6. $\frac{ax+b}{cx+d}$ **[NCERT Exemplar Q. 44, Page 241]**

Sol. Let $f(x) = \frac{ax+b}{cx+d}$

$$f(x+h) = \frac{a(x+h)+b}{c(x+h)+d}$$

$$\therefore \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{1}{h}[f(x+h)-f(x)]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{a(x+h)+b}{c(x+h)+d} - \frac{ax+b}{cx+d}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{ax+b+ah}{c(x+h)+d} - \frac{ax+b}{cx+d}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{(ax+ah+b)(cx+d)-(ax+b)\{c(x+h)+d\}}{\{c(x+h)+d\}(cx+d)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{(ax+ah+b)(cx+d)-(ax+b)(cx+ch+d)}{\{c(x+h)+d\}(cx+d)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{acx^2+achx+bcx+adx+adh+bd - \{acx^2+achx+adx+bcx+bch+bd\}}{\{c(x+h)+d\}(cx+d)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{acx^2+achx+bcx+adx+adh+bd - acx^2-achx-adx-bcx-bch-bd\}}{\{c(x+h)+d\}(cx+d)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{adh-bch}{\{c(x+h)+d\}(cx+d)}\right]$$

$$= \lim_{h\to 0}\frac{ac-bd}{\{c(x+h)+d\}(cx+d)} \qquad (h\neq 0)$$

$$= \frac{ac-bd}{(cx+d)^2}$$

Q. 7. $x^{2/3}$ **[NCERT Exemplar Q. 45, Page 438]**

Sol. Let $f(x) = x^{2/3}$

$$f(x+h) = (x+h)^{2/3}$$

Now, $\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{1}{h}\left[(x+h)^{2/3}-x^{2/3}\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[x^{2/3}\left(1+\frac{h}{x}\right)^{2/3}-x^{2/3}\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[x^{2/3}\left(1+\frac{h}{x}\cdot\frac{2}{3}+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\frac{h^2}{x^2}\cdots}{21}\right)\cdots 1\right]$

$\left[\because (1+x)^n = 1+nx+\frac{n(n-1)}{21}x^2+\cdots\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[x^{2/3}\left(\frac{2}{3}\cdot\frac{h}{x}-\frac{1}{9}\cdot\frac{h^2}{x^2}+\cdots\right)\right]$

$= \lim_{h\to 0}\frac{x^{2/3}}{h}\cdot\frac{2}{3}\frac{h}{x}\left(1-\frac{1}{6}\cdot\frac{h}{x}+\cdots\right)$

$= \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-1/3}$

Q. 8. ***x* cos *x*** **[NCERT Exemplar Q. 46, Page 438]**

Sol. Let $f(x) = x\cos x$

$\therefore$ $f(x+h) = (x+h)\cos(x+h)$

$\therefore$ $\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{1}{h}[(x+h)\cos(x+h)-x\cos x]$

$= \lim_{h\to 0}\frac{1}{h}[x\cos(x+h)+h\cos(x+h)-x\cos x]$

$= \lim_{h\to 0}\frac{1}{h}\left[x\left\{-2\sin\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right\}+h\cos(x+h)\right]$

$= \lim_{h\to 0}\left[-2x\sin\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{h}+\cos(x+h)\right]$ $\left[\because \cos C-\cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}\right]$

$= -2\lim_{h\to 0}x\sin\left(x+\frac{h}{2}\right)\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cdot\frac{1}{2}+\lim_{h\to 0}\cos(x+h)$

$= -2\cdot\frac{1}{2}x\sin x+\cos x$

$= \cos x - x\sin x$