CONTINUITY & DIFFERENTIABILITY

CHAPTER 5

Continuity

- **Continuity of a Function at a Point**
  A function is continuous at $x = k$ if the function is defined at $x = k$ and if the value of the function at $x = k$ equals the limit of the function at $x = k$, *i.e.*, a function $f(x)$ is continuous at $\lim_{x \to k^-} f(x) = \lim_{x \to k^+} f(x) = f(k)$

  If a function $f(x)$ is not continuous at a point $x = k$, then it is said to be discontinuous at that point. The point a is said to be a point of discontinuity.
- **Continuity of a Function in an Interval**

1. **Continuity on a closed interval:** A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ if and only if it is continuous in (a,b), right continuous at left end point and left continuous at right end point b.
2. **Continuity on an open interval:** A function $f(x)$ is said to be continuous on an open interval (a, b) if and only if it is continuous at every point on the interval (a, b).

- **Continuous function**
  A real function is said to be continuous if it is continuous at every point in the domain of f. A function $f(x)$ is said to be everywhere continuous if it is continuous $\forall$ R.

FORMULAE FOR LIMITS

(a) $\lim_{x\to 0} \cos x = 1$

(b) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(c) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

(d) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

(e) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

(f) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, (a > 0)$

(g) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(h) $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$

(i) $\lim_{x\to 0} \frac{x^n - a^n}{x - a} = na^{n-1}$

Differentiability

Differentiability can be defined as:

1. At a point $f(x)$ is differentiable at $x = c$ if

$$\lim_{\substack{x\to c-h \\ h\to 0}} \frac{f(c-h) - f(c)}{-h} = \lim_{\substack{x\to c+h \\ h\to 0}} \frac{f(c+h) - f(c)}{h} \text{ or } Lf'(c) = Rf'(c)$$

2. **In an interval**

(i) Open interval: If $f(x)$ is differentiable at every point of (a, b)

(ii) Closed interval: If $f(x)$ is differentiable in (a, b) and also at a and b.

- Left Hand Derivative of $f(x)$ at $x = m$,
  $Lf'(m) = \lim_{x\to m^-} \frac{f(x) - f(m)}{x - m}$ and,
- Right Hand Derivative of f(x) at x = m,
  $Rf'(m) = \lim_{x\to m^+} \frac{f(x) - f(m)}{x - m}$

  For a function to be differentiable at a point, LHD and RHD at that point should be equal.
- **Derivative of y *w.r.t.* x:** $\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x}$

  Also, for very-very small value

$$h, f'(x) = \frac{f(x+h) - f(x)}{h}, \text{ (as } h \to 0)$$

Relation between Continuity and Differentiability:

(i) If a function is differentiable at a point, it is continuous at that point as well.

(ii) If a function is not differentiable at a point, it may or may not be continuous at that point.

(iii) If a function is continuous at a point, it may or may not be differentiable at that point.

(iv) If a function is discontinuous at a point, it is not differentiable at that point.

SOME STANDARD DIFFERENTIATIONS:

Function	Derivatives
Derivative A constant (k)	0
x^n	nx^{n-1}
$\log_a x$	$\frac{1}{x \log_e a}$
$\log_e x$	$\frac{1}{x}$
a^x	$a^x \log_e a$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2 x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$
$\sec x$	$\sec x \tan x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

Continuity and Differentiability

Continuous Function

Suppose f is a real function on a subset of the real numbers and let 'c' be a point in the domain of f.

Then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$

A real function f is said to be continuous if it is continuous at every point in the domain of f.

eg: The function $f(x) = \frac{1}{x}$, $x \neq 0$ is continuous

Let 'c' be any non-zero real number, then $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$. For $c = 0, f(c) = \frac{1}{c}$ So $\lim_{x \to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of f.

Algebra of continuous functions

Suppose f and g are two real functions continuous at a real number c, then, $f+g$, $f-g$, $f.g$ and $\frac{f}{g}$ are continuous at $x = c[g(c) \neq 0]$.

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of f at c is $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

Every differentiable function is continuous, but the converse is not true.

Chain Rule

If $f = vou$, $t = u(x)$ and if both $\frac{dt}{dx}, \frac{dv}{dt}$ exists, then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$.

Logarithmic differentiation

Let $y = f(x) = [u(x)]^{v(x)}$

$\log y = v(x) \log[u(x)]$

$\frac{1}{y} \cdot \frac{dy}{dx} = v(x) \frac{1}{u(x)} u'(x) + v'(x) \log[u(x)]$

$\frac{dy}{dx} = y \left[\frac{v(x)}{u(x)} u'(x) + v'(x) \log[u(x)] \right]$

e.g. : Let $y = a^x$ Then $\log y = x \log a$

$\frac{1}{y} \cdot \frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a = a^x \log a$.

Derivatives of Implicit functions

If two variables are expressed by some relation then one will be the implicit function of other, is called Implicit function.

For example: Let $y = \cos x - \sin y$, then $\frac{dy}{dx} = \frac{d}{dx} \cos x - \frac{d}{dx} \sin y$

or, $\frac{dy}{dx} = -\sin x - \cos y . \frac{dy}{dx}$ or, $\frac{dy}{dx} = -\sin x / (1 + \cos y)$, where $y \neq (2n+1)\pi$

Derivatives of functions in parametric form

Let $x = f(t)$, $y = g(t)$ be two functions of parameter 't'.

Then, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\left(\frac{dx}{dy} \neq 0 \right)$

Thus, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ (provided $f'(t) \neq 0$)

eg : if $x = a\cos\theta$, $y = a\sin\theta$ then $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$, and so $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta$.

Second order derivative

Let $y = f(x)$ then $\frac{dy}{dx} = f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x)$ *i.e.*,

$\frac{d^2y}{dx^2} = f''(x)$ is the second order derivative of y *w.r.t.x.*

eg : if $y = 3x^2+2$, then $y' = 6x$ and $y'' = 6$.

Some Standard derivatives

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(vi) $\frac{d}{dx}(\mathrm{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

(vii) $\frac{d}{dx}(e^x) = e^x$

(viii) $\frac{d}{dx}(\log x) = \frac{1}{x}$

Trace the Mind Map

- First Level
- Second Level
- Third Level

Function	Derivatives
$\sec^{-1}x$	$\frac{1}{\lvert x\rvert\sqrt{1-x^2}}, \lvert x\rvert>1$
$\operatorname{cosec}^{-1}x$	$-\frac{1}{\lvert x\rvert\sqrt{1-x^2}}, -1\lvert x\rvert>1$
$\tan^{-1}x$	$\frac{1}{1+x^2}, x\in R$
$\cot^{-1}x$	$-\frac{1}{1+x^2}, x\in R$

Rules of Differentiation

(i) $\left(f(x)\pm g(x)\right)' = f'(x)\pm g'(x)$

(ii) Product Rule:

$$\frac{d}{x}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

or $[f(x)g(x)]' = f(x)g'(x) + g(x)f'(x)$

(iii) Quotient Rule:

$$\frac{d}{x}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x) - f(x)\frac{d}{dx}g(x)]}{[g(x)]^2}$$

or $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$

Derivatives of Implicit Functions

A variable y is said to be an explicit function of another independent variable x when the value of y is directly expressed in independent variable x. If $y = 5x^3 + 10x^2 + 9x + 7$, then y is an explicit function of x.

Implicit function

A variable y is said to be an implicit function of another independent variable x if y can not be expressed in terms of x. If $3x^2 + 5y^2 + 2axy + 5 = 0$, then the function is an implicit function because y cannot be expressed in independent variable x. Derivative of such function is obtained by differentiating both sides of the equation term wise and applying.

Derivatives of Inverse Trigonometric Functions

(i) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ **(ii)** $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$ **(iv)** $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

(v) $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$ **(vi)** $\frac{d}{dx}\operatorname{cosec}^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

Exponential Function:

A function f defined by $f(x) = k^x$, where k is any positive real number and x is any real number, is called the general exponential function with base e.

If base is 10, then it is called common exponential function with base e (an irrational number lying between 2 and 3), then the function $f(x) = e^x$, $x \in R$ is called natural exponential function.

Properties of Exponential Function:

(i) Domain of the exponential function is R, the set of real numbers.

(ii) Range C of the exponential function is the set of all positive real numbers.

(iii) The point (0, 1) always lies on the graph of the exponential function, [for any real $a > 1$]

(iv) Exponential function is always an increasing function.

(v) The graph of the exponential function approaches x-axis in the second quadrant but never meets it.

Logarithmic function:

(i) For real number $b > 1$, if if $b^x = a$, then logarithm of a to the base b is x. Thus, $\log_b a = x$ if $b^x = a$.

(ii) Domain of logarithmic function is positive real numbers. Range is real numbers.

(iii) Logarithmic functions with base 10 are called the 'Common Logarithms'.

(iv) Logarithmic functions with base e are called the 'Natural Logarithms'. Often natural logarithm is also denoted by ln.

Properties of logarithmic function:

(i) $\log_a b = \frac{\log_c b}{\log_c a}$ **(ii)** $\log_a xy = \log_a x + \log_a y$

(iii) $\log_a x^n = n\log_a x$ **(iv)** $\log_a \frac{x}{y} = \log_a x - \log_a y$

(v) $e^{\log_e f(x)} = f(x)$ **(vi)** $\log_e e = 1$

Logarithmic Differentiation

If $y = f(x)^{g(x)}$, where $f(x)$ and $g(x)$ are functions of x, we cannot find $\frac{dy}{dx}$ directly by any of the rules studied so far. To find derivative of this type of function, we take logarithm on both the sides, to get

$$\Rightarrow \log y = g(x)\log\left(f(x)\right)$$

$$\log y = \log\left(f(x)\right)^{g(x)}$$

Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = g(x)\frac{1}{f(x)}f'(x) + \log f(x)\cdot g'(x)$$

$$\therefore \frac{dy}{dx} = y\left[\frac{g(x)}{f(x)}f'(x) + g'(x)\log f(x)\right]$$

Alternatively, we may write

$$\Rightarrow y = e^{g(x)\log f(x)}$$

$$y = \left(f(x)\right)^{g(x)}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = e^{g(x)\log f(x)}\left[g(x)\cdot\frac{1}{f(x)}f'(x) + g'(x)\log f(x)\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(f(x)\right)^{g(x)}\left[\frac{g(x)}{f(x)}\cdot f'(x) + g'(x)\log f(x)\right]$$

Derivatives of Functions in Parametric Form

Sometimes the relation between two variables x and y can be expressed in terms of the third variable say t *i.e.,*

$$x = f(t) \text{ and } y = g(t).$$

This form of a function is called the parametric form and t is called the parameter.

To obtain $\frac{dy}{dx}$ in parametric form, we have by chain rule

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\left(\frac{dx}{dt}\neq 0\right)$$

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

NCERT CORNER

Exercise Questions

Exercise - 5.1

Q. 1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

Sol. **[NCERT Ex. 5.1, Q. 1, Page 116]**

$$f(x) = 5x - 3$$

$$\lim_{x\to 0^-} f(x) = \lim_{\substack{x\to 0-h\\ h\to 0}} 5(0-h) - 3 = -3$$

$$\lim_{x\to 0^+} f(x) = \lim_{\substack{x\to 0+h\\ h\to 0}} 5(0+h) - 3 = -3$$

$$\text{therefore,} \lim_{x\to 0} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\text{At } x = -3:$$

$$\text{We have, } f(-3) = 5(-3) - 3 = -18$$

$$\lim_{x\to -3^-} f(x) = \lim_{\substack{x\to 3-h\\ h\to 0}} [5(-3-h) - 3$$

$$= \lim_{\substack{x\to 3-h\\ h\to 0}} [-15 - 5h - 3] = -18$$

$$\lim_{x\to -3^+} f(x) = \lim_{\substack{x\to 3+h\\ h\to 0}} [5(-3+h) - 3]$$

$$= \lim_{\substack{x\to 3+h\\ h\to 0}} -15 + 5h - 3] = -18$$

$$\text{therefore } \lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} f(x) = f(-3)$$

Hence we can say that f is continuous at $x = -3$.

$$\text{At } x = 5: f(5) = 5(5) - 3 = 22$$

$$\lim_{x\to 5^-} f(x) = \lim_{\substack{x\to 5-h\\ h\to 0}} [5(5-h) - 3] = \lim_{\substack{x\to 5-h\\ h\to 0}} 25 - 5h - 3 = 22$$

$$\lim_{x\to -5^+} f(x) = \lim_{\substack{x\to 5+h\\ h\to 0}} [5(5+h) - 3] = \lim_{\substack{x\to 5+h\\ h\to 0}} 25 + 5h - 3 = 22$$

$$\text{therefore, } \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$$

Hence we can say that f is continuous at $x = 5$.

Q. 2. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$. **[NCERT Ex. 5.1, Q. 2, Page 116]**

Sol.

$$f(x) = 2x^2 - 1;$$

$$\text{RHL} = \lim_{x\to 3^+} f(x) = \lim_{\substack{x\to 3+h\\ h\to 0}} 2(3+h)^2 - 1$$

$$= \lim_{\substack{x\to 3+h\\ h\to 0}} 2(9 + 6h + h^2) - 1$$

$$= \lim_{\substack{x\to 3+h\\ h\to 0}} (18 + 12h + 2h^2) - 1$$

$$= \lim_{\substack{x\to 3+h\\ h\to 0}} (17 + 12h + 2h^2) = 17$$

$$\text{L.H.L} = \lim_{x\to 3^-} f(x) = \lim_{\substack{x\to 3-h\\ h-0}} 2(3-h)^2 - 1$$

$$= \lim_{\substack{x\to 3-h\\ h\to 0}} 2(9 - 6h + h^2) - 1$$

$$= \lim_{\substack{x\to 3-h\\ h\to 0}} (18 - 12h + 2h^2) - 1$$

$$= \lim_{\substack{x\to 3-h\\ h\to 0}} 2h^2 - 12h + 17 = 17$$

therefore, R.H.L. = L.H.L.

$$\text{Also, } f(3) = 2(3)^2 - 1 = 17$$

$$\text{therefore, } \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3)$$

Hence we can say that the given function $f(x) = 2x^2 - 1$ is continuous at $x = 3$.

Q. 3. Examine the following functions for continuity:

[NCERT Ex. 5.1, Q. 3, Page 116]

(a) $f(x) = x - 5$

(b) $f(x) = \dfrac{1}{x-5}, x \neq 5$

(c) $f(x) = \dfrac{x^2 - 25}{x+5}, x \neq -5$

(c) $f(x) = |x - 5|$

Sol. (a) $f(x) = x - 5$

$$\lim_{x\to a^+} f(x) = \lim_{\substack{x\to a+h\\ h\to 0}} (a+h) - 5 = a - 5$$

$$\lim_{x\to a^-} f(x) = \lim_{\substack{x\to a-h\\ h\to 0}} (a-h) - 5 = a - 5$$

$$\text{Also, } f(a) = a - 5 \text{ therefore, } \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$$

Hence we can say that the given function $f(x) = (x - 5)$ is continuous.

(b) $f(x) = \dfrac{1}{x-5}$

Let a be a real number, then

$$\lim_{x\to a^+} f(x) = \lim_{\substack{x\to a+h\\ h\to 0}} \frac{1}{a+h-5} = \frac{1}{a-5}$$

$$\lim_{x\to a^-} f(x) = \lim_{\substack{x\to a-h\\ h\to 0}} \frac{1}{a-h-5} = \frac{1}{a-5}$$

$$\text{Also, } f(a) = \frac{1}{a-5} \text{ therefore, } \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$$

Hence we can say that the given function $f(x) = \dfrac{1}{x-5}$ is continuous at all point except at $x = 5$.

(c) $f(x)=\frac{x^2-25}{x+5}=\frac{(x+5)(x-5)}{(x+5)}=x-5$

$\lim_{x\to a^+} f(x)=\lim_{\substack{x\to a+h\\h\to 0}}(a+h)-5=a-5$

and $\lim_{x\to a^-} f(x)=\lim_{\substack{x\to a-h\\h\to 0}}(a-h)-5=a-5$

Also, $f(a)=a-5$ therefore $\lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=f(a)$

Hence we can say that the given function $f(x)=\frac{1}{x-5}$ is continuous at every points of its domain.

(d) $f(x)=|x-5|$

$\lim_{x\to a^+} f(x)=\lim_{\substack{x\to a+h\\h\to 0}}|a+h-5|=|a-5|=a-5$

$\lim_{x\to a^-} f(x)=\lim_{\substack{x\to a-h\\h\to 0}}|a-h-5|=|a-5|=a-5$

Also, $f(a)=|a-5|=a-5$

therefore $\lim_{x\to a^+} f(x)=\lim_{x\to a-} f(x)=f(a)$

Hence we can say that the given function $f(x) = |x - 5|$ is continuous.

Q. 4. Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer

[NCERT Ex. 5.1, Q. 4, Page 116]

Sol. Given, $f(x) = x^n$, $n \in N$. So, $f(x)$ is a polynomial function and domain of f is R.

$\lim_{x\to n} f(x)^n=\lim_{x\to n} x^n=x^n=f(n)$

Hence it states that f is continuous at $n \in N$.

Q.5. Is the function f defined by

$$f(x)=\begin{cases}x, & \text{if } x\le 1\\ 5, & \text{if } x>1\end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

[NCERT Ex. 5.1, Q. 5, Page 116]

Sol.

$\lim_{x\to n} f(x)=\lim_{x\to n} x^n=x^n=f(n)$

$f(x)=\begin{cases}x, & \text{if } x\le 1\\ 5, & \text{if } x>1\end{cases}$

(i) At $x=0$, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(x)=0$

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x)=0$

Also, $f(0)=0$

Thus, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$

Hence we can say that f(x) is continuous at $x = 0$.

(ii) At $x=1$, $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x)=1$

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 5=5$

therefore, $\lim_{x\to 1^+} f(x)\ne\lim_{x\to 1^-}(x)$

Hence we can say that f is discontinuous at $x = 1$.

(iii) At $x=2$, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 5=5$

$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} 5=5$

Also, $f(2)=5$

Thus $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

Hence we can say that f(x) is continuous at $x = 2$.

Direction : For questions (6 - 12), find all points of discontinuity of function *f*, where if is defined by

Q. 6. $f(x)=\begin{cases}2x+3, & \text{if } x\le 2\\ 2x-3, & \text{if } x>2\end{cases}$ **[NCERT Ex. 5.1, Q.6, Page 116]**

Sol. For $x < 2$, function $f(x)= 2x + 3$ is polynomial and hence, continuous. For $x > 2$, function $f(x) = 2x-3$ is polynomial and hence, continuous.

For continuity at $x = 2$,

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2x+3)=\lim_{\substack{x\to 2-h\\h\to 0}}[2(2-h)+3]$

$=\lim_{\substack{x\to 2-h\\h\to 0}}(4-2h+3)=\lim_{\substack{x\to 2-h\\h\to 0}}(7-2h)=7$

$\lim_{x\to 2^+}, f(x)=\lim_{x\to 2^+}(2x-3)=\lim_{\substack{x\to 2+h\\h\to 0}}[2(2+h)-3]$

$=\lim_{\substack{x\to 2^+\\h\to 0}}(4+2h-3)=\lim_{\substack{x\to 2+h\\h\to 0}}(1+2h)=1$

Thus, $\lim_{x\to 2^-} f(x)\ne\lim_{x\to 2^+} f(x)$

Hence we can say that $f(x)$ is not continuous at $x = 2$.

So, the only point of discontinuity of the given function f is 2.

Q. 7. $f(x)=\begin{cases}|x|+3, & \text{if } x\le -3\\ -2x, & \text{if } -3<x<3\\ 6x+2, & \text{if } x\ge 3\end{cases}$

[NCERT Ex. 5.1, Q. 7, Page 116]

Sol.

At $x=3$:

$\lim_{x\to -3^-} f(x)=\lim_{x\to -3^-}|x|+3=\lim_{\substack{x\to 3-h\\h\to 0}}(|-3-h|+3)$

$=|-3-0|+3$

$=3+3=6$

$\lim_{x\to -3^+} f(x)=\lim_{x\to -3^+}(-2x)=\lim_{\substack{x\to 3+h\\h\to 0}}(-2(-3+h^3))$

$=-2(-3+0)=6$

$f(-3)=|-3|+3$

$=3+3=6$

Thus, $\lim_{x\to -3^-} f(x)=\lim_{x\to -3^+} f(x)=f(-3)$

therefore, f is continuous at $x = -3$.

At $x=3$:

$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(-2x)=\lim_{\substack{x\to 3-h\\h\to 0}}(-2(3-h))$

$=-2(3-0)=-6$

$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}(6x+2)=\lim_{\substack{x\to 3-h\\h\to 0}}(6(3+h)+2)$

$=6(3+0)+2=20$

Thus, $\lim_{x\to 3^+} f(x)\ne\lim_{x\to 3^+} f(x)$

Hence, we can say that $f(x)$ is discontinuous at $x = 3$

So, the only point of discontinuity of the given function f is 3.

Q. 8. $f(x)=\begin{cases}\dfrac{|x|}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ **[NCERT Ex. 5.1 ,Q. 8, Page 116]**

Sol. At $x=0:\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=\lim_{x\to 0^+}(1)=1$$

Thus, $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$

Hence we can say that $f(x)$ is discontinuous at $x = 0$.
So, the only point of discontinuity of f is 0.

Q. 9. $f(x)=\begin{cases}\dfrac{x}{|x|}, & \text{if } x<0\\ -1, & \text{if } x\geq 0\end{cases}$ **[NCERT Ex. 5.1, Q. 9, Page 116]**

Sol. At $x=0:\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{x}{|x|}=\lim_{x\to 0^+}\frac{x}{-x}$

$$=\lim_{x\to 0^-}(-1)=-1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(-1)=-1$$

Thus, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$

Hence we can say that $f(x)$ is continuous at $x= 0$.
So, $f(x)$ has no point of discontinuity.

Q. 10. $f(x)=\begin{cases}x+1, & \text{if } x\geq 1\\ x^2+1, & \text{if } x<1\end{cases}$

[NCERT Ex. 5.1 ,Q. 10, Page 116]

Sol. We observe that $f(x)$ is continuous at all real numbers $x < 1$ and $x > 1$ as it is a polynomial function.
Now, continuity at $x = 1$:

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x+1)=\lim_{x\to 1+h}(1+h)+1=2$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^2+1)=\lim_{x\to 1-h}(1-h)^2+1$$

$$=\lim_{\substack{x\to 1-h\\ h\to}}(1-2h+h^2)+1=2$$

Also, $f(1)=2$

$$\therefore \lim_{x\to 1^+} f(x)=\lim_{x\to 1^-} f(x)=f(1)$$

Hence we can say that, $f(x)$ is continuous at $x = 1$ and at all points.
So, $f(x)$ has no point of discontinuity.

Q. 11. $f(x)=\begin{cases}x^3-3, & \text{if } x\leq 2\\ x^2+1, & \text{if } x>2\end{cases}$

[NCERT Ex. 5.1, Q. 11, Page 116]

Sol. We observe that $f(x)$ is continuous at all real numbers $x < 2$ and $x > 2$ as it is polynomial function. Now, continuity at $x = 2$:

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(x^2+1)=\lim_{h\to 0}(2+h)^2+1$$

$$=\lim_{h\to 0}(4+4h+h^2)+1=5$$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(x^3-3)=\lim_{h\to 0}(2-h)^3-3$$

$$\lim_{\substack{x\to 2-h\\ h\to 0}}(8-12h+6h^2-h^3)-3=5$$

Also, $f(2)=8-3=5$

therfore, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

Hence we can say that $f(x)$ is continuous at $x = 2$ and at all points. So, f has no point of discontinuity.

Q. 12. $f(x)=\begin{cases}x^{10}-1, & \text{if } x\leq 1\\ x^2, & \text{if } x>1\end{cases}$

[NCERT Ex. 5.1, Q. 12, Page 116]

Sol. We observe that $f(x)$ is continuous at real numbers $x \leq 1$ and $x > 1$ as it is polynomial function. Now, continuity at $x = 1$:

$$\text{L.H.L.}=\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^{10}-1)$$

$$=\lim_{\substack{x\to 1-h\\ h\to 0}}\left[(1-h)^{10}-1\right]=0$$

$$\text{R.H.L.}=\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x^2)$$

$$=\lim_{\substack{x\to 1+h\\ h\to 0}}(1+h)^2=1$$

Also, $f(1)=1^{10}-1=0$

therfore, L.H.L $\neq$ R.H.L $\neq f(1)$

Hence we can say that f is discontinuous at $x = 1$.
So, the only point of discontinuity of $f(x)$ is $x = 1$.

Q. 13. Is the function defined by $f(x)=\begin{cases}x+5, & \text{if } x\leq 1\\ x-5, & \text{if } x>1\end{cases}$

a continuous function? [NCERT Ex. 5.1, Q. 13, Page 116]

Sol. We observe that $f(x)$ is continuous at all real numbers $x > 1$ and $x < 1$ as it is polynomial function. Now continuity at $x = 1$:

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(x-5)$$

$$=\lim_{\substack{x\to 1+h\\ h\to 0}}(1+h-5)$$

$$=\lim_{\substack{x\to 1+h\\ h\to 0}}(h-4)=-4$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x+5)$$

$$=\lim_{\substack{x\to 1-h\\ h\to 0}}(1-h+5)$$

$$=\lim_{\substack{x\to 1+h\\ h\to 0}}(6-h)=6$$

Thus, $\lim_{x\to 1^+} f(x)\neq \lim_{x\to 1^-} f(x)$

Hence we can say that $f(x)$ is not continuous at $x = 1$.

In Questions (14-16) Discuss the continuity of the function f, where f is defined by

Q.14. $f(x)=\begin{cases}3, & 0\leq x\leq 1\\ 4, & 1<x<3\\ 5, & 3\leq x\leq 10\end{cases}$

[NCERT Ex. 5.1, Q. 14, Page 117]

Sol.

At $x=1$:

$$\text{L.H.L.}=\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} 3=3 \text{ and}$$

$$\text{R.H.L. } \lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 4=4$$

therefore, L.H.L. $\neq$ R.H.L. at $x=1$

At $x=3$:

$$\text{L.H.L. } \lim_{x\to 3^-} f(x)=\lim_{x\to 3^-} 4=4 \text{ and}$$

R.H.L. $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 5 = 5$ therefore,

L.H.L. $\neq$ R.H.L. at $x = 3$

Hence we can say that function is not continuous at $x = 1$ and $x = 3$.

Q. 15. $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

[NCERT Ex.5.1, Q.15 Page 117]

Sol.

At $x = 0$:

L.H.L. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2(0) = 0$ and

R.H.L. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} 0 = 0$

Also, $f(0) = 0$ therefore, L.H.L. = R.H.L. = $f(0)$

So, $f(x)$ is continuous at $x = 0$. At $x = 1$:

L.H.L. = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 0 = 0$ and

R.H.L. = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4(1) = 4$

Also, $f(1)$=0 therefore, L.H.L. $\neq$ R.H.L.

Hence we can say that $f(x)$ is discontinuous at $x = 1$.

Q. 16. $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 \le x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$

[NCERT Ex. 5.1, Q. 16, Page 117]

Sol. At $x = -1$

R.H.L.= $\lim_{x\to -1^+} f(x)$= $\lim_{\substack{x\to -1+h \\ h\to 0}} 2(-1+h)$=$-2$

L.H.L.= $\lim_{x\to -1^-} f(x)$= $\lim_{\substack{x\to -1-h \\ h\to 0}} (-2)$=$-2$

Also, $f(-1)$=-2

therefore, $\lim_{x\to -1^-} f(x)$= $\lim_{x\to -1^+} f(x)$=$f(-1)$

Hence, $f(x)$ is continuous at x=-1.

Hence we can say that $f(x)$ is continuous at $x = -1$.

Q. 17. Find the relationship between *a* and *b* so that the function *f* defined by

$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+1, & \text{if } x > 3 \end{cases}$ **is continuous at $x = 3$.**

[NCERT Ex. 5.1, Q. 17, Page 117]

Sol. At $x = 3$:

$\lim_{x\to 3^-} f(x)$=$\lim_{x\to 3^-} (ax+1)$= $\lim_{\substack{x\to 3-h \\ h\to 0}} (a(3-h)+1)$

= $\lim_{x\to 3-h} (3a-ah+1)$=$3a+1$

$\lim_{x\to 3^+} f(x)$=$\lim_{x\to 3^+} (bx+3)$= $\lim_{\substack{x\to 3+h \\ h\to 0}} (b(3+h)+3)$

= $\lim_{\substack{x\to 3+h \\ h\to 0}} (3b+bh+3)$=$3b+3$

Also, $f(3)$=$3a+1$

Thus, $\lim_{x\to 3^-} f(x)$=$\lim_{x\to 3^+} f(x)$=$f(3)$

$f(x)$ is given as continuous at $x = 3$

$\Rightarrow 3b+3=3a+1 \Rightarrow 2=3(a-b) \Rightarrow a-b=\frac{2}{3}$

Therefore, this is the required relation between a and b.

Q. 18. For what value of λ is the function defined by $f(x) = \begin{cases} \lambda(x^2-2x), & \text{if } x \le 3 \\ 4x+1, & \text{if } x > 3 \end{cases}$ **continuous at $x = 0$?**

What about continuity at $x = 1$?

[NCERT Ex. 5.1, Q. 18, Page 117]

Sol. Since $f(x)$ is continuous at $x = 0$,

(i) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \lambda(x^2-2x) = \lambda(0-0) = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 4x+1 = 4(0)+1 = 1$

As L.H.L. $\neq$ R.H.L.

Hence we can say that for no value of λ. $f(x)$ is continuous at $x = 0$

(ii) At $x = 1$: $\lim_{x\to 1} f(x) = \lim_{x\to 1} 4x+1 = 4\times 1+1 = 5$ and

$f(1) = 4(1)+1 = 5$

Thus, $\lim_{x\to 1} f(x) = f(1)$ for any value of λ.

Hence we can say that $f(x)$ is continuous at $x = 1$ for any real value of λ.

Q. 19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here, $[x]$ denotes the greatest integer less than or equal to x.

[NCERT Ex. 5.1, Q. 19, Page 117]

Sol . Let $n \in I$. Then, $\lim_{x\to n^-} [x] = n-1$

and $g(n) = n - n = 0$.$[\because [n] = n$ because $n \in I]$

Now, $\lim_{x\to n^-} g(x) = \lim_{x\to n^-} (x-[x])$

$= \lim_{x\to n^-} x - \lim_{x\to n^-} [x] = n-(n-1) = 1$

and $\lim_{x\to n^+} g(x) = \lim_{x\to n^+} (x-[x])$

$= \lim_{x\to n^+} x - \lim_{x\to n^+} [x] = n - n = 0$

Thus, $\lim_{x\to n^-} g(x) \neq \lim_{x\to n^+} g(x)$.

Hence we can say that $g(x)$ is discontinuous at all integral points.

Q. 20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$? **[NCERT Ex. 5.1, Q. 20, Page 117]**

Sol. At $x = \pi$:

$\lim_{x\to \pi^+} f(x)$= $\lim_{\substack{x\to \pi+h \\ h\to 0}} (\pi+h)^2-\sin(\pi+h)+5$

=$\lim_{\substack{x\to \pi^+ \\ h\to 0}} [(\pi^2+h^2+2\pi h)+\sin h+5]$

=π^2+5

$\lim_{x\to \pi^-} f(x)$= $\lim_{\substack{x\to \pi-h \\ h\to 0}} (\pi-h)^2-\sin(\pi-h)+5$

= $\lim_{\substack{x\to \pi-h \\ h\to 0}} (\pi^2+h^2-2\pi h)-\sin h+5$

=π^2+5

Also, $f(\pi)$=π^2+5

Thus, R.H.L. = L.H.L. = $f(\pi)$.

Hence we can say that function is continuous at $x = \pi$.

Q. 21. Discuss the continuity of the following functions:
[NCERT Ex. 5.1, Q. 21, Page 117]

(a) $f(x) = \sin x + \cos x$

(b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \cdot \cos x$

Sol. (a) Let a be an arbitrary real number.
Then, $f(a) = \sin a + \cos a$

$$\lim_{x \to a^+} f(x) = \lim_{\substack{x \to a+h \\ h \to 0}} [\sin(a+h) + \cos(a+h)]$$
$$= \lim_{\substack{x \to a+h \\ h \to 0}} \{(\sin a \cos h + \cos a \sin h) + (\cos a \cos h - \sin a \sin h)\}$$
$$= \sin a(1) + \cos a(0) + \cos a(1) - \sin a(0)$$
$$= \sin a + \cos a$$
$$\lim_{x \to a^-} f(x) = \lim_{\substack{x \to a-h \\ h \to 0}} [\sin(a-h) + \cos(a-h)]$$
$$= \lim_{\substack{x \to \pi-h \\ h \to 0}} [(\sin a \cos h - \cos a \sin h) + (\cos a \cos h + \sin a \sin h)]$$
$$= \sin a(1) - \cos a(0) + \cos a(1) + \sin a(0)$$
$$= \sin a + \cos a.$$

$$+ (\cos a \cos h + \sin a \sin$$
$$= \sin a(1) - \cos a(0) + \cos a(1) + \sin a(0)$$
$$= \sin a + \cos a.$$

therefore, $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

Hence we can say that $f(x) = \sin x + \cos x$ is continuous everywhere.

(b) Let a be an arbitrary real number.

Then $f(a) = \sin a - \cos a$

$$\lim_{x \to a^+} f(x) = \lim_{\substack{x \to a+h \\ h \to 0}} [\sin(a+h) - \cos(a+h)]$$
$$= \lim_{\substack{x \to a+h \\ h \to 0}} \{(\sin a \cos h + \cos a \sin h) - (\cos a \cos h - \sin a \sin h)\}$$
$$= \sin a(1) + \cos a(0) - \cos a(1) + \sin a(0)$$
$$= \sin a - \cos a$$
$$\lim_{x \to a^-} f(x) = \lim_{\substack{x \to a-h \\ h \to 0}} [\sin(a-h) - \cos(a-h)]$$
$$= \lim_{\substack{x \to a-h \\ h \to 0}} [(\sin a \cos h - \cos a \sin h) - (\cos a \cos h + \sin a \sin h)]$$
$$= \sin a(1) - \cos a(0) - \cos a(1) + \sin a(0)$$
$$= \sin a - \cos a.$$

therefore, $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

$\Rightarrow f(x)$ is continuous at $x = a$.

Hence we can say that $f(x) = \sin x - \cos x$ is continuous everywhere.

(c) Let a be an arbitrary real number.

Then, $f(a) = \sin a \cos a$

$$\lim_{x \to a^+} f(x) = \lim_{\substack{x \to a+h \\ h \to 0}} [\sin(a+h)\cos(a+h)]$$
$$= \lim_{\substack{x \to a+h \\ h \to 0}} [(\sin a \cosh + \cos a \sinh)(\cos a \cosh - \sin a \sinh)]$$
$$= (\sin a(1) + \cos a(0))(\cos a(1) - \sin a(0))$$
$$= \sin a \cos a.$$
$$\lim_{x \to a^-} f(x) = \lim_{\substack{x \to a-h \\ h \to 0}} [\sin(a-h)\cos(a-h)]$$
$$= \lim_{\substack{x \to a-h \\ h \to 0}} (\sin a \cosh - \cos a \sinh)(\cos a \cosh + \sin a \sinh)$$
$$= (\sin a(1) - \cos a(0))\cos a(1) + \sin a(0)$$
$$= \sin a \cos a$$

therefore, $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$.

$\Rightarrow$ f(x) is continuous at $x = a$.

$\therefore f(x)$ is continuous at $x = a$.

Hence, we can say that $f(x) = \sin x . \cos x$ is continuous everywhere.

Q. 22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.
[NCERT Ex. 5.1, Q. 22, Page 117]

Sol. (a) $f(x) = \cos x$. Clearly, domain of $f = R$
Let a be an arbitrary real number, then $f(a) = \cos a$.

$$\lim_{x \to a^-} f(x) = \lim_{\substack{x \to a-h \\ h \to 0}} \cos(a-h)$$
$$= \lim_{h \to 0} (\cos a \cos h + \sin a \sin h) = \cos a$$
$$\lim_{x \to a^+} f(x) = \lim_{\substack{x \to a+h \\ h \to 0}} \cos(a+h)$$
$$= \lim_{h \to 0} (\cos a \cos h - \sin a \sin h) = \cos a$$

therefore, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

Hence we can say that $f(x) = \cos x$ is continuous at a for all $a \in R$.

(b) $f(x) = \operatorname{cosec} x \Rightarrow f(x) = \dfrac{1}{\sin x}$ and domain of $f = R - \{n\pi\}, n \in I$.

Also, $f(a) = \dfrac{1}{\sin a}$

$$\lim_{x \to a^+} \frac{1}{\sin x} = \frac{1}{\lim_{\substack{x \to a+h \\ x \to 0}} \sin(a+h)}$$
$$= \lim_{\substack{x \to a+h \\ x \to 0}} \frac{1}{\sin a \cos h + \cos a \sin h}$$
$$= \frac{1}{\sin a \cos 0 + \cos a \sin(0)}$$
$$= \frac{1}{\sin a(1) + \cos a(0)} = \frac{1}{\sin a + 0} = \frac{1}{\sin a}$$
$$\lim_{x \to a^-} f(x) = \lim_{\substack{x \to a-h \\ x \to 0}} \frac{1}{\sin(a-h)}$$
$$= \lim_{\substack{x \to a-h \\ h \to 0}} \frac{1}{\sin a \cos h - \cos a \sin h} = \frac{1}{\sin a}$$

therefore, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

Hence we can say that cosec x is continuous for all $a \in R - \{n\pi\}$ $n \in I$.

(c) $f(x) = \sec x \Rightarrow f(x) = \dfrac{1}{\cos x}$ Clearly, domain of

$f = R - \{(2n+1)\frac{\pi}{2}, n \in I\}$

Also, $f(a) = \frac{1}{\cos a}$

$$\lim_{x\to a^+} f(x) = \lim_{\substack{x\to a+h\\h\to 0}} \frac{1}{\cos(a+h)}$$

$$= \lim_{\substack{x\to a+h\\h\to 0}} \frac{1}{\cos a\cos h - \sin a\sin h}$$

$$= \frac{1}{\cos a\cos 0 - \sin a\sin 0}$$

$$= \frac{1}{\cos a\cos 0 - \sin a\sin 0}$$

$$= \frac{1}{\cos a(1) - \sin a(0)} = \frac{1}{\cos a}$$

$$\lim_{x\to a^-} f(x) = \lim_{\substack{x\to a-h\\h\to 0}} \frac{1}{\cos(a-h)}$$

$$= \lim_{\substack{x\to a-h\\h\to 0}} \frac{1}{\cos a\cos h + \sin a\sin h}$$

$$= \frac{1}{\cos a}$$

therefore, $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} f(x) = f(a)$

Hence we can say that sec x is continuous for all $a \in \mathrm{R} - \{(2n+1)\frac{\pi}{2}, n \in \mathrm{I}\}$

(d) $f(x) = \cot x$ $f(x) = \frac{1}{\tan x}$ and domain of $f = \mathrm{R} - \{n\pi, n \in \mathrm{I}\}$

$$\lim_{x\to a^+} f(x) = \lim_{\substack{x\to a+h\\h\to 0}} \frac{1}{\tan(a+h)}$$

$$= \lim_{\substack{x\to a+h\\h\to 0}} \frac{1}{\frac{\tan a + \tan h}{1 - \tan a \tanh}}$$

$$= \frac{1}{\frac{\tan a + 0}{1 - \tan a \tanh}} = \frac{1}{\frac{\tan a}{1-0}} = \frac{1}{\tan a}$$

$$\lim_{x\to a^-} f(x) = \lim_{\substack{x\to a-h\\h\to 0}} \frac{1}{\tan(a-h)}$$

$$= \lim_{\substack{x\to a-h\\h\to 0}} \frac{1}{\frac{\tan a - \tan h}{1 + \tan a \tanh}} = \frac{1}{\tan a}$$

Therefore, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

Hence we can say that cot x is continuous for all $a \in R - \{n\pi\}$ $n \in I$.

Q. 23. Find all points of discontinuity of f, where

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

[NCERT Ex. 5.1, Q. 23, Page 117]

Sol.

At, $x = 0, f(0) = 1$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{\substack{x\to 0-h\\h\to 0}} \frac{\sin(-h)}{-h} = 1$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{\substack{x\to 0+h\\h\to 0}} (h+1)$$

$$= 0 + 1 = 1$$

therefore, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Thus, $f(x)$ is continuous at $x = 0$.

When $x < 0$, sin x and x both are continuous. Therefore, $\frac{\sin x}{x}$ is also continuous.

When $x > 0$, $f(x) = x + 1$ is a polynomial.

Therefore, $f(x)$ is continuous.

Hence we can say that $f(x)$ is not discontinuous at any point.

Q. 24. Determine if f defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ a continuous function?

[NCERT Ex. 5.1, Q. 24, Page 117]

Sol. We have, $f(0) = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{\substack{x\to 0-h\\h\to 0}} (0-h)^2 \sin\frac{1}{(0-h)}$$

$$= \lim_{\substack{x\to 0-h\\h\to 0}} (-h^2 \sin\frac{1}{h})$$

$$\lim_{x\to 0^+} f(x) = \lim_{\substack{x\to 0+h\\h\to 0}} (0+h)^2 \frac{1}{(0+h)}$$

$$= \lim_{\substack{x\to 0+h\\h\to 0}} h^2 \sin\frac{1}{h} = 0$$

therefore, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \Rightarrow f(x)$ is continuous at $x = 0$.

For $x \uparrow 0$, $f(x)$ is continuous at every point.

Hence we can say that $f(x)$ is a continuous function.

Q. 25. Examine the continuity of f, where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$$

[NCERT Ex. 5.1, Q. 25, Page 118]

Sol. We have

$$\lim_{x\to 0^-} f(x) = \lim_{\substack{x\to 0-h\\h\to 0}} [\sin(0-h) - \cos(0-h)]$$

$$= \lim_{\substack{x\to 0-h\\h\to 0}} (-\sinh - \cosh)$$

$$= -(0) - 1 = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{\substack{x\to 0+h\\h\to 0}} [\sin(0+h) - \cos(0+h)]$$

$$= \lim_{h\to 0}(\sin h - \cos h)$$

$$= \sin 0 - \cos 0$$

$$= 0 - 1$$

$$= -1$$

Also, $f(0) = -1$

therefore, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

At $x < 0$, $f(x)$ = sin x – cos x is continuous

At $x > 0$, $f(x)$ = sin x – cos x is also continuous.

Hence we can say that $f(x)$ is continuous at all $x \in \mathrm{R}$.

Find the values of k so that the function f is continuous at the indicated point in questions 26 to 29.

Q. 26. $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$ **at** $x=\dfrac{\pi}{2}$.

[NCERT Ex. 5.1, Q. 26, Page 118]

Sol.

$$\lim_{x\to\frac{\pi}{2}^-} f(x)=\lim_{\substack{x\to\frac{\pi}{2}-h\\ h\to 0}}\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)}$$

$$=\lim_{\substack{x\to\frac{\pi}{2}-h\\ h\to 0}}\frac{k\sin h}{\pi-\pi+2h}$$

$$=\lim_{\substack{x\to\frac{\pi}{2}-h\\ h\to 0}}\frac{k\sin h}{2h}=\frac{k}{2}\lim_{\substack{x\to\frac{\pi}{2}-h\\ h\to 0}}\frac{\sin h}{h}=\frac{k}{2}$$

$$\lim_{x\to\frac{\pi}{2}^+} f(x)=\lim_{\substack{x\to\frac{\pi}{2}+h\\ h\to 0}}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=\lim_{\substack{x\to\frac{\pi}{2}+h\\ h\to 0}}\frac{-k\sin h}{-2h}$$

$$=\frac{k}{2}\lim_{\substack{x\to\frac{\pi}{2}+h\\ h\to 0}}\frac{\sin h}{h}=\frac{k}{2}$$

Also, $f\left(\dfrac{\pi}{2}\right)=3.$

For continuity at $x=\dfrac{\pi}{2}$, we have

$$\lim_{x\to\frac{\pi}{2}^-} f(x)=\lim_{x\to\frac{\pi}{2}^+} f(x)=f\left(\frac{\pi}{2}\right)$$

$\Rightarrow$ $\dfrac{k}{2}=3\Rightarrow k=6$

Q.27. $f(x)=\begin{cases}kx^2 & \text{if } x\leq 2\\ 3, & \text{if } x>2\end{cases}$ ***at*** $x=2.$

[NCERT Ex. 5.1, Q. 27, Page 118]

Sol.

We have, $f(2)=4k$

$$\lim_{x\to 2^+} f(x)=\lim_{\substack{x\to 2+h\\ h\to 0}} 3=3$$

$$\lim_{x\to 2^-} f(x)=\lim_{\substack{x\to 2-h\\ h\to 0}} (2-h)^2k=4k$$

For continuity at $x=2$, we have

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

$\Rightarrow 4k=3\Rightarrow k=\dfrac{3}{4}$

Q. 28. $f(x)=\begin{cases}kx+1 & \text{if } x\leq\pi\\ \cos x & \text{if } x>\pi\end{cases}$ **at** $x=\pi.$

[NCERT Ex. 5.1, Q. 28, Page 118]

Sol.

$$\lim_{x\to\pi^+} f(x)=\lim_{\substack{x\to\pi+h\\ h\to 0}} f(\pi+h)$$

$$=\lim_{\substack{x\to\pi+h\\ h\to 0}}\cos(\pi+h)$$

$$=\lim_{\substack{x\to\pi+h\\ h\to 0}} -\cos h=-\cos(0)=-1$$

$$\lim_{x\to\pi^-} f(x)=\lim_{\substack{x\to\pi-h\\ h\to 0}} k(\pi-h)+1$$

$$=k\pi+1 \text{ and } f(\pi)=k\pi+1$$

Since the given function is continuous at $x=\pi$,

therefore, $\lim_{x\to\pi^+} f(x)=\lim_{x\to\pi^-} f(x)=f(\pi)$

$\Rightarrow k\pi+1=-1\Rightarrow k\pi=-1-1$

$\Rightarrow k\pi=-2\Rightarrow k=\dfrac{-2}{\pi}$

Q. 29. $f(x)=\begin{cases}kx+1, & \text{if } x\leq 5\\ 3x-5, & \text{if } x>5\end{cases}$ **at** $x=5$

[NCERT Ex. 5.1, Q. 29, Page 118]

Sol.

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^-}(kx+1)$$

$$=\lim_{\substack{x\to 5-h\\ h\to 0}}(k(5-h)+1)=k(5-0)+1$$

$$=5k+1$$

$$\lim_{x\to 5^+} f(x)=\lim_{x\to 5^+}(3x-5)=\lim_{\substack{x\to 5+h\\ h\to 0}}(3(5+h)-5)$$

$$=3(5+0)-5=10$$

For continuity at $x=5$, $\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$

$\Rightarrow 5k+1=10$

$\Rightarrow 5k=9\Rightarrow k=\dfrac{9}{5}$

Q. 30. Find the values of a and b such that the function defined by $f(x)=\begin{cases}5, & \text{if } x\leq 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\geq 10\end{cases}$ **is a continuous function.**

[NCERT Ex. 5.1, Q. 30, Page 118]

Sol. Since f is continuous at all x, so f is continuous at $x=2, 10$.

At $x=2$: $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(5)=5$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(ax+b)$$

$$=\lim_{\substack{x\to 2+h\\ h\to 0}}(a(2+h)+b)=a(2+0)+b$$

$$=2a+b \text{ and } f(2)=5$$

For continuity, $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

$\Rightarrow 5=2a+b=5\Rightarrow 2a+b=5$ ...(i)

At $x=10$:

$$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^-}(ax+b)$$

$$=\lim_{\substack{(x\to 10-h)\\ h\to 0}}(a(10-h)+b)=a(10-0)+b$$

$$=10a+b$$

$$\lim_{x\to 10^+} f(x)=\lim_{x\to 10^+}(21)=21$$

$$\Rightarrow f(10)=21$$

For continuity, $\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$

$\Rightarrow 10a+b=21\Rightarrow 10a+b=21$ ...(ii)

Subtracting (i) from (ii), we get $8a=16\Rightarrow a=2$

Putting $a = 2$ in (i), we get $2(2) + b = 5$
$\Rightarrow b = 5 - 4 = 1$ Hence, $a = 2, b = 1$.

Q. 31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.
[NCERT Ex. 5.1, Q. 31, Page 118]

Sol. Let $f(x) = \cos(x^2)$.
Domain of $f = R$.
Let a be any arbitrary real number.

Then, $\lim_{x\to a^+} f(x) = \lim_{\substack{x\to a+h\\h\to 0}} \cos(a+h)^2 = \cos a^2$

$\Rightarrow \lim_{x\to a^-} f(x) = \lim_{\substack{x\to a-h\\h\to 0}} \cos(a-h)^2$
$= \cos a^2$ and $f(a) = \cos a^2$

Thus, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a) \forall a \in \text{R}$.

Therefore, $f(x) = \cos(x^2)$ is continuous at $a \forall a \in \text{R}$.

Q. 32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.
[NCERT Ex. 5.1, Q. 32, Page 118]

Sol. We know that cosine function is everywhere continuous and also modulus function is continuous. Therefore, $|\cos x|$ is everywhere continuous.

Q. 33. Examine that $\sin|x|$ is a continuous function.
[NCERT Ex. 5.1, Q. 33, Page 118]

Sol. Let $f(x) = |x|$ and $g(x) = \sin x$.
Then, $(gof)(x) = g[f(x)] = g(|x|) = \sin |x|$
Now, f and g being continuous, it follows that their composite function (gof) is continuous.

Q. 34. Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$.
[NCERT Ex. 5.1, Q. 34, Page 118]

Sol.
We have,

$$f(x) = \begin{cases} -(x)-[-(x+1)], & \text{if } x < -1 \\ -(x)-(x+1), & \text{if} -1 \le x < 0 \\ (x)-(x+1), & \text{if } x \ge 0 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} 1, & \text{if } x < -1 \\ -2x-1, & \text{if} -1 \le x < 0 \\ -1, & \text{if } x \ge 0 \end{cases}$$

At $x = -1$:

$$\lim_{x\to -1^-} f(x) = 1$$
$$\lim_{x\to -1^+} f(x) = \lim_{\substack{x\to -1+h\\h\to 0}} (-2(-1+h)-1) = 1$$
$$f(-1) = -2(-1)-1 = 1$$

Thus, $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$
$\Rightarrow f$ is continuous at $x = -1$

At $x = 0$:
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-2x-1) = \lim_{\substack{x\to 0-h\\h\to 0}} (-2(-h)-1) = -1$$

$$\lim_{x\to 0^+} = (-1)$$

Also, $f(0) = -1$
Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow f$ is continuous at $x = 0$.
Also, f being a constant is continuous when $x < -1$ or when $x > 0$. Therefore, f is continuous for all $x \in \text{R}$.
Hence we can say that there is no point of discontinuity.

Exercise - 5.2

Differentiate the functions with respect to x in Questions 1 to 8.

Q. 1. $\sin(x^2 + 5)$ **[NCERT Ex. 5.2, Q. 1, Page 122]**

Sol. Let $y = \sin(x^2+5)$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\sin(x^2+5) = \cos(x^2+5)\frac{d}{dx}(x^2+5)$$
$$= \cos(x^2+5)(2x+0) = 2x\cos(x^2+5)$$

therefore

$$\frac{dy}{dx} = \frac{d}{dx}\cos(x^2+5)(2x+0) = 2x\cos(x^2+5)$$

Q. 2. $\cos(\sin x)$ **[NCERT Ex. 5.2, Q. 2, Page 122]**

Sol. Let $y = \cos(\sin x)$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\cos(\sin x) = -\sin(\sin x)\frac{d}{dx}\sin x$$
$$= -\sin(\sin x)\cos x$$

therefore $\frac{dy}{dx} = -\sin(\sin x)\cos x$

Q. 3. $\sin(ax + b)$ **[NCERT Ex. 5.2, Q. 3, Page 122]**

Sol. Let $y = \sin(ax + b)$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\sin(ax+b) = \cos(ax+b)\frac{d}{dx}(ax+b)$$
$$= \cos(ax+b)(a+0)$$

therefore $\frac{dy}{dx} = a\cos(ax+b)$

Q. 4. $\sec(\tan(\sqrt{x}))$ **[NCERT Ex. 5.2, Q. 4, Page 122]**

Sol. Let $y = \sec\{(\tan(\sqrt{x})\}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\sec(\tan\sqrt{x})$$
$$= \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})\frac{d}{dx}\tan\sqrt{x}$$
$$= \sec(\tan\sqrt{x})\cdot\tan(\tan\sqrt{x})\cdot\sec^2\sqrt{x}\frac{d}{dx}(\sqrt{x})$$
$$= \sec(\tan\sqrt{x})\cdot\tan(\tan\sqrt{x})\cdot\sec^2\sqrt{x}\cdot\frac{1}{2}x^{\frac{1}{2}-1}$$
$$= \sec(\tan\sqrt{x})\cdot\tan(\tan\sqrt{x})\cdot\sec^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

Therefore,

$$\frac{dy}{dx} = \sec(\tan\sqrt{x})\cdot\tan(\tan\sqrt{x})\cdot\sec^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

Q. 5. $\frac{\sin(ax+b)}{\cos(cx+d)}$ **[NCERT Ex. 5.2, Q. 5, Page 122]**

Sol. Let $y = \frac{\sin(ax+b)}{\cos(cx+d)}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin(ax+b)}{\cos(cx+d)}\right)$$

$$= \frac{\cos(cx+d)\frac{d}{dx}\sin(ax+b) - \sin(ax+b)\frac{d}{dx}\cos(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{a\cos(cx+d)\cos(ax+b) + c\sin(ax+d)\sin(cx+d)}{\cos^2(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b).\tan(cx+d) \sec(cx+d)$$

Therefore, $\frac{dy}{dx} = a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\cdot\tan(cx+d) \sec(cx+d)$

Q. 6. $\cos x^3 \cdot \sin^2(x^5)$ **[NCERT Ex. 5.2, Q. 6, Page 122]**

Sol. Let $y = \cos x^3 \cdot \sin^2(x^5)$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[\cos x^3 \cdot \sin^2(x^5)]$$

$$= \cos x^3 \frac{d}{dx}\sin^2(x^5) + \sin^2(x^5)\frac{d}{dx}\cos x^3$$

$$= \cos x^3 \cdot 2\sin(x^5)\frac{d}{dx}\sin(x^5) + \sin^2(x^5)(-\sin x^3)\frac{d}{dx}(x^3)$$

$$= \cos x^3 .2\sin(x^5)\cos(x^5)\frac{d}{dx}(x^5) + \sin^2(x^5)(-\sin x^3)(3x^2)$$

$$= 10x^4\cos x^3\sin(x^5)\cos(x^5) - 3x^2\sin^2(x^5)\sin x^3.$$

Therefor $\frac{dy}{dx} = 10x^4\cos x^3\sin(x^5)\cos(x^5) - 3x^2\sin^2(x^5)\sin x^3$

Q. 7. $2\sqrt{\cot(x^2)}$ **[NCERT Ex. 5.2, Q. 7, Page 122]**

Sol. Let $y = 2\sqrt{\cot(x^2)}$

$$\Rightarrow \frac{dy}{dx} = 2\frac{d}{dx}\sqrt{\cot(x^2)} = 2\cdot\frac{1}{2}\{\cot(x^5)\}^{-1/2}\cdot\frac{d}{dx}\cot(x^2)$$

$$= \frac{1}{\sqrt{\cot(x^2)}}\{-\text{cosec}^2(x^2)\}\frac{d}{dx}(x^2)$$

$$= \frac{1}{\sqrt{\cot(x^2)}}\{-\text{cosec}^2(x^2)\}(2x)$$

$$= \frac{-2x\,\text{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}$$

$$= \frac{-2x}{\sin^2 x^2\sqrt{\cot x^2}}$$

$$\Rightarrow \frac{-2x}{(\sin x^2)^2\sqrt{\cot x^2}}$$

Therefore, $\frac{dy}{dx} = \frac{-2x}{(\sin x^2)^2\sqrt{\cot x^2}}$

$$= \frac{-2\sqrt{2}x}{\sin x^2\sqrt{2\sin x^2\cos x^2}}$$

$$= \frac{-2\sqrt{2}x}{\sin x^2\sqrt{\sin 2x^2}}$$

Q. 8. $\cos(\sqrt{x})$ **[NCERT Ex. 5.2, Q. 8, Page 122]**

Sol. Let $y = \cos(\sqrt{x})$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\cos(\sqrt{x}) = -\sin\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x})$$

$$= -\sin\sqrt{x}\cdot\frac{1}{2}(x)^{-1/2} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

Therefore, $\frac{dy}{dx} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$

Q. 9. Prove that the function f given by $f(x) = |x - 1|$, $x \in$ R is not differentiable at $x = 1$.

[NCERT Ex. 5.2, Q. 9, Page 122]

Sol. We have, $f(x) = |x-1|$

$\Rightarrow$ $f(1) = |1-1| = 0$

$$Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{|1+h-1|-0}{h}$$

$$= \lim_{h\to 0}\frac{|h|}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

and $$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{|1-h-1|-0}{-h}$$

$$= \lim_{h\to 0}\frac{|-h|}{h} = \lim_{h\to 0}\frac{h}{-h} = -1$$

Thus, $Rf'(1) \neq Lf'(1)$

This shows that $f(x)$ is not differentiable at $x = 1$.

Q. 10. Prove that the greatest integer function defined by $f(x) = [x]$, $0 < x < 3$ is not differentiable at $x = 1$ and $x = 2$.

[NCERT Ex. 5.2, Q. 10, Page 122]

Sol. At $x = 1$:

$$Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{[1+h]-[1]}{h}$$

$= 0$ since $[1+h] = 1$ and $[1] = 1]$

and $$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{[1-h]-[1]}{-h} = \infty$$

Thus, $Rf'(1) = Lf'(1)$

Hence $f(x) = [x]$ is not differentiable at $x = 1$.

At $x = 2$

$$Rf'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{[2+h]-[2]}{h} = 0$$
$$Lf'(2) = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$
$$= \lim_{h\to 0}\frac{[2-h]-[2]}{-h}$$
$$= \frac{1-2}{0} = \infty$$

Therefore, $Rf'(2) \neq Lf'(2)$

Hence we can say that $f(x) = [x]$ is not differentiable at $x = 2$.

Exercise - 5.3

Find $\frac{dy}{dx}$ in the following:

Q. 1. $2x + 3y = \sin x$ [NCERT Ex. 5.3, Q. 1, Page 125]

Sol. Given that,

$$2x + 3y = \sin x \quad ...(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$2 + 3\frac{dy}{dx} = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

Q. 2. $2x + 3y = \sin y$ [NCERT Ex. 5.3, Q. 2, Page 125]

Sol. Given that,

$$2x + 3y = \sin y \quad ...(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$2 + 3\frac{dy}{dx} = \cos y\frac{dy}{dx}$$
$$\Rightarrow \cos y\frac{dy}{dx} - 3\frac{dy}{dx} = 2$$
$$\Rightarrow \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

Q. 3. $ax + by^2 = \cos y$ [NCERT Ex. 5.3, Q. 3, Page 125]

Sol. Given that,

$$ax + by^2 = \cos y \quad (i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$a + b\left[2y\frac{dy}{dx}\right] = -\sin y\frac{dy}{dx}$$
$$\Rightarrow a + 2by\frac{dy}{dx} + \sin y\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx}[2by + \sin y] = -a$$
$$\Rightarrow \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

Q. 4. $xy + y^2 = \tan x + y$

[NCERT Ex. 5.3 ,Q. 4,Page 125]

Sol. Given that,

$$xy + y^2 = \tan x + y \quad ...(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$
$$\Rightarrow x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x - y$$
$$\Rightarrow \frac{dy}{dx}[x + 2y - 1] = \sec^2 x - y$$
$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$$

Q. 5. $x^2 + xy + y^2 = 100$ [NCERT Ex. 5.3, Q. 5, Page 125]

Sol. Given that,

$$x^2 + xy + y^2 = 100 \quad ...(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx}[2 + 2y] = -(2x + y)$$
$$\Rightarrow \frac{dy}{dx} = \frac{-(2x+y)}{(x+2y)}$$

Q. 6. $x^3 + x^2y + xy^2 + y^3 = 81$

[NCERT Ex. 5.3, Q. 6, Page 125]

Sol. We are given that,

$$x^3 + x^2y + xy^2 + y^3 = 81 \quad (i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$3x^2 + x^2\frac{dy}{dx} + y(2x) + y^2 + x\left(2y\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx}[x^2 + 2xy + 3y^2] = -(3x^2 + 2xy + y^2)$$
$$\Rightarrow \frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}$$

Q. 7. $\sin^2 y + \cos xy = k$

[NCERT Ex. 5.3, Q. 7, Page 125]

Sol. Given that,

$$\sin^2 y + \cos xy = k \quad ...(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$2\sin y\frac{d}{dx}(\sin y) + (-\sin xy)\frac{d}{dx}(xy) = 0$$
$$\Rightarrow 2\sin y\cos y\frac{dy}{dx} + (-\sin xy)\left[x\frac{dy}{dx} + y\right] = 0$$
$$\Rightarrow 2\sin y\cos y\frac{dy}{dx} - x\sin xy\frac{dy}{dx} - y\sin xy = 0$$
$$\Rightarrow \frac{dy}{dx}[2\sin y\cos y - x\sin xy] = y\sin xy$$
$$\Rightarrow \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Q. 8. $\sin^2 x + \cos^2 y = 1$ [NCERT Ex. 5.3, Q. 8, Page 125]

Sol. Given that,

$$\sin^2 x + \cos^2 y = 1 \quad ...(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$2\sin x\frac{d}{dx}(\sin x) + 2\cos y\frac{d}{dx}(\cos y) = 0$$
$$\Rightarrow 2\sin x\ \cos x + 2\cos\ y(-\sin y)\frac{dy}{dx} = 0$$
$$\Rightarrow \sin 2x - \sin 2y\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

Q. 9. $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ **[NCERT Ex. 5.3, Q. 9, Page 125]**

Sol. $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Putting $x = \tan\theta$, we get

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$
$$= 2\theta = 2\tan^{-1} x$$

Therefore, $\frac{dy}{dx} = \frac{2}{1+x^2}$

Q. 10. $y = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

[NCERT Ex. 5.3, Q. 10, Page 125]

Sol. $y = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$$

$\Rightarrow y = \tan^{-1}(\tan 3\theta)$

$\Rightarrow y = 3\theta$

$\Rightarrow y = 3\tan^{-1} x$

$\Rightarrow \frac{dy}{dx} = \frac{3}{1+x^2}$

Q. 11. $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1.$

[NCERT Ex. 5.3, Q. 11, Page 125]

Sol. $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, where, $0 < x < 1$

Putting $x = \tan\theta$, we get

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \Rightarrow y = \cos^{-1}(\cos 2\theta)$$

$\Rightarrow y = 2\theta$

$\Rightarrow y = 2\tan^{-1} x$

$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$

Q. 12. $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1.$

[NCERT Ex. 5.3, Q. 12, Page 125]

Sol. Putting $x = \tan\theta$, we get

$$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \Rightarrow y = \sin^{-1}(\cos 2\theta)$$

$\Rightarrow y = \sin^{-1}\left\{\cos\left(\frac{\pi}{2} - 2\theta\right)\right\}$

$\Rightarrow y = \frac{\pi}{2} - 2\theta$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1} x$

$\Rightarrow \frac{dy}{dx} = 0 - \frac{2}{1+x^2}$

$\Rightarrow \frac{dy}{dx} = -\frac{2}{1+x^2}$

Q. 13. $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1.$

[NCERT Ex. 5.3, Q. 13, Page 125]

Sol. Putting $x = \tan\theta$, we get

$$y = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \Rightarrow y = \cos^{-1}(\sin 2\theta)$$

$\Rightarrow y = \cos^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$

$\Rightarrow y = \frac{\pi}{2} - 2\theta$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1} x$

$\Rightarrow \frac{dy}{dx} = -\frac{2}{1+x^2}$

Q. 14. $y = \sin^{-1}(2x\sqrt{1-x^2}), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

[NCERT Ex. 5.3, Q. 14, Page 125]

Sol. Putting $x = \sin\theta$, we get

$$y = \sin^{-1}[2\sin\theta\sqrt{1-\sin^2\theta}]$$

$\Rightarrow y = \sin^{-1}(\sin 2\theta)$

$\Rightarrow y = 2\theta$

$\Rightarrow y = 2\sin^{-1} x$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

Q. 15. $y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right), 0 < x < \frac{1}{\sqrt{2}}.$

[NCERT Ex. 5.3, Q. 15, Page 125]

Sol. Putting $x = \cos\theta$, we get

$$y = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right)$$

$\Rightarrow y = \cos^{-1}(2\cos^2\theta - 1)$

$\Rightarrow y = \cos^{-1}(\cos 2\theta)$

$\Rightarrow y = 2\theta$

$\Rightarrow y = 2\cos^{-1} x$

$\Rightarrow \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$

Exercise - 5.4

Differentiate the following w.r.t. x

Q. 1. $\dfrac{e^x}{\sin x}$ **[NCERT Ex. 5.4, Q. 1, Page 130]**

Sol. Let $y = \dfrac{e^x}{\sin x}$

Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x \cdot e^x - e^x \cdot \cos x}{\sin^2 x}$$

$$= \frac{e^x(\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in Z$$

Q. 2. $e^{\sin^{-1} x}$ **[NCERT Ex. 5.4, Q. 2, Page 130]**

Sol. Let $e^{\sin^{-1} x} = y$

Therefore, $\dfrac{dy}{dx} = \dfrac{d}{dx}(e^{\sin^{-1} x})\dfrac{d}{dx}(\sin^{-1} x)$

$$= e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}, x \in (-1,1)$$

Q. 3. e^{x^3} **[NCERT Ex. 5.4, Q. 3, Page 130]**

Sol. Let $y = e^{x^3}$

Therefore, $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(e^{x^3}\right) = e^{x^3}\dfrac{d}{dx}(x^3)$

$$= e^{x^3} \cdot 3x^2 = 3e^{x^3} \cdot x^2$$

Q. 4. $\sin(\tan^{-1} e^{-x})$ **[NCERT Ex. 5.4, Q. 4, Page 130]**

Sol. Let $y = \sin(\tan^{-1} e^{-x})$

Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\sin(\tan^{-1}(e^{-x})))$$

$$= \cos(\tan^{-1} e^{-x})\frac{d}{dx}(\tan^{-1}(e^{-x}))$$

$$= \cos(\tan^{-1} e^{-x}) \cdot \frac{1}{(1+e^{-2x})} \cdot \frac{d}{dx}e^{-x}$$

$$= \cos(\tan^{-1} e^{-x}) \cdot \frac{-e^{-x}}{(1+e^{-2x})} = \frac{-e^{-x}\cos(\tan^{-1} e^{-x})}{1+e^{-2x}}$$

Q. 5. $\log(\cos e^x)$ **[NCERT Ex. 5.4, Q. 5, Page 130]**

Sol. Let $y = \log(\cos e^x)$

Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}\log(\cos e^x)$$

$$= \frac{1}{\cos e^x}\frac{d}{dx}(\cos e^x)$$

$$= \frac{1}{\cos e^x} \cdot (-\sin e^x)\frac{d}{dx}(e^x)$$

$$= -\tan e^x \cdot e^x$$

$$= -e^x \tan e^x,$$

where, $e^x \neq (2n+1)\dfrac{\pi}{2}, n \in N$

Q. 6. $e^x + e^{x^2} + ... + e^{x^5}$ **[NCERT Ex. 5.4, Q. 6, Page 130]**

Sol. Let $y = e^x + e^{x^2} + ... + e^{x^5}$

Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(e^x + e^{x^2} + ... + e^{x^5})$$

$$= \frac{d}{dx}(e^x) + \frac{d}{dx}(e^{x^2}) + \frac{d}{dx}(e^{x^3}) + \frac{d}{dx}(e^{x^4}) + \frac{d}{dx}(e^{x^5})$$

$$= e^x + e^{x^2}(2x) + e^{x^3}(3x^2) + e^{x^4}(4x^3) + e^{x^5}(5x^4)$$

$$= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

Q. 7. $\sqrt{e^{\sqrt{x}}}, x > 0,$ **[NCERT Ex.5. 4, Q. 7, Page 130]**

Sol. Let $y = \sqrt{e^{\sqrt{x}}}$

Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{e^{\sqrt{x}}}\right) = \frac{d}{dx}(e^{\sqrt{x}})^{1/2}$$

$$= \frac{1}{2}(e^{\sqrt{x}})^{-1/2} \cdot \frac{d}{dx}e^{\sqrt{x}}$$

$$= \frac{1}{2}(e^{\sqrt{x}})^{-1/2} \cdot e^{\sqrt{x}}\frac{d}{dx}\sqrt{x}$$

$$= \frac{1}{2}(e^{\sqrt{x}})^{-1/2} \cdot e^{\sqrt{x}} \cdot \frac{1}{2}(x)^{-1/2}$$

$$= \frac{e^{\sqrt{x}}}{4\sqrt{e^x}\sqrt{x}} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x > 0$$

Q. 8. $\log(\log x)$, $x > 1$ **[NCERT Ex. 5.4, Q. 8, Page 130]**

Sol. Let $y = \log(\log x), x > 1$

Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}\log(\log x)$$

$$= \frac{1}{(\log x)} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x} \cdot \frac{1}{x}$$

$$= \frac{1}{x \log x}, x > 1$$

Q. 9. $\dfrac{\cos x}{\log x}, x > 0$ **[NCERT Ex. 5.4, Q. 9, Page 130]**

Sol. Let $y = \dfrac{\cos x}{\log x}$

Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{\log x}\right)$$

$$= \frac{\log x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{\log x(-\sin x) - \cos x\left(\frac{1}{x}\right)}{(\log x)^2}$$

$$= -\left[\frac{\sin x \log x + \frac{1}{x}\cos x}{(\log x)^2}\right]$$

$$= \frac{-(x\sin x \cdot \log x + \cos x)}{x(\log x)^2}$$

Q. 10. $\cos(\log x + e^x), x > 0$

[NCERT Ex. 5.4, Q. 10, Page 130]

Sol. Let $y = \cos(\log x + e^x)$

Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}\{\cos(\log x + e^x)\}$$

$$= -\sin(\log x + e^x)\frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x)\left[\frac{1}{x} + e^x\right]$$

$$= -\left(\frac{1}{x} + e^x\right)\sin(\log x + e^x), x > 0$$

Exercise - 5.5

Differentiate the functions given in Questions 1 to 11 w.r.t. x

Q. 1. $\cos x \cdot \cos 2x \cdot \cos 3x$

[NCERT Ex. 5.5, Q. 1, Page 134]

Sol. Let $y = \cos x \cdot \cos 2x \cdot \cos 3x$

By taking log on both sides, we get

$$\log y = \log(\cos x. \cos 2x.\cos 3x)$$

Then,

$$\log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x) \quad ...(i)$$

On differentiating (i) both sides w.r.t. x, we get

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) + \frac{1}{\cos 2x}(-\sin 2x)(2) + \frac{1}{\cos 3x}(-\sin 3x)(3)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\tan x - 2\tan 2x - 3\tan 3x$$

therefore,

$$\frac{dy}{dx} = -\cos x \cdot \cos 2x \cdot \cos 3x \ (\tan x + 2\tan 2x + 3\tan 3x)$$

Q. 2. $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

[NCERT Ex. 5.5, Q. 2, Page 134]

Sol. Let $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ ...(i)

By taking log on both sides of (i), we get

$$\log y = \frac{1}{2}\left[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)\right] \quad ...(ii)$$

Now, differentiating (ii) on both sides w.r.t, x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{(x-1)} + \frac{1}{(x-2)} - \frac{1}{(x-3)} - \frac{1}{(x-4)} - \frac{1}{(x-5)}\right]$$

therefore,

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \times \left[\frac{1}{(x-1)} + \frac{1}{(x-2)} - \frac{1}{(x-3)} - \frac{1}{(x-4)} - \frac{1}{(x-5)}\right]$$

Q. 3. $(\log x)^{\cos x}$ **[NCERT Ex. 5.5, Q. 3, Page 134]**

Sol. Let $y = (\log x)^{\cos x}$ ...(i)

By taking log on both sides of (i), we get

$$\log y = \cos x \log(\log x) \quad ...(ii)$$

On differentiating (ii) on both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}\cos x$$

$$= \cos x \cdot \frac{1}{\log x}\frac{1}{x} + \log(\log x)(-\sin x)$$

$$= \frac{\cos x}{x\log x} - \sin x\log(\log x)$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x\log(\log x)\right]$$

Q. 4. $x^x - 2^{\sin x}$ **[NCERT Ex. 5.5, Q. 4, Page 134]**

Sol. $y = x^x - 2^{\sin x}$

$\Rightarrow$ $y = u - v$, where $u = x^x$

and $v = 2^{\sin x}$

therefore, $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$ ...(i)

Now $u = x^x$

By taking log on both sides, we get

$$\log u = x\log x$$

Differentiating w.r.t. x, we get

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{1}{x} + \log x$$

$$\Rightarrow \frac{du}{dx} = x^x[1 + \log x]$$

$v = 2^{\sin x}$ ...(ii)

By taking log on both sides, we get $\log v = \sin x \log 2$

Differentiating w.r.t. x, we get

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \log 2(\cos x) \Rightarrow \frac{dv}{dx} = 2^{\sin x}(\cos x \cdot \log 2) \quad ...(iii)$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = x^x[1 + \log x] - 2^{\sin x}(\cos x \cdot \log 2)$$

Q. 5. $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

[NCERT Ex. 5.5, Q. 5, Page 134]

Sol. Let $y = (x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

By taking log on both sides, we get

$$\log y = \log\left[(x+3)^2(x+4)^3(x+5)^4\right]$$

$$\Rightarrow \quad \log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5) \quad \text{...(i)}$$

Differentiating (i) on both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 2\cdot\frac{1}{(x+3)} + 3\cdot\frac{1}{(x+4)} + 4\cdot\frac{1}{(x+5)}$$

$$\Rightarrow \quad \frac{dy}{dx} = (x+3)^2\cdot(x+4)^3\cdot(x+5)^4 \left[\frac{2}{x+3}+\frac{2}{x+4}+\frac{2}{x+5}\right]$$

$$= (x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$$

Q. 6. $\left(x+\frac{1}{x}\right)^x + x^{\left(x+\frac{1}{x}\right)}$ **[NCERT Ex. 5.5, Q. 6, Page 134]**

Sol. Let,

$$y = \left(x+\frac{1}{x}\right)^x + x^{\left(x+\frac{1}{x}\right)} = u+v$$

where, $u = \left[x+\frac{1}{x}\right]^x$ and $v = x^{\left(x+\frac{1}{x}\right)}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \quad \text{...(i)}$$

Now, $u = \left(x+\frac{1}{x}\right)^x$

$$\Rightarrow \quad \log u = x\log\left(x+\frac{1}{x}\right) \quad \text{...(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log\left(x+\frac{1}{x}\right) + \log\left(x+\frac{1}{x}\right)(1)$$

$$= \frac{x}{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right) + \log\left(x+\frac{1}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x}{x+\frac{1}{x}}\left(x-\frac{1}{x^2}\right)+\log\left(x+\frac{1}{x}\right)\right] \text{....(iii)}$$

Also, $v = x^{\left(x+\frac{1}{x}\right)}$

Taking log on both the sides, we get

$$\Rightarrow \log v = \left(x+\frac{1}{x}\right)\log x \quad \text{...(iv)}$$

Differentiating (iv) on both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left(x+\frac{1}{x}\right)\frac{d}{dx}\log x + \log x\frac{d}{dx}\left(x+\frac{1}{x}\right)$$

$$= \left(x+\frac{1}{x}\right)\frac{1}{x} + \log x\left(1-\frac{1}{x^2}\right)$$

$$\Rightarrow \frac{dv}{dx} = x^{\left(x+\frac{1}{x}\right)}\left[\left(x+\frac{1}{x}\right)\frac{1}{x}+\log x\left(1-\frac{1}{x^2}\right)\right] \quad \text{...(v)}$$

Substituting the values of (iii) and (v) in (i), we get

$$\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x}{x+\frac{1}{x}}\left(x-\frac{1}{x^2}\right)+\log\left(x+\frac{1}{x}\right)\right]$$

$$+x^{\left(x+\frac{1}{x}\right)}\left[\left(x+\frac{1}{x}\right)\frac{1}{x}+\log x\left(1\frac{-1}{x^2}\right)\right]$$

$$= \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]$$

$$+x^{\left(x+\frac{1}{x}\right)}\left[\frac{x+1-\log x}{x^2}\right]$$

Q. 7. $(\log x)^x + x^{\log x}$ **[NCERT Ex. 5.5, Q. 7, Page 134]**

Sol. Let $\quad y = (\log x)^x + x^{\log x}$

$$\Rightarrow \quad y = u+v,$$

where $\quad u = (\log x)^x, v = x^{\log x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \quad \text{...(i)}$$

Now, $\quad u = (\log x)^x$

By taking log on both sides, we get therefore,

$$\log u = x\log(\log x)$$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}\log(\log x) + \log(\log x)$$

$$= x\cdot\frac{1}{\log x}\cdot\frac{1}{x} + \log(\log x)$$

$$= \frac{1}{\log x} + \log(\log x)$$

therefore, $\quad \frac{du}{dx} = (\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]$...(ii)

Also, $\quad v = x^{\log x}$

By taking log on both sides, we get

$$\log v = \log x\log x = (\log x)^2$$

Differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = 2\log x\cdot\frac{1}{x}.$$

therefore, $\quad \frac{dv}{dx} = x^{\log x}\left[\frac{2\log x}{x}\right] \quad$ ...(iii)

From (i), (ii), & (iii), we get

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right] + x^{\log x}\left[\frac{2\log x}{x}\right]$$

$$= (\log x)^{x-1}\left[1+\log x\cdot\log(\log x)\right] + 2x^{\log x-1}\cdot\log x$$

Q. 8. $(\sin x)^x + \sin^{-1}\sqrt{x}$ **[NCERT Ex. 5.5, Q. 8, Page 134]**

Sol. Let $\quad y = (\sin x)^x + \sin^{-1}\sqrt{x} = u+v,$

where, $\quad u = (\sin x)^x, v = \sin^{-1}\sqrt{x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \quad \text{...(i)}$$

Now, $\quad u = (\sin x)^x$

By taking log on both sides, we get

$$\log u = x\log\sin x$$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}(\log\sin x) + \log\sin x$$

$$= x\cdot\frac{1}{\sin x}\cdot\cos x + \log\sin x$$

therefore,

$$\frac{du}{dx} = (\sin x)^x [x\cot x + \log\sin x] \qquad ...(ii)$$

Also,

$$v = \sin^{-1}\sqrt{x} \Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} \qquad ...(iii)$$

From (i), (ii) & (iii), we get

$$\frac{dy}{dx} = (\sin x)^x[x\cot x + \log\sin x] + \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$$

$$= (\sin x)^x[x\cot x + \log\sin x] + \frac{1}{2\sqrt{x-x^2}}$$

Q. 9. $x^{\sin x} + (\sin x)^{\cos x}$ [NCERT Ex. 5.5, Q. 9, Page 134]

Sol. Let $y = x^{\sin x} + (\sin x)^{\cos x} = u + v,$

$$u = x^{\sin x} \text{ and } v = (\sin x)^{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now, $u = x^{\sin x}$

By taking log on both sides, we get

$$\log u = \sin x \log x \qquad ...(ii)$$

Differentiating (ii) w.r.t. x, we get

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$$

$$= \sin x \cdot \frac{1}{x} + \log x \cos x$$

therefore,

$$\frac{du}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cos x\right] \qquad ...(iii)$$

Also, $v = (\sin x)^{\cos x}$

By taking log on both sides, we get

$$\log v = \cos x \log(\sin x) \qquad ...(iv)$$

Differentiating (iv) w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \cos x \frac{d}{dx}\log\sin x + \log\sin x \frac{d}{dx}\cos x$$

$$= \cos x \frac{1}{\sin x}\cos x + \log\sin x(-\sin x)$$

$$= \cos x \cot x - \sin x \log\sin x$$

therefore,

$$\frac{dv}{dx} = (\sin x)^{\cos x}[\cos x \cot x - \sin x \log\sin x] \qquad ...(v)$$

Substituting the values of (iii) & (v) in (i), we get

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x\cos x\right] + (\sin x)^{\cos x}[\cos x\cot x - \sin x\log\sin x]$$

Q. 10. $x^{x\cos x} + \frac{x^2+1}{x^2-1}$ [NCERT Ex. 5.5, Q. 10, Page 134]

Sol. Let $y = x^{x\cos x} + \frac{x^2+1}{x^2-1} = u + v$

where

$$u = x^{\cos x} \text{ and } v = \frac{x^2+1}{x^2-1} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now, $u = x^{x\cos x}$

By taking log on both sides, we get

$$\log u = x\cos x \cdot \log x \qquad ...(ii)$$

Differentiating (ii) w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x\cos x\frac{d}{dx}(\log x) + x\log x\frac{d}{dx}\cos x + \cos x\log x\frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = x\cos x\cdot\frac{1}{x} + x\log x(-\sin x) + \cos x\log x$$

$$= \cos x - x\sin x\log x + \cos x\log x$$

therefore,

$$\frac{du}{dx} = x^{x\cos x}[\cos x - x\sin x\log x + \cos x\log x] \qquad ...(iii)$$

Also, $v = \frac{x^2+1}{x^2-1}$ ...(iv)

Differentiating (iv) w.r.t x, we get

$$\Rightarrow \frac{dv}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$= \frac{2x[x^2-1-x^2-1]}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2} \qquad ...(v)$$

Substituting the values of (iii) & (v) in (i), we get

$$\frac{dy}{dx} = x^{x\cos x}[\cos x - x\sin x\log x + \cos x\log x] - \frac{4x}{(x^2-1)^2}$$

$$= x^{x\cos x}[\cos x(1+\log x) - x\sin x\log x] - \frac{4x}{(x^2-1)^2}$$

Q. 11. $(x\cos x)^x + (x\sin x)^{1/x}$ [NCERT Ex.5.5, Q.11, Page 134]

Sol. Let $y = (x\cos x)^x + (x\sin x)^{1/x} = u + v$

where, $u = (x\cos x)^x, v = (x\sin x)^{1/x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now, $u = (x\cos x)^x$.

By taking log on both sides, we get

$$\log u = x\log(x\cos x) \qquad ...(ii)$$

Differentiating (ii) on both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}[\log(x\cos x)] + \log(x\cos x)$$

$$= x\left[\frac{1}{x\cos x}\times\frac{d}{dx}(x\cos x)\right] + \log(x\cos x)$$

$$= \sec x(-x\sin x + \cos x) + \log(x\cos x)$$

$$= -x\tan x + 1 + \log(x\cos x)$$

$$\frac{du}{dx}=(x\cos x)^x[1-x\tan x+\log(x\cos x)] \quad ...(iii)$$

Now, $v=(x\sin x)^{1/x}$

By taking log on both sides, we get

$$\log v=\frac{1}{x}\log(x\sin x) \quad ...(iv)$$

Differentiating (iv) w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx}=\frac{1}{x}\frac{d}{dx}\log(x\sin x)+\log(x\sin x)\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$=\frac{1}{x}\cdot\frac{1}{x\sin x}\frac{d}{dx}(x\sin x)+\log(x\sin x)\cdot\left(\frac{-1}{x^2}\right)$$

$$=\frac{1}{x^2\sin x}\left[x\frac{d}{dx}(\sin x)+\sin x\right]-\frac{\log(x\sin x)}{x^2}$$

$$=\frac{\cot x}{x}+\frac{1}{x^2}-\frac{\log(x\sin x)}{x^2}$$

$$=\frac{x\cot x+1-\log(x\sin x)}{x^2}$$

therefore,

$$\frac{dv}{dx}=(x\sin x)^{1/x}\left[\frac{x\cot x+1-\log(x\sin x)}{x^2}\right] \quad ...(v)$$

Substituting the values of (iii) and (v) in (i), we get

$$\frac{dy}{dx}=(x\cos x)^x[1-x\tan x+\log(x\cos x)]$$

$$+(x\sin x)^{1/x}\left[\frac{x\cot x+1-\log(x\sin x)}{x^2}\right]$$

Find $\frac{dy}{dx}$ of the functions given in questions 12 to 15.

Q. 12. $x^y + y^x = 1$ [NCERT Ex. 5.5, Q. 12, Page 134]

Sol. $x^y + y^x = 1$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{d}{dx}(x^y)+\frac{d}{dx}(y^x)=0 \quad ...(ii)$$

Let $u = x^y$ therefore, $\log u = y\log x$

Differentiating w.r.t. x, we get

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=y\cdot\frac{1}{x}+\log x\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx}=x^y\left[\frac{y}{x}+\log x\frac{dy}{dx}\right] \quad ...(iii)$$

Let $v=y^x \Rightarrow \log v = x\log y$

Differentiating w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx}=x\cdot\frac{1}{y}\cdot\frac{dy}{dx}+\log y$$

therefore, $$\frac{dv}{dx}=y^x\left(\frac{x}{y}\times\frac{dy}{dx}+\log y\right) \quad ...(iv)$$

Substituting the values of (iii) and (iv) in (ii), we get

$$x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)+y^x\left(\frac{x}{y}\times\frac{dy}{dx}+\log y\right)=0$$

$$\Rightarrow \left(x^y\log x+xy^{x-1}\right)\frac{dy}{dx}=-\left(y^x\log y+yx^{y-1}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=-\left(\frac{y^x\log y+yx^{y-1}}{x^y\log x+xy^{x-1}}\right)$$

Q. 13. $y^x = x^y$ [NCERT Ex. 5.5, Q. 13, Page 134]

Sol. We are given that, $y^x = x^y$

By taking log on both sides, we get

$$x\log y = y\log x \quad ...(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$x\frac{d}{dx}\log y+\log y\cdot 1=y\frac{d}{dx}\log x+\log x\frac{dy}{dx}$$

$$\Rightarrow x\cdot\frac{1}{y}\cdot\frac{dy}{dx}+\log y=\frac{y}{x}+\log x\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[\log x-\frac{x}{y}\right]=\log y-\frac{y}{x} \Rightarrow \frac{dy}{dx}=\frac{y(x\log y-y)}{x(y\log x-x)}$$

Q. 14. $(\cos x)^y = (\cos y)^x$ [NCERT Ex. 5.5, Q. 14, Page 134]

Sol. We have, $(\cos x)^y = (\cos y)^x$

By taking log on both sides, we get

$$y\log(\cos x)=x\log(\cos y) \quad ...(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$y\frac{d}{dx}\log(\cos x)+\log(\cos x)\frac{dy}{dx}$$

$$= x\frac{d}{dx}(\log(\cos y))+\log(\cos y)$$

$$\Rightarrow y\frac{1}{\cos x}(\sin x)+\log(\cos x)\frac{dy}{dx}$$

$$= x\cdot\frac{1}{\cos y}(\sin y)\frac{dy}{dx}+\log(\cos y)$$

$$\Rightarrow \frac{dy}{dx}[\log(\cos x)+x\tan y]=\log(\cos y)+y\tan x$$

$$\Rightarrow \frac{dy}{dx}=\frac{\log(\cos y)+y\tan x}{\log(\cos x)+x\tan y}$$

Q. 15. $xy = e^{(x-y)}$ [NCERT Ex. 5.5, Q. 15, Page 134]

Sol. We have, $xy = e^{(x-y)}$

By taking log on both sides, we get

$\log(xy) = \log e^{(x-y)}$ or $\log x + \log y = x - y$ (i)

Differentiating (i) on both sides w.r.t. x, we get

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=\left(1-\frac{dy}{dx}\right)\Rightarrow\frac{dy}{dx}\left(\frac{1}{y}+1\right)=1-\frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

Q. 16. Find the derivative of the function given by $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ and hence find $f'(1)$.

[NCERT Ex. 5.5, Q. 16, Page 134]

Sol. Let $f(x) = y$

$$\Rightarrow \quad y=(1+x)(1+x^2)(1+x^4)(1+x^8)$$

By taking log on both sides, we get

$$\log y=\log\left[(1+x)(1+x^2)(1+x^4)(1+x^8)\right]$$

$$\Rightarrow \log y=\log(1+x)+\log(1+x^2)$$

$$+\log(1+x^4)+\log(1+x^8)$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{(1+x)}+\frac{1}{(1+x^2)}(2x)+\frac{1}{(1+x^4)}(4x^3)+\frac{1}{(1+x^8)}(8x^7)$$

$$\Rightarrow \frac{dy}{dx}=y\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{(1+x^4)}+\frac{8x^7}{(1+x^8)}\right]$$

therefore,

$$f'(x)=(1+x)(1+x^2)+(1+x^4)+(1+x^8)\times\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

$$\Rightarrow f'(x)=(1+1)(1+1)(1+1)(1+1)\left[\frac{1}{1+1}+\frac{2(1)}{1+1}+\frac{4(1)}{1+1}+\frac{8(1)}{1+1}\right]$$

$$\Rightarrow f'(x)=(2)(2)(2)(2)\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$

$$\Rightarrow f'(1)=16\left\{\frac{1+2+4+8}{2}\right\}$$

$$=\frac{16}{2}(15)=8(15)=120$$

Q. 17. Differentiate $(x^2-5x+8)(x^3+7x+9)$ in three ways mentioned below:

(i) by using product rule

(ii) by expanding the product to obtain a single polynomial,

(iii) by logarithmic differentiation.

Do they all give the same answer.

[NCERT Ex. 5.5, Q. 17, Page 134]

Sol. (i) By using product rule,

Let $f(x)=(x^2-5x+8)(x^3+7x+9)$

$$f'(x)=(x^2-5x+8)\frac{d}{dx}(x^3+7x+9)+(x^3+7x+9)\frac{d}{dx}(x^2-5x+8)$$

$$\Rightarrow f'(x)=(x^2-5x+8)(3x^2+7)+(x^3+7x+9)(2x-5)$$

$$\Rightarrow f'(x)=3x^4-15x^3+24x^2+7x^2-35x+56+2x^4+14x^2+18x-5x^3-35x-45$$

$$\Rightarrow f'(x)=5x^4-20x^3+45x^2-52x+11$$

(ii) By expanding the product to obtain a single polynomial,

$$f(x)=(x^2-5x+8)(x^3+7x+9)$$

$$\Rightarrow f(x)=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$$

$$\Rightarrow f(x)=x^5-5x^4+15x^3-26x^2+11x+72$$

therefore, $f'(x)=5x^4-20x^3+45x^2-52x+11$

(iii) By logarithmic differentiation,

Let $f(x)=y\Rightarrow y=(x^2-5x+8)(x^3+7x+9)$

Taking log on both the sides, we get

$$\log y=\log\{(x^2-5x+8)(x^3+7x+9)\}$$

$$\log y=\log(x^2-5x+8)+\log(x^3+7x+9)$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{(x^2-5x+8)}(2x-5)+\frac{1}{(x^3+7x+9)}(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx}=y\left\{\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right\}$$

$$\Rightarrow \frac{dy}{dx}=(x^2-5x+8)(x^3+7x+9)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$=(2x-5)(x^3+7x+9)+(3x^2+7)(x^2-5x+8)$$

therefore, $\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$

Yes, the answer is same in all the three cases.

Q. 18. If u, v and w are functions of x, then show that $\frac{d}{dx}(u\cdot v\cdot w)=\frac{du}{dx}v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\frac{dw}{dx}$ in two ways– first by repeated application of product rule, second by logarithmic differentiation.

[NCERT Ex. 5.5, Q. 18, Page 134]

Sol.(i) Let $y=u\cdot v\cdot w=u\cdot(vw)$ ...(i)

Differentiating (i) on both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}u\cdot(vw)-u\frac{d}{dx}(vw)$$

$$=u'\cdot(vw)+u[v'w+vw']$$

$$=u'\cdot v\cdot w+uv'w+uvw'$$

$$=\frac{du}{dx}\cdot v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\cdot\frac{dw}{dx}$$

(ii) $y=u\cdot v\cdot w$

By taking log on both sides, we get

$\log y=\log u+\log v+\log w$ ....(ii)

Differentiating (ii) on both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1du}{udx}+\frac{1dv}{vdx}+\frac{1dw}{wdx}$$

$$\Rightarrow \frac{dy}{dx}=y\left(\frac{1du}{udx}+\frac{1dv}{vdx}+\frac{1dw}{wdx}\right)$$

$$=uvw\left(\frac{1du}{udx}+\frac{1dv}{vdx}+\frac{1dw}{wdx}\right)$$

$$=vw\frac{du}{dx}+uw\frac{dv}{dx}+uv\frac{dw}{dx}$$

$$=\frac{du}{dx}\cdot v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\cdot\frac{dw}{dx}.$$

Exercise - 5.6

If x and y are connected parametrically by the equations given in questions 1 to 10, without eliminating the parameter, find $\frac{dy}{dx}$.

Q. 1. $x=2at^2, y=at^4$ **[NCERT Ex. 5.6, Q. 1, Page 137]**

Sol. Here, $x=2at^2$ ...(i)

and $y = at^4$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{dt} = 2a(2t) = 4at \text{ and } \frac{dy}{dt} = 4at^3$$

therefore, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4at^3}{4at} = t^2$

Q. 2. $x = a\cos\theta, y = b\cos\theta$

[NCERT Ex. 5.6, Q. 2, Page 137]

Sol. Here, $x = a\cos\theta$ ...(i)

and $y = b\cos\theta$ ...(ii)

Differentiating (i) & (ii) w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(-\sin\theta) = -a\sin\theta$$

and $\frac{dy}{d\theta} = b(-\sin\theta) = -b\sin\theta$

therefore, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$

Q. 3. $x = \sin t, y = \cos 2t$

[NCERT Ex. 5.6, Q. 3, Page 137]

Sol. Here, $x = \sin t$ ...(i)

$y = \cos 2t$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{dt} = \cos t \text{ and}$$

$$\frac{dy}{dt} = -\sin 2t \cdot 2 = -2\sin 2t$$

therefore, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$= \frac{-2\sin 2t}{\cos t}$$

$$= \frac{-2\cdot 2\sin t\cos t}{\cos t} = -4\sin t$$

Q. 4. $x = 4t, y = \frac{4}{t}$ **[NCERT Ex. 5.6, Q. 4, Page 137]**

Sol. Here, $x = 4t$ ...(i)

and $y = \frac{4}{t}$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{dt} = 4 \text{ and } \frac{dy}{dt} = \frac{-4}{t^2}$$

therefore, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4}{t^2}\times\frac{1}{4} = \frac{-1}{t^2}$

Q. 5. $x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$

[NCERT Ex. 5.6, Q. 5, Page 137]

Sol. Here, $x = \cos\theta - \cos 2\theta$ ...(i)

and $y = \sin\theta - \sin 2\theta$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, θ, we get

$$\frac{dx}{d\theta} = -\sin\theta - (-\sin 2\theta)\cdot 2$$

$$\frac{dy}{d\theta} = \cos\theta - \cos 2\theta \cdot 2$$

$$= \cos\theta - 2\cos 2\theta$$

therefore, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$= \frac{\cos\theta - 2\cos 2\theta}{2\sin 2\theta - \sin\theta}$$

Q. 6. $x = a(\theta - \sin\theta), y = a(1+\cos\theta)$

[NCERT Ex.5.6, Q.6, Page 137]

Sol. Here, $x = a(\theta - \sin\theta)$ ...(i)

and $y = a(1 + \cos\theta)$ ...(ii)

Differentiating (i) & (ii) w.r.t. θ, we get

$$\frac{dx}{d\theta} = a[1-\cos\theta] \text{ and}$$

$$\frac{dy}{d\theta} = a[-\sin\theta] = -a\sin\theta$$

therefore, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$= \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-\sin\theta}{1-\cos\theta}$$

$$= \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$= -\cot\frac{\theta}{2}$$

Q. 7. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

[NCERT Ex. 5.6, Q. 7, Page 137]

Sol. Here, $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$ ....(i)

and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{dt} = \frac{\sqrt{\cos 2t}\frac{d}{dt}(\sin^3 t) - \sin^3 t\frac{d}{dt}(\sqrt{\cos 2t})}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\,3\sin^2 t\cos t - \sin^3 t\,\frac{1}{2\sqrt{\cos 2t}}.(-\sin 2t).2}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\,3\sin^2 t\cos t + \frac{\sin^3 t\sin 2t}{\sqrt{\cos 2t}}}{\cos 2t}$$

$$= \frac{3\cos 2t\sin^2 t\cos t + \sin^3 t\sin 2t}{(\cos 2t)^{3/2}}$$

$$\frac{dy}{dt} = \frac{\sqrt{\cos 2t}\frac{d}{dt}(\cos^3 t) - \cos^3 t\frac{d}{dt}(\sqrt{\cos 2t})}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}.3\cos^2 t(-\sin t) - \cos^3 t.\frac{1}{2\sqrt{\cos 2t}}.(-\sin 2t).2}{\cos 2t}$$

$$= \frac{-3\cos^2 t\cdot\sin t\cdot\sqrt{\cos 2t} + \frac{\cos^3 t\sin 2t}{\sqrt{\cos 2t}}}{\cos 2t}$$

$$= \frac{\cos^3 t\sin 2t - 3\cos^2 t\cdot\sin t\cos 2t}{(\cos 2t)^{3/2}}$$

therefore, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$= \frac{\cos^3 t \sin 2t - 3\cos^2 t \cdot \sin t \cos 2t}{3\cos 2t \sin^2 t \cos t + \sin^3 t \sin 2t} = -\cot 3t$$

Q. 8. $x = a\left(\cos t + \log \tan \frac{t}{2}\right), y = a \sin t$

[NCERT Ex. 5.6, Q. 8, Page 137]

Sol. Here, $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ ...(i)

$y = a \sin t$ ...(ii)

Differentiating (i) & (ii) w.r.t. t, we get

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \frac{d}{dt}\left(\tan \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan \frac{t}{2}} \sec^2 \frac{t}{2} \cdot \frac{1}{2}\right]$$

$$= \frac{a \cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a \cos t$$

therefore, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$= \frac{a \cos t \sin t}{a \cos^2 t} = \tan t$$

Q. 9. $x = a \sec \theta, y = b \tan \theta$

[NCERT Ex. 5.6, Q. 9, Page 137]

Sol. Here, $x = a \sec \theta$ ...(i)

and $y = b \tan \theta$ ...(ii)

Differentiating (i) & (ii) w.r.t. θ, we get

$$\frac{dx}{d\theta} = a \sec \theta \tan \theta$$

and $\frac{dy}{d\theta} = b \sec^2 \theta$

therefore, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$= \frac{b \sec^2 \theta}{a \sec \theta \tan \theta}$$

$$= \frac{b \sec \theta}{a \tan \theta} = \frac{b}{a} \operatorname{cosec} \theta$$

Q. 10. $x = a(\cos \theta + \theta \sin \theta), y = a(\sin \theta - \theta \cos \theta)$

[NCERT Ex. 5.6, Q. 10, Page 137]

Sol. Here,

$x = a(\cos\theta + \theta\sin\theta)$ ...(i)

and $y = a(\sin\theta - \theta\cos\theta)$ ...(ii)

Differentiating (i) & (ii) w.r.t. θ, we get

$$\frac{dx}{d\theta} = a[-\sin\theta + \theta \cdot \cos\theta + \sin\theta]$$

$$= a\theta \cos\theta$$

$$\frac{dy}{d\theta} = a[\cos\theta - (\theta(-\sin\theta) + \cos\theta)]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta] = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Q. 11. If $x = \sqrt{a^{\sin^{-1} t}}$ **and** $y = \sqrt{a^{\cos^{-1} t}}$**, show that** $\frac{dy}{dx} = -\frac{y}{x}$

[NCERT Ex. 5.6, Q. 11, Page 137]

Sol. Given: $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{1}{2} \cdot \frac{1}{\sqrt{a^{\sin^{-1} t}}} \cdot \frac{d}{dt} a^{\sin^{-1} t}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{a^{\sin^{-1} t}}} a^{\sin^{-1} t} . \log a . \frac{d}{dt} \sin^{-1} t$$

$$= \frac{\sqrt{a^{\sin^{-1} t}}}{2} . \log a \frac{1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dt} = \frac{1}{2} \cdot \frac{1}{\sqrt{a^{\cos^{-1} t}}} \cdot \frac{d}{dt} a^{\cos^{-1} t}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{a^{\cos^{-1} t}}} . a^{\cos^{-1} t} . \log a \frac{-1}{\sqrt{1-t^2}}$$

$$= \frac{\sqrt{a^{\cos^{-1} t}}}{2} \log a \frac{-1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\frac{\sqrt{a^{\cos^{-1} t}}}{2} . \log a . \frac{-1}{\sqrt{1-t^2}}}{\frac{\sqrt{a^{\sin^{-1} t}}}{2} . \log a . \frac{1}{\sqrt{1-t^2}}}$$

$$= -\frac{\sqrt{a^{\cos^{-1} t}}}{\sqrt{a^{\sin^{-1} t}}} = \frac{-y}{x}$$

Exercise - 5.7

Find the second order derivatives of the functions given in questions 1 to 10.

Q. 1. $x^2 + 3x + 2$ **[NCERT Ex. 5.7, Q. 1, Page 139]**

Sol. Let $y = x^2 + 3x + 2$

$\Rightarrow$ $\frac{dy}{dx} = 2x + 3$

therefore, $\frac{d^2y}{dx^2} = 2$

Q. 2. x^{20} **[NCERT Ex. 5.7, Q. 2, Page 139]**

Sol. Let $y = x^{20}$

$$\frac{dy}{dx}=20x^{19}$$

therefore, $\frac{d^2y}{dx^2}=20\times19x^{18}=380x^{18}$

Q. 3. x.cos x **[NCERT Ex. 5.7, Q. 3, Page 139]**

Sol. Let $y = x\cos x$

$\Rightarrow \frac{dy}{dx} = x.(-\sin x) + \cos x = -x\sin x + \cos x$

$\Rightarrow \frac{d^2y}{dx^2} = -[x\cos x + \sin x] - \sin x$

$= -(x\cos x + 2\sin x)$

Q. 4. log x **[NCERT Ex. 5.7, Q. 4, Page 139]**

Sol. Let $y = \log x$

$\Rightarrow \frac{dy}{dx}=\frac{1}{x}$

therefore, $\frac{d^2y}{dx^2}=-\frac{1}{x^2}$

Q. 5. x^3 log x **[NCERT Ex. 5.7, Q. 5, Page 139]**

Sol. Let $y = x^3\log x$

$\Rightarrow\frac{dy}{dx}=x^3.\frac{1}{x}+\log x.3x^2=x^2+3x^2\log x$

$\Rightarrow\frac{d^2y}{dx^2}=2x+3\left[x^2.\frac{1}{x}+\log x.2x\right]$

$= 2x + 3[x + 2x\log x] = 2x + 3x + 6x\log x$

$= 5x + 6x\log x = x(5 + 6\log x)$

Q. 6. e^x sin 5x **[NCERT Ex. 5.7, Q. 6, Page 139]**

Sol. Let $y = e^x\sin 5x$

$\Rightarrow \frac{dy}{dx} = e^x.\cos 5x.5 + \sin 5x.e^x$

$= e^x[5\cos 5x + \sin 5x]$

therefore,

$\frac{d^2y}{dx^2} = e^x[5(-\sin 5x).5 + \cos 5x.5]$

$+ [5\cos 5x + \sin 5x]\,e^x$

$= e^x[-25\sin 5x + 5\cos 5x + 5\cos 5x + \sin 5x]$

$= e^x[10\cos 5x - 24\sin 5x]$

$= 2e^x[5\cos 5x - 12\sin 5x]$

Q. 7. e^{6x} cos 3x **[NCERT Ex. 5.7, Q. 7, Page 139]**

Sol. Let $y = e^{6x}\cos 3x$

$\Rightarrow \frac{dy}{dx} = e^{6x}(-\sin 3x).3 + \cos 3x.e^{6x}.6$

$= 6e^{6x}\cos 3x - 3e^{6x}\sin 3x$

$\frac{d^2y}{dx^2} = 6[e^{6x}(-\sin 3x).3 + \cos 3x.e^{6x}.6]$

$- 3[e^{6x}\cos 3x.3 + (\sin 3x)e^{6x}.6]$

$= -18e^{6x}\sin 3x + 36e^{6x}\cos 3x$

$- 9e^{6x}\cos 3x - 18e^{6x}\sin 3x$

$= 27e^{6x}\cos 3x - 36e^{6x}\sin 3x$

$= 9e^{6x}(3\cos 3x - 4\sin 3x)$

Q. 8. $\tan^{-1}x$ **[NCERT Ex. 5.7, Q. 8, Page 139]**

Sol. Let $y = \tan^{-1}x$

$\Rightarrow \frac{dy}{dx}=\frac{1}{(1+x^2)}$

therefore, $\frac{d^2y}{dx^2}=\frac{(1+x^2).0-1.(2x)}{(1+x^2)^2}$

$=\frac{-2x}{(1+x^2)^2}$

Q. 9. log(log x) **[NCERT Ex. 5.7, Q. 9, Page 139]**

Sol. Let $y = \log(\log x)$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\log x}\frac{1}{x}$

therefore, $\frac{d^2y}{dx^2}=\frac{1}{\log x}\left(-\frac{1}{x^2}\right)+\frac{1}{x}\frac{d}{dx}\left(\frac{1}{\log x}\right)$

$=\frac{-1}{x^2\log x}+\frac{1}{x}\left[\frac{\log x.0-1.\frac{1}{x}}{(\log x)^2}\right]$

$=\frac{-1}{x^2\log x}+\frac{1}{x}\left[\frac{-\frac{1}{x}}{(\log x)^2}\right]$

$=\frac{-1}{x^2\log x}-\frac{-1}{x^2(\log x)^2}$

$=\frac{-1}{x^2\log x}\left[1+\frac{1}{\log x}\right]=\frac{-(1+\log x)}{(x\log x)^2}$

Q. 10. sin(log x) **[NCERT Ex. 5.7, Q. 10, Page 139]**

Sol. Let $y = \sin(\log x)$

$\Rightarrow \frac{dy}{dx}=\cos(\log x)\frac{1}{x}$

therefore, $\frac{d^2y}{dx^2}=\cos(\log x).\left(-\frac{1}{x^2}\right)$

$+\frac{1}{x}.\{-\sin(\log x)\}.\frac{1}{x}$

$=\frac{-\cos(\log x)}{x^2}-\frac{\sin(\log x)}{x^2}$

$=-\frac{1}{x^2}[\cos(\log x)+\sin(\log x)]$

Q. 11. If $y = 5\cos x - 3\sin x$, then prove that $\frac{d^2y}{dx^2}+y=0$.

[NCERT Ex. 5.7, Q. 11, Page 140]

Sol. We have, $y = 5\cos x - 3\sin x$

$\Rightarrow \frac{dy}{dx} = 5(-\sin x) - 3(\cos x) -5\sin x - 3\cos x$

$\Rightarrow \frac{d^2y}{dx^2} = -5\cos x - 3(-\sin x)$

$= -5\cos x + 3\sin x$

$\Rightarrow \frac{d^2y}{dx^2} = -(5\cos x + 3\sin x)$

$\frac{d^2y}{dx^2} = -y$

$\frac{d^2y}{dx^2}+y = 0$ **Hence proved.**

Q. 12. If $y = \cos^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

[NCERT Ex. 5.7, Q. 12, Page 140]

Sol. $y = \cos^{-1}x \Rightarrow x = \cos y$ ...(i)

Differentiating (i) w.r.t. x, we get

$$1 = -\sin y \frac{dy}{dx}$$

$\Rightarrow \quad \frac{dy}{dx} = -\text{cosec } y$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \text{cosec} y \cot y \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\text{cosec}^2 y \cot y$$

Q. 13. If $y = 3\cos(\log x) + 4\sin(\log x)$, then show that $x^2y_2 + xy_1 + y = 0$

[NCERT Ex. 5.7, Q. 13, Page 140]

Sol. We have, $y = 3\cos(\log x) + 4\sin(\log x)$, ...(i)

Differentiating (i) w.r.t. x, we get

$\frac{dy}{dx} = 3[-\sin(\log x)]\frac{1}{x} + 4[\cos(\log x)]\frac{1}{x}$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 3\left[\left(-\sin(\log x)\left(-\frac{1}{x^2}\right)\right) + \frac{1}{x}(-\cos(\log x))\right]$$

$$+4\left[\cos(\log x).\left(\frac{-1}{x^2}\right) + \frac{1}{x}\{-\sin(\log x)\} \times \frac{1}{x}\right]$$

$$= \frac{1}{x^2}\begin{bmatrix} 3\sin(\log x) - 3\cos(\log x) \\ -4\cos(\log x) - 4\sin(\log x) \end{bmatrix}$$

$$x^2\frac{d^2y}{dx^2} = \begin{bmatrix} 3\sin(\log x) - 4\cos(\log x) \\ -\{3\cos(\log x) + 4\sin(\log x)\} \end{bmatrix}$$

$$x^2\frac{d^2y}{dx^2} = -x\frac{dy}{dx} - y$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

$\Rightarrow \quad x^2y_2 + xy_1 + y = 0$ **Hence proved.**

Q. 14. If $y = Ae^{mx} + Be^{nx}$, then show that

$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0.$$

[NCERT Ex. 5.7, Q. 14, Page 140]

Sol. Let $y = Ae^{mx} + Be^{nx}$ ...(i)

Differentiating (i) w.r.t. x, we get

$\frac{dy}{dx} = Ae^{mx}.m + Be^{nx}.n = Ame^{mx} + Bne^{nx}$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$\frac{d^2y}{dx^2} = \text{A}me^{mx}.m + \text{B}ne^{nx}.n = \text{A}m^2e^{mx} + \text{B}n^2e^{nx}$ ...(iii)

Now,

$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = \text{A}m^2e^{mx} + \text{B}n^2e^{nx}$$
$$-\left[(m+n)(\text{A}me^{mx} + \text{B}ne^{nx})\right] + mn\left(\text{A}e^{mx} + \text{B}e^{nx}\right)$$

[from (i), (ii) and (iii)]

$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx}$
$\quad - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx} = 0$

Hence proved.

Q. 15. If $y = 500e^{7x} + 600e^{-7x}$, then show that $\frac{d^2y}{dx^2} = 49y$.

[NCERT Ex. 5.7, Q. 15, Page 140]

Sol. Let $y = 500\, e^{7x} + 600\, e^{-7x}$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 500.e^{7x}.7 + 600.e^{-7x}.(-7)$$

$= 3500e^{7x} - 4200e^{-7x}$ ...(ii)

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 3500.7.e^{7x} - 4200.(-7)e^{-7x}$$
$$= 24500\, e^{7x} + 29400\, e^{-7x}$$
$$= 49(500\, e^{7x} + 600\, e^{-7x}) = 49\, y$$

therefore, $\frac{d^2y}{dx^2} = 49y$

Q. 16. If $e^y(x + 1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

[NCERT Ex. 5.7, Q. 16, Page 139]

Sol. $xe^y + e^y = 1$ ...(i)

Differentiating (i) w.r.t. x, we get

$$xe^y\frac{dy}{dx} + e^y + e^y\frac{dy}{dx} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{x+1}$ ...(ii)

From (ii),

$\left(\frac{dy}{dx}\right)^2 = \left(\frac{-1}{x+1}\right)^2 = \frac{1}{(x+1)^2}$ ...(iii)

Differentiating (ii) w.r.t. x, we get

$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$ ...(iv)

From (iii) and (iv), we get

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

Hence proved.

Q. 17. If $y = (\tan^{-1}x)^2$, show that
$(x^2 + 1)^2y_2 + 2x(x^2 + 1)y_1 = 2.$

[NCERT Ex. 5.7, Q. 17, Page 140]

Sol. Let $y = (\tan^{-1}x)^2$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1}x.\frac{1}{(1+x^2)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 2\left[\tan^{-1}x\frac{(\{1+x^2\}.0 - 2x)}{(1+x^2)^2} + \frac{1}{(1+x^2)}.\frac{1}{(1+x^2)}\right]$$

$$= 2\left[\frac{-2x\tan^{-1}x}{(1+x^2)^2} + \frac{1}{(1+x^2)^2}\right]$$

$$= 2\left[\frac{-2x\tan^{-1}x + 1}{(1+x^2)^2}\right]$$

Now, $(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx}$

$$= (x^2+1)^2.2\left[\frac{-2x\tan^{-1}x + 1}{(1+x^2)^2}\right]$$

$$+2x(x^2+1).2\tan^{-1}x\frac{1}{(1+x^2)}$$

$$= -4x\tan^{-1}x + 2 + 4x\tan^{-1}x = 2$$

Hence proved.

Miscellaneous Exercise

Direction: Differentiate w.r.t x the function in questions 1 to 11.

Q. 1. $(3x^2 - 9x + 5)^9$ **[NCERT Misc., Q. 1, Page 144]**

Sol. Let $y = (3x^2 - 9x + 5)^9$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 9(3x^2 - 9x + 5)^8.(6x - 9)$$
$$= 27(3x^2 - 9x + 5)^8.(2x - 3)$$

Q. 2. $\sin^3 x + \cos^6 x$ **[NCERT Misc., Q.2, Page 144]**

Sol. Let $y = \sin^3 x + \cos^6 x$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 3\sin^2 x\cos x + 6\cos^5 x(-\sin x)$$
$$= 3\sin x\cos x(\sin x - 2\cos^4 x)$$

Q.3. $(5x)^{3\cos 2x}$ **[NCERT Misc., Q. 3, Page 144]**

Sol. Let $y = (5x)^{3\cos 2x}$

By taking log on both sides, we get

$$\log y = 3\cos 2x\log(5x) = 3\cos 2x[\log 5 + \log x]$$
$$\log y = 3\cos 2x\log 5 + 3\cos 2x\log x \quad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = 3\log 5(-\sin 2x)\cdot 2 + \frac{3\cos 2x}{x} - 3\log x\cdot(2\cdot\sin 2x)$$
$$= -6\log 5\sin 2x + \frac{3\cos 2x}{x} - 6\log x\sin 2x$$

therefore,

$$\frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6[\log 5 + \log x]\sin 2x\right]$$
$$= (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\log 5x\sin 2x\right]$$

Q. 4. $\sin^{-1}\left(x\sqrt{x}\right), 0 \le x \le 1$. **[NCERT Misc., Q. 4, Page 144]**

Sol. Let $y = \sin^{-1}\left(x\sqrt{x}\right)$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^3}}\cdot\frac{d}{dx}x\sqrt{x}$$
$$= \frac{1}{\sqrt{1-x^3}}\frac{d}{dx}x^{3/2}$$
$$= \frac{3}{2\sqrt{1-x^3}}x^{1/2}$$
$$= \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Q.5. $\dfrac{\cos^{-1}\left(\frac{x}{2}\right)}{\sqrt{2x+7}}, -2 < x < 2.$ **[NCERT Misc., Q.5, Page 144]**

Sol. Let $y = \left(\cos^{-1}\frac{x}{2}\right)(2x+7)^{-1/2}$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \cos^{-1}\frac{x}{2}\left[\frac{d}{dx}(2x+7)^{-1/2}\right] + (2x+7)^{-1/2}\left(\frac{d}{dx}\cos^{-1}\frac{x}{2}\right)$$
$$= \cos^{-1}\frac{x}{2}\left[\frac{-1}{2}(2x+7)^{-3/2}(2)\right] + (2x+7)^{-1/2}\left(\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\right)\times\frac{1}{2}$$
$$\Rightarrow \frac{dy}{dx} = -\cos^{-1}\frac{x}{2}(2x+7)^{-3/2} + (2x+7)^{-1/2}\left[\frac{-1}{2^x\sqrt{1-\left(\frac{x^2}{4}\right)}}\right]$$
$$= -\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}} + \frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{3/2}}\right]$$

Q. 6. $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \dfrac{\pi}{2}.$

[NCERT Misc., Q.6, Page 144]

Sol. Let $y = \cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \dfrac{\pi}{2}$

$$\Rightarrow y = \cos^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$\Rightarrow y = \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

$$\Rightarrow y = \cot^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right]$$

$$\Rightarrow y = \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right]$$

$$\Rightarrow y = \cot^{-1}\left[\cot\frac{x}{2}\right]$$

$$\Rightarrow y = \frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

Q.7. $(\log x)^{\log x}, x > 1$ [NCERT Misc., Q.7, Page 144]

Sol. Let $y = (\log x)^{\log x}$

By taking log on both sides, we get

$$\log y = \log x \log(\log x) \quad ...(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \log x \cdot \frac{1}{\log x}\cdot\frac{1}{x} + \log(\log x)\frac{1}{x}$$

$$= \frac{1}{x}\cdot[1+\log(\log x)]$$

therefore, $\frac{dy}{dx} = (\log x)^{\log x}\cdot\frac{1}{x}\cdot[1+\log(\log x)]$.

Q. 8. cos (a cos x + $b$sin x), for some constant a and b. [NCERT Misc., Q. 8, Page 144]

Sol. Let $y = \cos(a\cos x + b\sin x)$ ...(i)

Differentiating (i) on both sides w.r.t. x, we get

$$\frac{dy}{dx} = -\sin(a\cos x + b\sin x)[a(-\sin x) + b\cos x]$$

$$= -\sin(a\cos x + b\sin x)[-a\sin x + b\cos x]$$

$$= (a\sin x - b\cos x)\sin(a\cos x + b\sin x)$$

Q. 9. $(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$.

[NCERT Misc., Q. 9, Page 144]

Sol. Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

By taking log on both sides, we get

$$\log y = (\sin x - \cos x)\log(\sin x - \cos x) \quad ...(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x)\frac{(\cos x + \sin x)}{(\sin x - \cos x)} + \log(\sin x - \cos x)\cdot(\cos x + \sin x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\cos x + \sin x) + \log(\sin x - \cos x)\cdot(\cos x + \sin x)$$

$$\Rightarrow \frac{dy}{dx} = y(\cos x + \sin x)[1 + \log(\sin x - \cos x)]$$

therefore,

$$\frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}[(\cos x + \sin x)(1 + \log(\sin x - \cos x))],\ \sin x > \cos x$$

Q. 10. $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$. [NCERT Misc., Q. 10, Page 144]

Sol. Let us assume that: $A = x^x$ ...(I)

$B = x^a$ ...(II)

$C = a^x$ ...(III)

$D = a^a$ ...(IV)

I. Let

$$A = x^x \Rightarrow \log A = \log x^x \Rightarrow \log A = x\log x \quad .....(i)$$

Differentiating (i) on both sides w.r.t. x, we get

$$\frac{1}{A}\frac{dA}{dx} = x\left(\frac{1}{x}\right) + \log x \Rightarrow \frac{dA}{dx} = A[1+\log x] \quad ...(i)$$

$$\Rightarrow \frac{dA}{dx} = x^x(1+\log x) \quad ...(ii)$$

II. Let

$$B = x^a \Rightarrow \log B = \log x^a \Rightarrow \log B = a\log x \quad ....(iii)$$

Differentiating (iii) on both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{B}\frac{dB}{dx} = a\frac{1}{x} \Rightarrow \frac{dB}{dx} = B\left(\frac{a}{x}\right)$$

$$\Rightarrow \frac{dB}{dx} = x^a\left(\frac{a}{x}\right) = x^a\cdot a\cdot x^{-1} = ax^{a-1} \quad ....(iv) \quad ...(iv)$$

III. Let

$$C = a^x \Rightarrow \log C = \log a^x \Rightarrow \log C = x\log a \quad ...(v)$$

Differentiating (v) on both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{C}\frac{dy}{dx} = \log a(1) \Rightarrow \frac{dy}{dx} = C\log a = a^x\log a \quad ...(vi)$$

IV. Let $D = 1 = a^a \Rightarrow \log D = \log a^a \Rightarrow \log D = a\log a$

Differentiating (vii) on both sides w.r.t. x, we get

$$\Rightarrow \frac{1}{D}\frac{dy}{dx} = a(0) \Rightarrow \frac{dD}{dx} = 0 \quad ....(viii)$$

therefore, $\frac{dy}{dx} = x^x(1+\log x) + ax^{a-1} + a^x\log a$

[from (ii), (iv), (vi) and (viii)]

Q. 11. $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$.

[NCERT Misc., Q. 11, Page 145]

Sol. Let

$$x^{x^2-3} + (x-3)^{x^2} = u + v,$$

where $u = x^{x^2-3}$,

and $v = (x-3)^{x^2}$

Let $u = x^{x^2-3}$

On taking log on both sides, we get

$$\log u = (x^2-3)\log x \quad ...(i)$$

Differentiating (i) w.r.t. x on both sides, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{(x^2-3)}{x} + \log x(2x)$$

therefore, $\frac{du}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right]$ ...(ii)

and $v = (x-3)^{x^2}$

By taking log on both sides, we get

$$\log v = x^2\log(x-3) \quad ...(iii)$$

Differentiating (iii) w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \frac{x^2}{x-3} + \log(x-3)\cdot(2x)$$

therefore, $\frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$ ...(iv)

So, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$$= x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right]$$

$$+ (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

[from (ii)&(iv)]

Q. 12. Find

$$\frac{dy}{dx}, \text{ if } y = 12(1-\cos t), x = 10(t-\sin t), -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

[NCERT Misc., Q.12, Page 145]

Sol. Here, $y = 12(1-\cos t)$ ...(i)

$x = 10(t - \sin t)$ ...(ii)

Differentiating (i), (ii) w.r.t. t, we get

$$\frac{dy}{dt} = 12[-(-\sin t)] = 12\sin t$$

and $\frac{dx}{dt} = 10(1-\cos t)$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12\sin t}{10(1-\cos t)}$$

$$= \frac{6\sin t}{5(1-\cos t)}$$

$$= \frac{6}{5}\left[\frac{2\sin(t/2)\cos(t/2)}{2\sin^2(t/2)}\right]$$

$$= \frac{6}{5}\cot(t/2)$$

Q. 13. Find $\frac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$, $-1 \le x \le 1$.

[NCERT Misc., Q. 13, Page 145]

Sol. Here,

$$y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2} = u + v$$

Let $u = \sin^{-1}x$

and $v = \sin^{-1}\sqrt{1-x^2}$

$\Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ and for $v = \sin^{-1}\sqrt{1-x^2}$

Putting $x = \cos\theta$, we get

$$v = \sin^{-1}\sqrt{1-\cos^2\theta} = \sin^{-1}\sqrt{\sin^2\theta}$$

$$= \sin^{-1}(\sin\theta) = \theta = \cos^{-1}x$$

therefore, $\frac{dv}{dx} = \frac{-1}{\sqrt{1-x^2}}$

So, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$

Q. 14. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$,

prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

[NCERT Misc., Q. 14, Page 144]

Sol. We have,

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x} \Rightarrow x^2(1+y) = y^2(1+x)$$

$$\Rightarrow (x^2-y^2) + xy(x-y) = 0 \Rightarrow y = \frac{-x}{x+1}$$

therefore,

$$\frac{dy}{dx} = \frac{(x+1)(-1)-(-x)(1)}{(x+1)^2}$$

$$= \frac{-x-1+x}{(x+1)^2} = \frac{-1}{(x+1)^2}.$$

Q. 15. If, $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b.

[NCERT Misc., Q. 15, Page 145]

Sol. $(x-a)^2 + (y-b)^2 = c^2$ ...(i)

Differentiate w.r.t. x

$$\Rightarrow 2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow (x-a) + (y-b)\frac{dy}{dx} = 0 \quad ...(ii)$$

Squaring both side

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left(-\left(\frac{x-a}{y-b}\right)\right)^2$$

Adding 1 to both side

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{x-a}{y-b}\right)^2$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \frac{(y-b)^2+(x-a)^2}{(y-b)^2}$$

$$\Rightarrow \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \left\{\left(\frac{c}{y-b}\right)^2\right\}^{\frac{3}{2}}$$

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \left(\frac{c}{y-b}\right)^3 \quad ...(iii)$$

diff. eq. (ii) w.r.t. x

$$1 + (y-b)\frac{d^2y}{dx^2} + \frac{dy}{dx}.\frac{dy}{dx} = 0$$

$$\frac{d^2y}{dx^2} = \frac{-\left[1+\left(\frac{dy}{dx}\right)^2\right]}{(y-b)} \quad ...(iv)$$

dividing eq. (iii) with (iv)

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\frac{c^3}{(y-b)^2}}{\frac{-\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{(y-b)}}$$

$$= \frac{\frac{c^3}{(y-b)^2}}{\left\{\frac{c^2}{(y-b)^2}\right\}}$$

$$= -c$$

Q. 16. If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

[NCERT Misc., Q. 16, Page 145]

Sol. We have,

$\cos y = x\cos(a+y)$

$\Rightarrow \quad x = \frac{\cos y}{\cos(a+y)}$

Differentiating both sides with respect to y

$$\Rightarrow \frac{dx}{dy} = \frac{\cos(a+y)(-\sin y) - \cos y(-\sin(a+y))}{\cos^2(a+y)}$$

$$= \frac{\cos y\sin(a+y) - \sin y\cos(a+y)}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$

therefore,

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Q. 17. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$.

[NCERT Misc., Q. 17, Page 145]

Sol. Here,

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t)$$

therefore,

$$\frac{dx}{dt} = at\cos t$$

$$\Rightarrow \quad \frac{dy}{dt} = a[\cos t - \{t(-\sin t) + \cos t(1)\}]$$

$$\Rightarrow \quad \frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$$

therefore,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = at\sin t \times \frac{1}{at\cos t} = \tan t$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{1}{at\cos t} = \frac{\sec^2 t}{at\cos t}$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

Q. 18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it **[NCERT Misc., Q. 18, Page 145]**

Sol. Case I. When $x \geq 0$. Here, $f(x) = |x|^3 = x^3$

Therefore, $f'(x) = 3x^2$ and $f''(x) = 6x$

Case II. When $x < 0$. Here, $f(x) = (-x)^3 = -x^3$

Therefore, $f'(x) = -3x^2$ and $f''(x) = -6x$

Hence, we can study that $f''(x)$ exist for all real x.

Q. 19. Using the fact that sin(A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

[NCERT Misc., Q. 19, Page 145]

Sol. $\sin(A+B) = \sin A\cos B + \cos A\sin B$ ...(i)

Consider A and B as function of t and differentiating both sides of (i) w.r.t. t, we get

$$\cos(A+B)\left(\frac{dA}{dt} + \frac{dB}{dt}\right)$$

$$= \sin A(-\sin B)\frac{dB}{dt} + \cos B\left[\cos A\frac{dA}{dt}\right]$$

$$+ \cos A\cos B\frac{dB}{dt} + \sin B(-\sin A)\frac{dA}{dt}$$

$$\Rightarrow \cos(A+B)\left(\frac{dA}{dt} + \frac{dB}{dt}\right) = -\sin A\sin B\left(\frac{dB}{dt} + \frac{dA}{dt}\right)$$

$$+ \cos A\cos B\left(\frac{dA}{dt} + \frac{dB}{dt}\right)$$

$$\Rightarrow \cos(A+B)\left(\frac{dA}{dt} + \frac{dB}{dt}\right)$$

$$= (\cos A\cos B - \sin A\sin B)\left(\frac{dA}{dt} + \frac{dB}{dt}\right)$$

$$\Rightarrow \cos(A+B) = \cos A\cos B - \sin A\sin B$$

Q. 20. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer. **[NCERT Misc., Q. 20, Page 145]**

Sol. Let the function be

$f(x) = |x-1| + |x-2|$.

We redefine $f(x)$ as:

$$f(x) = \begin{cases} -(x-1)-(x-2); & \text{if } x < 1 \\ ((x-1)-(x-2); & \text{if } 1 \leq x \leq 2 \\ (x-1)+(x-2); & \text{if } x > 2 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} -2x+3; & \text{if } x < 1 \\ 1; & \text{if } 1 \leq x \leq 2 \\ (2x-3); & \text{if } x > 2 \end{cases}$$

$f(x)$ is clearly continuous at all x except at $x = 1, 2$. At $x = 1$:

$$\lim_{x \to 1^-} f(x) = \lim_{\substack{x \to 1-h \\ h \to 0}} (-2(1-h)+3)$$

$$= -2+3 = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1) = 1.$$

Also, $f(1) = 1$

Thus, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

At $x = 2$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 1 = 1$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(2x-3)$$
$$= \lim_{\substack{x\to 2+h \\ h\to 0}}(2(2+h)-3)$$
$$= 2(2)-3 = 1$$

Also, $f(2) = 1$

Thus, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} f(x) = f(2)$

Hence, $f(x)$ is continuous at $x = 2$.
Hence, $f(x)$ is continuous at $x \in$ R.
Differentiability at $x = 1$:

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$
$$= \lim_{h\to 0}\frac{-2(1-h)+3-1}{-h}$$
$$= \lim_{h\to 0}\frac{2h}{-h} = \lim_{h\to 0}(-2) = -2$$

Thus, $Lf'(1) \neq Rf' \Rightarrow f$ is not derivable at $x = 1$

Derivability at $x = 2$:

$$lf'(2) = \lim_{h\to 0}\frac{f(2-h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{(1)-1)}{-h} = 0$$
$$Rf'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{2(2+h)-3-1}{h}$$
$$= \lim_{h\to 0}\frac{2h}{h} = \lim_{h\to 0} 2 = 2$$

Thus, $Lf'(2) \neq Rf'(2) \Rightarrow f$ is not derivable at $x = 2$.
Hence $f(x) = |x-1| + |x-2|$ is continuous everywhere and differentiable at all x except at 1 and 2.

Q. 21. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, **prove that**

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}.$$

[NCERT Misc., Q. 21, Page 145]

Sol. We have, $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

$$\therefore \frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}(f(x)) & \frac{d}{dx}(g(x)) & \frac{d}{dx}(h(x)) \\ l & m & n \\ a & b & c \end{vmatrix}$$
$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$
$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Hence proved.

Q. 22. If $y = e^{a\cos^{-1} x}, -1 \leq x \leq 1$

show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$

[NCERT Misc., Q. 22, Page 145]

Sol. We have, $y = e^{a\cos^{-1} x}$ ...(i)

Differentiating (i) on both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{a\cos^{-1}x}\frac{d}{dx}(a\cos^{-1}x)$$
$$= e^{a\cos^{-1}x}\left(\frac{-a}{\sqrt{1-x^2}}\right) = \frac{-ay}{\sqrt{1-x^2}} \quad \text{...(ii)}$$

Differentiating (ii) on both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -a\left[\frac{\sqrt{1-x^2}\frac{dy}{dx} - y\frac{d}{dx}\sqrt{1-x^2}}{(1-x^2)}\right]$$
$$\Rightarrow \frac{d^2y}{dx^2} = -a\left[\frac{\sqrt{1-x^2}\frac{dy}{dx} - \frac{y}{2\sqrt{1-x^2}}(-2x)}{(1-x^2)}\right]$$
$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = -a\left[-ay + \frac{xy}{\sqrt{1-x^2}}\right] \quad \text{[From (ii)]}$$
$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = -a\left[-ay + x\cdot\left(\frac{-1}{a}\cdot\frac{dy}{dx}\right)\right]$$
$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = a^2y + x\frac{dy}{dx}$$
$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

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