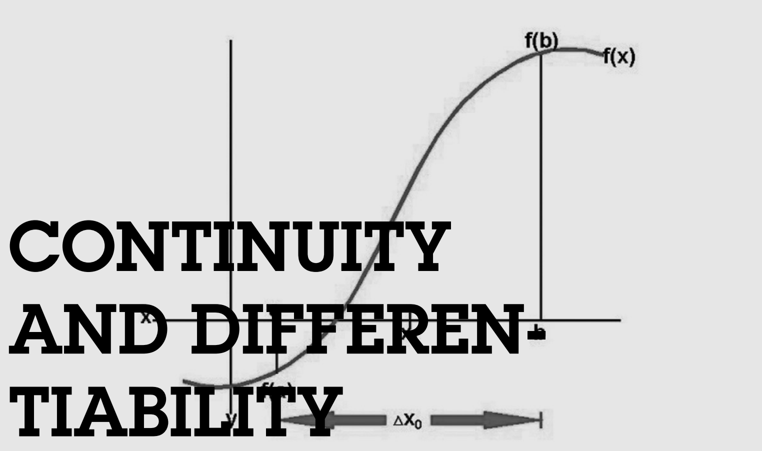
CHAPTER

5 CONTINUITY AND DIFFERENTIABILITY



Chapter Objectives

This chapter will help you understand :

- ***Continuity and differentiability :*** *Continuity of a function at a point; Continuity in an interval; Geometrical meaning of continuity; Discontinuity and Continuity of composite function; Differentiability and Differentiability in interval.*
- ***Derivatives :*** *Algebra of derivatives; Derivatives of composite function; Derivatives of implicit function; Derivatives of trigonometric function; Derivatives of inverse trigonometric function; Exponential function; Derivatives of exponential function; Logarithmic function; Logarithmic rules and its differentiation; Derivative of function in parametric forms and Second order of derivative.*
- ***Rolle's theorem and MVT :*** *Rolle's theorem and Mean value theorem.*

TOPIC-1
Continuity and Differentiability

Quick Review

- Continuity of a function at a point : Let f be a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if $\lim_{x\to c} f(x) = f(c)$. In other words, if the left-hand limit, right-hand limit and the value of the function at $x = c$ exist and are equal to each other, $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$ then f is said to be continuous at $x = c$.
- Continuity in an interval :
  - The function f is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.
  - The function f is said to be continuous in closed interval $[a, b]$ if f is continuous in open interval (a, b), $\lim_{x\to a^+} f(x) = f(a)$, $\lim_{x\to b^-} f(x) = f(b)$.
- Geometrical meaning of continuity : The geometrical meaning of a continuous at c of a function f is that there is no break in the graph of the function at the point $[c, f(c)]$
- Discontinuity : The function f will be discontinuous at $x = a$ in any of the following cases :
  - $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$
  - $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \neq f(a)$
  - $f(a)$ is not defined.
- Continuity of composite function : Let f and g be real valued functions such that (fog) is defined at a. If g is continuous at a and f is continuous at $g(a)$, then (fog) is continuous at a.

TIPS...

- *The constant function $f(x) = k$ is continuous for all real values of x.*
- *The identity function $f(x) = x$ is continuous for all real values of x.*
- *The modulus function $f(x) = |x|$ is continuous for all real values of x.*
- *The polynomial function $f(x) = a_n x^n + a^{n-1} + a_{n-2}x^{n-2} .... + a_2x^2 + a_1x^1 + a_0x^0$ is continuous for all real values of x.*
- *The greatest integer function $f(x) = [x]$ is continuous for all real values of x except at integral values of x.*

TRICKS...

- *All trigonometrical functions are continuous in their respective domains.*
- *Exponential and Logarithmic Functions are continuous for all real values of x.*

❖ Differentiability : If a function f is differentiable at a point c in its domain if both LHD (left-hand derivative) and RHD (right-hand derivative) are finite and equal, it means : $LHD = \lim_{h \to 0^-} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = RHD.$

❖ Differentiability in an interval : The function $y = f(x)$ is said to be differentiable in an open interval (a, b) if it is differentiable at every point of (a, b). And, the function $y = f(x)$ is said to be differentiable in the closed interval $[a, b]$ if LHD and RHD exist and $f'(x)$ exists for every point of (a, b). Every differentiable function is continuous, but the converse is not true.

Multiple Choice Questions **(1 mark each)**

Q. 1. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$ then which of the following can be a discontinuous function?

(a) $f(x) + g(x)$ **(b)** $f(x) - g(x)$

(c) $f(x).g(x)$ **(d)** $\frac{g(x)}{f(x)}$

[NCERT Exemp. Ex. 5.3, Q. 83, Page 113]

Ans. Correct option : (d)

Explanation : Since $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$ are continuous functions, then by using the algebra of continuous functions , the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x).g(x)$ are also continuous functions but $\frac{g(x)}{f(x)}$ is discontinuous function at $x = 0$.

Q. 2. The function $f(x) = \frac{4 - x^2}{4x - x^3}$

(a) discontinuous at only one point

(b) discontinuous at exactly two points

(c) discontinuous at exactly three points

(d) none of these

[NCERT Exemp. Ex. 5.3, Q. 84, Page 113]

Ans. Correct option : (c)

Explanation : Given that,

$f(x) = \frac{4 - x^2}{4x - x^3}$, then it is discontinuous if

$\Rightarrow \quad 4x - x^3 = 0$

$\Rightarrow \quad x(4 - x^2) = 0$

$\Rightarrow x(2 + x)(2 - x) = 0$

$\Rightarrow \quad x = 0, -2, 2$

Thus, the given function is discontinuous at exactly three points.

Q. 3. The set of points where the function f given by $f(x) = |2x - 1| \sin x$ is differentiable is

(a) R

(b) $R - \left\{\frac{1}{2}\right\}$

(c) $(0, \infty)$

(d) none of these

[NCERT Exemp. Ex. 5.3, Q. 85, Page 113]

Ans. Correct option : (c)

Explanation : Given that,

$f(x) = |2x - 1| \sin x$

The function $\sin x$ is differentiable.

The function $|2x - 1|$ is differentiable, except

$2x - 1 = 0$

$\Rightarrow x = \frac{1}{2}$

Thus, the given function is differentiable $R - \left\{\frac{1}{2}\right\}$.

Q. 4. The function $f(x) = \cot x$ is discontinuous on the set

(a) $\{x = n\pi ; n \in Z\}$

(b) $\{x = 2n\pi ; n \in Z\}$

(c) $\left\{x = (2n + 1)\frac{\pi}{2} ; n \in Z\right\}$

(d) $\left\{x = \frac{n\pi}{2} ; n \in Z\right\}$

[NCERT Exemp. Ex. 5.3, Q. 86, Page 114]

Ans. Correct option : (a)

Explanation : Given that,

$f(x) = \cot x = \frac{\cos x}{\sin x}$

It is discontinuous at

$\sin x = 0$

$\Rightarrow x = n\pi, n \in Z$

Thus, the given function is discontinuous at $\{x = n\pi : n \in Z\}$.

Q. 5. The function $f(x) = e^{|x|}$ is

(a) continuous everywhere but not differentiable at $x = 0$

(b) continuous and differentiable everywhere

(c) not continuous at $x = 0$

(d) none of these

[NCERT Exemp. Ex. 5.3, Q. 87, Page 114]

Ans. Correct option : (a)

Explanation : Given that,

$f(x) = e^{|x|}$

The functions e^x and $|x|$ are continuous functions for all real value of x.

Since e^x is differentiable everywhere but $|x|$ is non-differentiable at $x = 0$.

Thus, the given functions $f(x)=e^{|x|}$ is continuous everywhere but not differentiable at $x = 0$.

Q. 6. If $f(x)=x^2\sin\frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is

(a) 0 **(b) –1**

(c) 1 **(d) none of these**

[NCERT Exemp. Ex. 5.3, Q. 88, Page 114]

Ans. Correct option : (a)

Explanation : Given that,

$$f(x)=x^2\sin\frac{1}{x}$$

Thus,

$$f(0)=\lim_{x\to 0}\left(x^2\sin\frac{1}{x}\right)$$

$$\Rightarrow f(0)=0\times\begin{pmatrix}\text{an oscillating value}\\ \text{between }-1\text{ and }1\end{pmatrix}$$

$$\Rightarrow f(0)=0$$

Q. 7. If $f(x)=\begin{cases} mx+1 \text{ if } x\le\frac{\pi}{2} \\ \sin x+n, \text{ if } x>\frac{\pi}{2}\end{cases}$, is continuous at $x=\frac{\pi}{2}$ then

(a) $m = 1, n = 0$ **(b) $m=\frac{n\pi}{2}+1$**

(c) $n=\frac{m\pi}{2}$ **(d) $m=n=\frac{\pi}{2}$**

[NCERT Exemp. Ex. 5.3, Q. 89, Page 114]

Ans. Correct option : (c)

Explanation : Given that,

$$f(x)=\begin{cases} mx+1 \text{ if } x\le\frac{\pi}{2} \\ \sin x+n, \text{ if } x>\frac{\pi}{2}\end{cases}$$ is continuous function at $x=\frac{\pi}{2}$, then

$$LHL = RHL$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} f(x)=\lim_{x\to\frac{\pi}{2}^+} f(x)$$

$$\Rightarrow \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$$

$$\Rightarrow \lim_{h\to 0} m\left(\frac{\pi}{2}-h\right)+1=\lim_{h\to 0}\sin\left(\frac{\pi}{2}+h\right)+n$$

$$\Rightarrow \lim_{h\to 0} m\left(\frac{\pi}{2}-h\right)+1=\lim_{h\to 0}\cos h+n$$

$$\Rightarrow m\left(\frac{\pi}{2}\right)+1=1+n$$

$$\Rightarrow n=\frac{m\pi}{2}$$

Q. 8. Let $f(x)=|\sin x|$, then

(a) f is everywhere differentiable

(b) f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$.

(c) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}$, $n \in Z$.

(d) none of these

[NCERT Exemp. Ex. 5.3, Q. 90, Page 114]

Ans. Correct option : (b)

Explanation : Given that,

$$f(x)=|\sin x|$$

The functions $|x|$ and $\sin x$ are continuous function for all real value of x.

Thus, the function $f(x)=|\sin x|$ is continuous function everywhere.

Now, $|x|$ is non-differentiable function at $x = 0$.

Since $f(x)=|\sin x|$ is non-differentiable function at $\sin x = 0$

Thus, f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$.

Q. 9. Fill in the blanks :

An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is __________

[NCERT Exemp. Ex. 5.3, Q. 97, Page 116]

Ans. $f(x)=|x^2-4|$

Explanation : The function $f(x)=|x^2-4|$ is continuous everywhere but it is non-differentiable at $x^2-4=0$

$$\Rightarrow x=\pm 2$$

Q. 10. State True or False for the statement :

If f is continuous on its domain D, then $|f|$ is also continuous on D.

[NCERT Exemp. Ex. 5.3, Q. 103, Page 116]

Ans. True

Explanation : Let a function $f(x) = x$ which is continuous in its domain R, then the function $|f(x)| = |x|$ is also a continuous function in its domain.

Q. 11. State True or False for the statement :

The composition of two continuous functions is a continuous function.

[NCERT Exemp. Ex. 5.3, Q. 104, Page 116]

Ans. True

Explanation : The composition of two continuous functions is a continuous function.

Q. 12. State True or False for the statement :

Trigonometric and inverse-trigonometric functions are differentiable in their respective domain.

[NCERT Exemp. Ex. 5.3, Q. 105, Page 116]

Ans. True

Explanation : Trigonometric and inverse-trigonometric functions are differentiable in their respective domain.

Q. 13. State True or False for the statement :

If $f.g$ is continuous at $x = a$, then f and g are separately continuous at $x = a$.

[NCERT Exemp. Ex. 5.3, Q. 106, Page 116]

Ans. False

Explanation : Let $f(x)=\sin x$ and $g(x)=\cot x$.

Thus,

$$f(x)\times g(x)=\sin x\times\cot x=\sin x\times\frac{\cos x}{\sin x}=\cos x.$$

It is continuous function at $x = 0$ but $g(x)=\cot x$ is not continuous function at $x = 0$.

Very Short Answer Type Question (1 or 2 mark)

Q. 1. Given the function $f(x)=\frac{1}{x+2}$. Find the points of discontinuity of the composite function $y = f(f(x))$.
[NCERT Exemp. Ex. 5.3, Q. 17, Page 108]

Ans. Since the given function is not defined at $x = -2$, the function is discontinuous function at $x = -2$.

Now $f(f(x))=\frac{1}{f(x)+2}$

$$=\frac{1}{\left(\frac{1}{x+2}\right)+2}$$

$$=\frac{x+2}{2x+5}$$

[1]

Thus, the function $f(f(x))=\frac{x+2}{2x+5}$ is discontinuous at $x = -2$ and $-\frac{5}{2}$. **[1]**

Q. 2. Find all points of discontinuity of the function $f(t)=\frac{1}{t^2+t-2}$, where $t=\frac{1}{x-1}$.
[NCERT Exemp. Ex. 5.3, Q. 18, Page 109]

Ans. Given, that,

$$f(t)=\frac{1}{t^2+t-2}$$

Put $t=\frac{1}{x-1}$, we have

$$f\left(\frac{1}{x-1}\right)=\frac{1}{\left(\frac{1}{x-1}\right)^2+\left(\frac{1}{x-1}\right)-2}$$

$$=\frac{(x-1)^2}{(2x-1)(2-x)}$$

[1]

Since it is not defined at $(2x-1)(2-x) = 0$ $x = 2, \frac{1}{2}$, then

The given function is discontinuous function at $x = 2$ and $\frac{1}{2}$.

Short Answer Type Questions (3 or 4 marks each)

Q. 1. Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.
[NCERT Exemp. Ex. 5.3, Q. 1, Page 107]

Ans. Given function is $f(x) = x^3 + 2x^2 - 1$.

$$\text{LHL (at } x=1) = \lim_{x\to1^-} f(x)$$
$$=\lim_{h\to0} f(1-h)$$
$$=\lim_{h\to0}(1-h)^3+2(1-h)^2-1$$
$$=1+2-1$$
$$=2$$

[1]

$$\text{RHL (at } x=1) = \lim_{x\to1^+} f(x)$$
$$=\lim_{h\to0} f(1+h)$$
$$=\lim_{h\to0}(1+h)^3+2(1+h)^2-1$$
$$=1+2-1$$
$$=2$$

[1]

And, $f(1)=1+2-1=2$

Since LHL = RHL = $f(1)=2$, then the given function is continuous at $x = 1$. **[1]**

Q. 2. Is the function $f(x)=\begin{cases}3x+5, & \text{if } x\ge2\\ x^2, & \text{if } x<2\end{cases}$ continuous or discontinuous at $x = 2$?
[NCERT Exemp. Ex. 5.3, Q. 2, Page 107]

Ans. Given function is $f(x)=\begin{cases}3x+5, & \text{if } x\ge2\\ x^2, & \text{if } x<2\end{cases}$

$$\text{LHL (at } x=2) = \lim_{x\to2^-} f(x)$$
$$=\lim_{h\to0} f(2-h)$$
$$=\lim_{h\to0}(2-h)^2$$
$$=4$$

[1½]

$$\text{RHL (at } x=2) = \lim_{x\to2^+} f(x)$$
$$=\lim_{h\to0} f(2+h)$$
$$=\lim_{h\to0}3(2+h)+5$$
$$=11$$

[1½]

Since LHL ≠ RHL, then the given function is discontinuous at $x = 2$.

Q. 3. Is the function $f(x)=\begin{cases}\frac{1-\cos 2x}{x^2}, & \text{if } x\ne0\\ 5, & \text{if } x=0\end{cases}$ continuous or discontinuous at $x = 0$?
[NCERT Exemp. Ex. 5.3, Q. 3, Page 107]

Ans. Given function is $f(x)=\begin{cases}\frac{1-\cos 2x}{x^2}, & \text{if } x\ne0\\ 5, & \text{if } x=0\end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to0^-} f(x)$$
$$=\lim_{h\to0} f(0-h)$$
$$=\lim_{h\to0} f(-h)$$
$$=\lim_{h\to0}\left[\frac{1-\cos 2(-h)}{(-h)^2}\right]$$

$$= \lim_{h\to 0}\left[\frac{1-\cos 2h}{h^2}\right]$$
$$= \lim_{h\to 0}\left[\frac{2\sin^2 h}{h^2}\right]$$
$$= \lim_{h\to 0}\left[2\left(\frac{\sin h}{h}\right)^2\right]$$
$$= 2$$ **[1½]**

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[\frac{1-\cos 2(h)}{(h)^2}\right]$$
$$= \lim_{h\to 0}\left[\frac{1-\cos 2h}{h^2}\right]$$
$$= \lim_{h\to 0}\left[\frac{2\sin^2 h}{h^2}\right]$$
$$= \lim_{h\to 0}\left[2\left(\frac{\sin h}{h}\right)^2\right]$$
$$= 2$$ **[1½]**

And, $f(0)=5$

Since $\text{LHL} = \text{RHL} \neq f(0) = 5$, then the given function is discontinuous at $x = 0$.

Q. 4. Is the function $f(x)=\begin{cases}\dfrac{|x-4|}{2(x-4)}, & \text{if } x\neq 4\\ 0, & \text{if } x=4\end{cases}$ **continuous or discontinuous at $x = 4$?**

[NCERT Exemp. Ex. 5.3, Q. 5, Page 107]

Ans. Given function is $f(x)=\begin{cases}\dfrac{|x-4|}{2(x-4)}, & \text{if } x\neq 4\\ 0, & \text{if } x=4\end{cases}$.

$$\text{LHL (at } x=4) = \lim_{x\to 4^-} f(x)$$
$$= \lim_{h\to 0} f(4-h)$$
$$= \lim_{h\to 0}\left[\frac{|4-h-4|}{2(4-h-4)}\right]$$
$$= \lim_{h\to 0}\left[\frac{|-h|}{-2h}\right]$$
$$= \lim_{h\to 0}\left[\frac{h}{-2h}\right]$$
$$= \lim_{h\to 0}\left[-\frac{1}{2}\right]$$
$$= -\frac{1}{2}$$ **[1]**

$$\text{RHL (at } x=4) = \lim_{x\to 4^+} f(x)$$
$$= \lim_{h\to 0} f(4+h)$$
$$= \lim_{h\to 0}\left[\frac{|4+h-4|}{2(4+h-4)}\right]$$
$$= \lim_{h\to 0}\left[\frac{|h|}{2h}\right]$$
$$= \lim_{h\to 0}\left[\frac{h}{2h}\right]$$
$$= \lim_{h\to 0}\left[\frac{1}{2}\right]$$
$$= \frac{1}{2}$$ **[1]**

Since LHL ≠ RHL, then the given function is discontinuous at $x = 4$. **[1]**

Q. 5. Is the function $f(x)=\begin{cases}|x|\cos\dfrac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ **continuous or discontinuous at $x = 0$?**

[NCERT Exemp. Ex. 5.3, Q. 6, Page 107]

Ans. Given function is $f(x)=\begin{cases}|x|\cos\dfrac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[|-h|\cos\frac{1}{(-h)}\right]$$
$$= \lim_{h\to 0}\left[h\cos\frac{1}{h}\right]$$
$$= 0\times\left[\begin{matrix}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{matrix}\right]$$
$$= 0$$ **[1]**

$$\text{RHL (at } x=0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[|h|\cos\frac{1}{(h)}\right]$$
$$= \lim_{h\to 0}\left[h\cos\frac{1}{h}\right]$$
$$= 0\times\left[\begin{matrix}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{matrix}\right]$$
$$= 0$$ **[1]**

And, $f(0)=0$

Since $\text{LHL}=\text{RHL}=f(0)=0$, then the given function is continuous at $x = 0$. **[1]**

Q. 6. Is the function $f(x)=\begin{cases}|x-a|\sin\dfrac{1}{x-a}, & \text{if } x\neq a\\ 0, & \text{if } x=a\end{cases}$ **continuous or discontinuous at $x = a$?**

[NCERT Exemp. Ex. 5.3, Q. 7, Page 107]

Ans. Given function is $f(x)=\begin{cases}|x-a|\sin\dfrac{1}{x-a}, & \text{if } x\neq a\\ 0, & \text{if } x=a\end{cases}$.

$$\text{LHL (at } x = a) = \lim_{x\to a^-} f(x)$$
$$= \lim_{h\to 0} f(a-h)$$
$$= \lim_{h\to 0}\left[|a-h-a|\sin\frac{1}{(a-h-a)}\right]$$
$$= \lim_{h\to 0}\left[|-h|\sin\frac{1}{(-h)}\right]$$
$$= \lim_{h\to 0}\left[-h\sin\frac{1}{h}\right]$$
$$= 0\times\left[\begin{array}{l}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{array}\right]$$
$$= 0$$
[1]

$$\text{RHL (at } x = a) = \lim_{x\to a^+} f(x)$$
$$= \lim_{h\to 0} f(a+h)$$
$$= \lim_{h\to 0}\left[|a+h-a|\sin\frac{1}{(a+h-a)}\right]$$
$$= \lim_{h\to 0}\left[|h|\sin\frac{1}{(h)}\right]$$
$$= \lim_{h\to 0}\left[h\sin\frac{1}{h}\right]$$
$$= 0\times\left[\begin{array}{l}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{array}\right]$$
$$= 0$$
[1]

And, $f(a) = 0$

Since LHL = RHL = $f(0) = 0$, then the given function is continuous at $x = a$. **[1]**

Q. 7. Is the function $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ **continuous or discontinuous at $x = 0$?**

[NCERT Exemp. Ex. 5.3, Q. 8, Page 107]

Ans. Given function is $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

$$\text{LHL (at } x = 0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[\frac{e^{\frac{1}{-h}}}{1+e^{\frac{1}{-h}}}\right]$$
$$= \left[\frac{e^{-\infty}}{1+e^{-\infty}}\right]$$
$$= \frac{0}{1+0}$$
$$= 0$$
[1]

$$\text{RHL (at } x = 0) = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[\frac{e^{\frac{1}{h}}}{1+e^{\frac{1}{h}}}\right]$$
$$= \lim_{h\to 0}\left[\frac{1}{1+e^{-\frac{1}{h}}}\right]$$
$$= \frac{1}{1+e^{-\infty}}$$
$$= \frac{1}{1+0}$$
$$= 1$$
[1]

And, $f(0) = 0$

Since LHL ≠ RHL = $f(0) = 0$, then the given function is discontinuous at $x = 0$. **[1]**

Q. 8. Is the function $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ **continuous or discontinuous at x = 1?**

[NCERT Exemp. Ex. 5.3, Q. 9, Page 107]

Ans. Given function is $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$.

$$\text{LHL (at } x = 1) = \lim_{x\to 1^-} f(x)$$
$$= \lim_{h\to 0} f(1-h)$$
$$= \lim_{h\to 0}\left[\frac{(1-h)^2}{2}\right]$$
$$= \frac{1}{2}$$
[1]

$$\text{RHL (at } x = 1) = \lim_{x\to 1^+} f(x)$$
$$= \lim_{h\to 0} f(1+h)$$
$$= \lim_{h\to 0}\left[2(1+h)^2 - 3(1+h) + \frac{3}{2}\right]$$
$$= 2(1)^2 - 3(1) + \frac{3}{2}$$
$$= \frac{1}{2}$$
[1]

And, $f(1) = \frac{1}{2}$

Since LHL = RHL = $f(1) = \frac{1}{2}$, then the given function is continuous at $x = 1$. **[1]**

Q. 9. Is the function $f(x) = |x| + |x-1|$ continuous or discontinuous at $x = 1$?

[NCERT Exemp. Ex. 5.3, Q. 10, Page 107]

Ans. Given function is $f(x) = |x| + |x-1|$.

$$\text{LHL (at } x=1) = \lim_{x \to 1^-} f(x)$$
$$= \lim_{h \to 0} f(1-h)$$
$$= \lim_{h \to 0} \left[|1-h| + |1-h-1| \right]$$
$$= \lim_{h \to 0} \left[|1-h| + |-h| \right]$$
$$= 1 + 0$$
$$= 1$$
[1]

$$\text{RHL (at } x=1) = \lim_{x \to 1^+} f(x)$$
$$= \lim_{h \to 0} f(1+h)$$
$$= \lim_{h \to 0} \left[|1+h| + |1+h-1| \right]$$
$$= \lim_{h \to 0} \left[|1+h| + |h| \right]$$
$$= 1 + 0$$
$$= 1$$
[1]

And, $f(1) = 1$

Since LHL = RHL = $f(1) = 1$, then the given function is continuous at $x = 1$. **[1]**

Q. 10. Find the value of k in $f(x) = \begin{cases} 3x-8, & \text{if } x \le 5 \\ 2k, & \text{if } x > 5 \end{cases}$ so that the function f is continuous at $x = 5$?

[NCERT Exemp. Ex. 5.3, Q. 11, Page 108]

Ans. Given function is $f(x) = \begin{cases} 3x-8, & \text{if } x \le 5 \\ 2k, & \text{if } x > 5 \end{cases}$.

$$\text{LHL (at } x=5) = \lim_{x \to 5^-} f(x)$$
$$= \lim_{h \to 0} f(5-h)$$
$$= \lim_{h \to 0} \left[3(5-h) - 8 \right]$$
$$= 3(5) - 8$$
$$= 7$$
[1]

$$\text{RHL (at } x=5) = \lim_{x \to 5^+} f(x)$$
$$= \lim_{h \to 0} f(5+h)$$
$$= \lim_{h \to 0} [2k]$$
$$= 2k$$
[1]

And, $f(5) = 7$

Since the given function is continuous at $x = 5$, then LHL = RHL = $f(5) = 7$.

LHL=RHL

$\Rightarrow 7 = 2k$

$\Rightarrow k = \frac{7}{2}$

[1]

Q. 11. Find the value of k in $f(x) = \begin{cases} \frac{2^{x+2}-16}{4^x-16}, & \text{if } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}$ so that the function f is continuous at $x = 2$?

[NCERT Exemp. Ex. 5.3, Q. 12, Page 108]

Ans. Given function is

$$f(x) = \begin{cases} \frac{2^{x+2}-16}{4^x-16}, & \text{if } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \frac{4 \times 2^x - 16}{4^x - 16}, & \text{if } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}.$$

Since the given function is continuous at $x = 2$, then LHL = RHL = $f(2) = k$. **[1]**

$$k = \lim_{x \to 2} f(x)$$
$$= \lim_{x \to 2} \left(\frac{2^{x+2}-16}{4^x-16} \right)$$
$$= \lim_{x \to 2} \frac{4(2^x-4)}{(2^x-4)(2^x+4)}$$
$$= \lim_{x \to 2} \frac{4}{(2^x+4)}$$
$$= \frac{4}{4+4} = \frac{1}{2}$$
[2]

Q. 12. Prove that the function f defined by

$$f(x) = \begin{cases} \frac{x}{|x| + 2x^2}, & \text{if } x \ne 0 \\ k, & \text{if } x = 0 \end{cases}$$

remains discontinuous at $x = 0$, regardless the choice of k.

[NCERT Exemp. Ex. 5.3, Q. 15, Page 108]

Ans. Given function is $f(x) = \begin{cases} \frac{x}{|x| + 2x^2}, & \text{if } x \ne 0 \\ k, & \text{if } x = 0 \end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x \to 0^-} f(x)$$
$$= \lim_{h \to 0} f(0-h)$$
$$= \lim_{h \to 0} f(-h)$$
$$= \lim_{h \to 0} \left[\frac{-h}{|-h| + 2(-h)^2} \right]$$
$$= \lim_{h \to 0} \left[\frac{-h}{h + 2h^2} \right]$$
$$= \lim_{h \to 0} \left[\frac{-1}{1 + 2h} \right]$$
$$= -1$$
[1½]

$$\text{RHL (at } x=0) = \lim_{x \to 0^+} f(x)$$
$$= \lim_{h \to 0} f(0+h)$$
$$= \lim_{h \to 0} f(h)$$
$$= \lim_{h \to 0} \left[\frac{h}{|h| + 2(h)^2} \right]$$
$$= \lim_{h \to 0} \left[\frac{h}{h + 2h^2} \right]$$
$$= \lim_{h \to 0} \left[\frac{1}{1 + 2h} \right]$$
$$= 1$$
[1½]

Since LHL $\ne$ RHL, then the given function is discontinuous at $x = 0$ regardless the choice of k.

Q. 13. Find the values of a and b such that the function f defined by $f(x)=\begin{cases}\frac{x-4}{|x-4|}+a, \text{if } x<4\\ a+b, \quad \text{if } x=4\\ \frac{x-4}{|x-4|}+b, \text{if } x>4\end{cases}$ is a continuous function at $x = 4$.

[NCERT Exemp. Ex. 5.3, Q. 16, Page 108]

Ans. Given that $f(x)=\begin{cases}\frac{x-4}{|x-4|}+a, \text{ if } x<4\\ a+b, \quad \text{ if } x=4\backslash\\ \frac{x-4}{|x-4|}+b, \text{ if } x>4\end{cases}$

$$\text{LHL (at } x=4)=\lim_{x\to 4^{-1}} f(x)$$
$$=\lim_{h\to 0} f(4-h)$$
$$=\lim_{h\to 0}\left[\frac{4-h-4}{|4-h-4|}+a\right]$$
$$=\lim_{h\to 0}\left[\frac{-h}{h}+a\right]$$
$$=[a-1]$$

[1½]

$$\text{RHL (at } x=4)=\lim_{x\to 4^{+}} f(x)$$
$$=\lim_{h\to 0} f(4+h)$$
$$=\lim_{h\to 0}\left[\frac{4+h-4}{|4+h-4|}+b\right]$$
$$=\lim_{h\to 0}\left[\frac{h}{h}+b\right]$$
$$=[b+1]$$

And, $f(4)=a+b$

Since the given function is continuous at $x = 4$, then LHL = RHL $= f(4)=a+b$.

Thus, $a-1=b+1=a+b \Rightarrow a=1$ and $b=-1$. **[1½]**

Q. 14. Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

[NCERT Exemp. Ex. 5.3, Q. 19, Page 109]

Ans. Given function is $f(x)=|\sin x+\cos x|$.

$$\text{LHL (at } x=\pi)=\lim_{x\to \pi^{-}} f(x)$$
$$=\lim_{h\to 0} f(\pi-h)$$
$$=\lim_{h\to 0}\left[\left|\sin(\pi-h)+\cos(\pi-h)\right|\right]$$
$$=\lim_{h\to 0}\left[\left|\sin h-\cos h\right|\right]$$
$$=|0-1|$$
$$=1$$

[1½]

$$\text{RHL (at } x=\pi)=\lim_{x\to \pi^{+}} f(x)$$
$$=\lim_{h\to 0} f(\pi+h)$$
$$=\lim_{h\to 0}\left[\left|\sin(\pi+h)+\cos(\pi+h)\right|\right]$$
$$=\lim_{h\to 0}\left[\left|-\sin h-\cos h\right|\right]$$
$$=|-0-1|$$
$$=1$$

And, $f(\pi)=|-1|=1$

Since LHL = RHL $= f(\pi)=1$, then the given function is continuous at $x = \pi$. **[1½]**

Q. 15. A function $f : R \to R$ satisfies the equation $f(x + y) = f(x) f(y)$ for all $x, y \in R$, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

[NCERT Exemp. Ex. 5.3, Q. 24, Page 109]

Ans. Since the given function is differentiable at $x = 0$, then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{x+h-x}$$
$$=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$$
$$=f(x)\left[\lim_{h\to 0}\frac{f(h)-1}{h}\right]$$

[1½]

Since

$$f(0+0)=f(0)f(0)$$
$$\Rightarrow \quad [f(0)]^2-f(0)=0$$
$$\Rightarrow \quad f(0)[f(0)-1]=0$$
$$\Rightarrow \quad f(0)=1$$

and $f'(0) = 2$, then

$$f'(x)=f(x)\left[\lim_{h\to 0}\frac{f(h)-f(0)}{h}\right]$$
$$f'(x)=f(x)f'(0)$$
$$\Rightarrow \quad f'(x)=2f(x)$$

[1½]

Q. 16. Find the values of p and q so that $f(x)=\begin{cases}x^2+3x+p, \text{ if } x\le 1\\ qx+2, \quad \text{ if } x>1\end{cases}$ is differentiable at $x = 1$.

[NCERT Exemp. Ex. 5.3, Q. 79, Page 112]

Ans. Given that $f(x)=\begin{cases}x^2+3x+p, \text{ if } x\le 1\\ qx+2, \quad \text{ if } x>1\end{cases}$ is differentiable at $x = 1$.

Thus, it is also a continuous function at $x = 1$.

$\Rightarrow \quad$ LHL = RHL $= f(1)$

$\Rightarrow \quad 1+3+p=q+2=1+3+p$

$\Rightarrow \quad p-q=-2$ …(i)

[1½]

Since $f'(x)=\begin{cases}2x+3, \text{ if } x<1\\ q, \quad \text{ if } x>1\end{cases}$ is differentiable at $x = 1$.

Thus,

$\Rightarrow$ LHL = RHL

$\Rightarrow 2+3=q$

$\Rightarrow \quad q=5$

From equation (i), $p = 3$

Thus, $p = 3$ and $q = 5$. **[1½]**

Long Answer Type Questions

(5 or 6 marks each)

Q. 1. Is the function $f(x)=\begin{cases}\dfrac{2x^2-3x-2}{x-2}, & \text{if } x\neq 2\\ 5, & \text{if } x=2\end{cases}$ **continuous or discontinuous at $x = 2$?**

[NCERT Exemp. Ex. 5.3, Q. 4, Page 107]

Ans. Given function is $f(x)=\begin{cases}\dfrac{2x^2-3x-2}{x-2}, & \text{if } x\neq 2\\ 5, & \text{if } x=2\end{cases}$.

$$\begin{aligned}\text{LHL (at } x=2) &= \lim_{x\to 2^-} f(x)\\ &= \lim_{h\to 0} f(2-h)\\ &= \lim_{h\to 0}\left[\frac{2(2-h)^2-3(2-h)-2}{(2-h)-2}\right]\\ &= \lim_{h\to 0}\left[\frac{8+2h^2-8h-6+3h-2}{-h}\right]\\ &= \lim_{h\to 0}\left[\frac{2h^2-5h}{-h}\right]\\ &= \lim_{h\to 0}\left[\frac{h(2h-5)}{-h}\right]\\ &= \lim_{h\to 0}[5-2h]\\ &= 5\end{aligned}$$

[2]

$$\begin{aligned}\text{RHL (at } x=2) &= \lim_{x\to 2^+} f(x)\\ &= \lim_{h\to 0} f(2+h)\\ &= \lim_{h\to 0}\left[\frac{2(2+h)^2-3(2+h)-2}{(2+h)-2}\right]\\ &= \lim_{h\to 0}\left[\frac{8+2h^2+8h-6-3h-2}{h}\right]\\ &= \lim_{h\to 0}\left[\frac{2h^2+5h}{h}\right]\\ &= \lim_{h\to 0}\left[\frac{h(2h+5)}{h}\right]\\ &= \lim_{h\to 0}[5+2h]\\ &= 5\end{aligned}$$

[2]

And, $f(2)=5$

Since LHL = RHL = $f(2)=5$, then the given function is continuous at $x = 2$. **[1]**

Q. 2. Find the value of k in

$$f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x\le 1\end{cases}$$

so that the function f is continuous at $x = 0$?

[NCERT Exemp. Ex. 5.3, Q. 13, Page 108]

Ans. Given function is

$$f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x\le 1\end{cases}.$$

[1]

$$\begin{aligned}\text{LHL(at } x=0) &= \lim_{x\to 0^-} f(x)\\ &= \lim_{h\to 0} f(0-h)\\ &= \lim_{h\to 0} f(-h)\\ &= \lim_{h\to 0}\left[\frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}\right]\\ &= \lim_{h\to 0}\left[\frac{\sqrt{1+kh}-\sqrt{1-kh}}{h}\times\frac{\sqrt{1+kh}+\sqrt{1-kh}}{\sqrt{1+kh}+\sqrt{1-kh}}\right]\\ &= \lim_{h\to 0}\left[\frac{1+kh-1+kh}{h\left(\sqrt{1+kh}+\sqrt{1-kh}\right)}\right]\\ &= \lim_{h\to 0}\left[\frac{2kh}{h\left(\sqrt{1+kh}+\sqrt{1-kh}\right)}\right]\\ &= \lim_{h\to 0}\left[\frac{2k}{\left(\sqrt{1+kh}+\sqrt{1-kh}\right)}\right]\\ &= \frac{2k}{2}\\ &= k\end{aligned}$$

[2]

$$\begin{aligned}\text{RHL (at } x=0) &= \lim_{x\to 0^+} f(x)\\ &= \lim_{h\to 0} f(0+h)\\ &= \lim_{h\to 0} f(h)\\ &= \lim_{h\to 0}\left[\frac{2h+1}{h-1}\right]\\ &= -1\end{aligned}$$

And, $f(0)=-1$

Since the given function is continuous at $x = 0$, then LHL = RHL = $f(0)=-1$.

LHL = RHL

$\Rightarrow \quad k=-1$ **[1]**

Q. 3. Find the value of k in

$$f(x)=\begin{cases}\dfrac{1-\cos kx}{x\sin x}, & \text{if } x\neq 0\\ \dfrac{1}{2}, & \text{if } x=0\end{cases}$$

so that the function f is continuous at $x = 0$?

[NCERT Exemp. Ex. 5.3, Q. 14, Page 108]

Ans. Given function is $f(x)=\begin{cases}\dfrac{1-\cos kx}{x\sin x}, & \text{if } x\neq 0\\ \dfrac{1}{2}, & \text{if } x=0\end{cases}$.

$$\text{LHL (at } x=0) = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0}\left[\frac{1-\cos k(-h)}{(-h)\sin(-h)}\right]$$
$$= \lim_{h\to 0}\left[\frac{1-\cos kh}{h\sin h}\right]$$
$$= \lim_{h\to 0}\left[\frac{2\sin^2\left(\frac{kh}{2}\right)}{h\sin h}\right]$$
$$= \lim_{h\to 0}\left[\frac{2\left(\frac{\sin\frac{kh}{2}}{\frac{kh}{2}}\right)^2\times\frac{(kh)^2}{4}}{h\sin h}\right]$$
$$= \lim_{h\to 0}\left[\frac{\frac{k^2}{2}\left(\frac{\sin\frac{kh}{2}}{\frac{kh}{2}}\right)^2}{\left(\frac{\sin h}{h}\right)}\right]$$
$$= \frac{k^2}{2}$$ **[3]**

And, $f(0)=\frac{1}{2}$

Since the given function is continuous at $x = 0$, then $\text{LHL} = \text{RHL} = f(0) = \frac{1}{2}$

$\text{LHL} = f(0)$

$\Rightarrow \quad \frac{k^2}{2}=\frac{1}{2}$

$\Rightarrow \quad k=\pm 1$ **[2]**

Q. 4. Examine the differentiability of *f*, where *f* is defined by

$$f(x)=\begin{cases} x[x], & \textbf{if } 0\le x<2\\ (x-1)x, & \textbf{if } 2\le x<3\end{cases} \textbf{ at x = 2.}$$

[NCERT Exemp. Ex. 5.3, Q. 20, Page 109]

Ans. Given function is $f(x)=\begin{cases} x[x], & \text{if } 0\le x<2\\ (x-1)x, & \text{if } 2\le x<3\end{cases}$

$$\text{LHD (at } x=2) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h\to 0}\frac{f(2-h)-f(2)}{2-h-2}$$
$$= \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$
$$= \lim_{h\to 0}\frac{(2-h)[2-h]-2}{-h}$$
$$= \lim_{h\to 0}\frac{(2-h)-2}{-h}$$
$$= \lim_{h\to 0}\left(\frac{-h}{-h}\right)$$
$$=1$$ **[2]**

$$\text{RHD (at } x=2) = \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h\to 0}\frac{f(2+h)-f(2)}{2+h-2}$$
$$= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{(2+h-1)(2+h)-2}{h}$$
$$= \lim_{h\to 0}\frac{h^2+3h}{h}$$
$$=3$$ **[2]**

Since $\text{LHD}\ne\text{RHD}$, then the given function is not differentiable at $x = 2$. **[1]**

Q. 5. Examine the differentiability of *f*, where *f* is defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \textbf{if } x\ne 0\\ 0, & \textbf{if } x=0\end{cases}$$

at $x = 0$. [NCERT Exemp. Ex. 5.3, Q. 21, Page 109]

Ans. Given function is $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\ne 0\\ 0, & \text{if } x=0\end{cases}$

$$\text{LHD (at } x=0) = \lim_{x\to 2^-}\frac{f(x)-f(0)}{x-0}$$
$$= \lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h}$$
$$= \lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$
$$= \lim_{h\to 0}\frac{(-h)^2\sin\frac{1}{(-h)}}{(-h)}$$
$$= \lim_{h\to 0}\left(h\sin\frac{1}{h}\right)$$
$$= 0\times\left[\begin{matrix}\text{An oscillating number}\\ \text{between } -1 \text{ and } 1\end{matrix}\right]$$
$$=0$$ **[2]**

$$\text{RHD (at } x=0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$
$$= \lim_{h\to 0} \frac{f(0+h)-f(0)}{0+h}$$
$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$
$$= \lim_{h\to 0} \frac{(-h)^2 \sin\frac{1}{(-h)}}{-h}$$
$$= \lim_{h\to 0} \left(h \sin\frac{1}{h}\right)$$
$$= 0 \times \begin{bmatrix}\text{An oscillating number} \\ \text{between } -1 \text{ and } 1\end{bmatrix}$$
$$= 0$$ **[2]**

Since LHD = RHD, then the given function is differentiable at $x = 0$. **[1]**

Q. 6. Examine the differentiability of *f*, where *f* is defined by

$$f(x) = \begin{cases} \mathbf{1+x}, & \textbf{if } x \le 2 \\ \mathbf{5-x}, & \textbf{if } x > 2 \end{cases}$$

at x = 2. [NCERT Exemp. Ex. 5.3, Q. 22, Page 109]

Ans. Given function is $f(x) = \begin{cases} 1+x, & \text{if } x \le 2 \\ 5-x, & \text{if } x > 2 \end{cases}$

$$\text{LHD (at } x=2) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h\to 0} \frac{f(2-h)-f(2)}{2-h-2}$$
$$= \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$$
$$= \lim_{h\to 0} \frac{1+(2-h)-1-2}{-h}$$
$$= \lim_{h\to 0} \left(\frac{-h}{-h}\right)$$
$$= 1$$ **[2]**

$$\text{RHD (at } x=2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h\to 0} \frac{f(2+h)-f(2)}{2+h-2}$$
$$= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0} \frac{5-(2+h)-3}{h}$$
$$= \lim_{h\to 0} \left(\frac{-h}{h}\right)$$
$$= -1$$ **[2]**

Since LHD ≠ RHD, then the given function is differentiable at $x = 2$. **[1]**

Q. 7. Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

[NCERT Exemp. Ex. 5.3, Q. 23, Page 109]

Ans. We know that the modulus function is continuous function for all real values. Thus, the given function $f(x) = |x - 5|$ is continuous at $x = 5$. **[1]**

Now, $f(x) = |x-5| = \begin{cases} x-5, & \text{if } x \ge 5 \\ 5-x, & \text{if } x < 5 \end{cases}$

$$\text{LHD (at } x=5) = \lim_{x\to 5^-} \frac{f(x)-f(5)}{x-5}$$
$$= \lim_{h\to 0} \frac{f(5-h)-f(5)}{5-h-5}$$
$$= \lim_{h\to 0} \frac{f(5-h)-f(5)}{-h}$$
$$= \lim_{h\to 0} \frac{5-(5-h)}{-h}$$
$$= \lim_{h\to 0} \left(\frac{h}{-h}\right)$$
$$= -1$$ **[2]**

$$\text{RHD (at } x=5) = \lim_{x\to 5^+} \frac{f(x)-f(5)}{x-5}$$
$$= \lim_{h\to 0} \frac{f(5+h)-f(5)}{5+h-5}$$
$$= \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$$
$$= \lim_{h\to 0} \frac{5+h-5}{h}$$
$$= \lim_{h\to 0} \left(\frac{h}{h}\right)$$
$$= 1$$ **[2]**

Since LHD ≠ RHD, then the given function is differentiable at $x = 5$.

TOPIC-2
Derivatives

Quick Review

❖ Algebra of derivatives : If u and v are the function of x, then

$$\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$$

$$\frac{d}{dx}(uv) = u\frac{d}{dx}(v) \pm v\frac{d}{dx}(u)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

❖ Derivatives of composite function : Let f be a real valued function which is a composite of two functions u and v; *i.e.*, $f = v \circ u$. Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, then $\frac{df}{dx} = \frac{dv}{dt}\frac{dt}{dx}$.

> **TRICKS...**
>
> *The differentiation of a function f(x) = x^n with respect to x is* $\frac{d}{dx}(x^n) = nx^{n-1}$.
>
> *The differentiation of a constant function f(x) = c with respect to x is* $\frac{d}{dx}(c) = 0$.

❖ Derivatives of implicit function : When a relationship between x and y is expressed in a way that it is easy to solve for y and write $y = f(x)$, we say that y is given as an explicit function of x. When a relationship between x and y is expressed in a way that it is not necessary that functions are always expressed in this form. It does not seem that there is an easy way to solve for y**.** There is no doubt about the dependence of y on x in either of the cases. In that case, differentiate the given function of x and y with respect to x and find the value of $\frac{dy}{dx}$. Hence, we get the derivative of implicit function.

❖ Derivatives of trigonometric function : Following are some of the standard derivatives (in appropriate domains) :

$\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\tan x) = \sec^2 x$,

$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$, $\frac{d}{dx}(\sec x) = \sec x \tan x$, $\frac{d}{dx}(\text{cosec} x) = -\text{cosec} x \cot x$

❖ Derivatives of inverse trigonometric function : Following are some of the standard derivatives (in appropriate domains) :

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$,

$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, $\frac{d}{dx}(\cos ec^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

❖ Exponential function : The exponential function with positive base $b > 1$ is the function $y = f(x) = b^x$. Its domain is R, the set of all real numbers and range is the set of all positive real numbers. Exponential function with base 10 is called the common exponential function and with base e is called the natural exponential function.

❖ Derivatives of exponential function : The derivative of $y = f(x) = e^x$ with respect to x is $\frac{dy}{dx} = e^x$.

❖ Logarithmic function : Let $b > 1$ be a real number. Then we say logarithm of a to base b is x if $b^x = a$, Logarithm of a to the base b is denoted by $\log_b a$. If the base $b = 10$, we say it is common logarithm and if $b = e$, then we say it is natural logarithms. $f(x) = \log x$ denotes the logarithm function to base e. The domain of logarithm function is R^+, the set of all positive real numbers and the range is the set of all real numbers.

❖ Logarithmic rules : The properties of logarithmic function to any base $b > 1$ are as below.

$\log_b(xy) = \log_b(x) + \log_b(y)$, $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$, $\log_b(x^n) = n\log_b(x)$,

$\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$, $\log_b(x) = \frac{1}{\log_x(b)}$, $\log\ (\ )$ and $\log_b(1) = 0$,

❖ Derivatives of logarithmic function : The derivative of $y = f(x) = \log x$ with respect to x is $\frac{dy}{dx} = \frac{1}{x}$.

- Derivative of function in parametric forms : If a relation expressed between two variables x and y in the form $x = f(t)$, $y = g(t)$ is said to be parametric form with t as a parameter. In order to find derivative of function in such form, we use $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left(\frac{dy}{dt}\right) \times \left(\frac{dt}{dx}\right)$.
- Second order of derivative : If a function $y = f(x)$ is differentiate with respect to x, then $\frac{dy}{dx} = f'(x)$ and if it is again differentiate with respect to x, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[f'(x)] \Rightarrow \frac{d^2y}{dx^2} = f''(x)$. It is known as second order of derivative of $f(x)$.

Multiple Choice Questions (1 mark each)

Q. 1. If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{4x^3}{1-x^4}$ **(b)** $\frac{-4x}{1-x^4}$

(c) $\frac{1}{4-x^4}$ **(d)** $\frac{-4x^3}{1-x^4}$

[NCERT Exemp. Ex. 5.3, Q. 91, Page 114-115]

Ans. Correct option : (b)

Explanation : Given that,

$$y = \log\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow y = \log(1-x^2) - \log(1+x^2).$$

Differentiate with respect to x, we have

$$\frac{dy}{dx} = \frac{d}{dx}[\log(1-x^2)] - \frac{d}{dx}[\log(1+x^2)]$$

$$= \frac{-2x}{1-x^2} - \frac{2x}{1+x^2}$$

$$= -2x\left(\frac{2}{(1-x^2)(1+x^2)}\right)$$

$$= -\frac{4x}{1-x^4}$$

Q. 2. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\cos x}{2y-1}$ **(b)** $\frac{\cos x}{1-2y}$

(c) $\frac{\sin x}{1-2y}$ **(d)** $\frac{\sin x}{2y-1}$

[NCERT Exemp. Ex. 5.3, Q. 92, Page 115]

Ans. Correct option : (a)

Explanation : Given that,

$$y = \sqrt{\sin x + y}$$

$$\Rightarrow y^2 = \sin x + y.$$

Differentiate with respect to x, we have

$$\Rightarrow 2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow (2y-1)\frac{dy}{dx} = \cos x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

Q. 3. The derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\cos^{-1}x$ is

(a) 2 **(b)** $\frac{-1}{2\sqrt{1-x^2}}$

(c) $\frac{2}{x}$ **(d)** $1-x^2$

[NCERT Exemp. Ex. 5.3, Q. 93, Page 115]

Ans. Correct option : (a)

Explanation : Let

$$u = \cos^{-1}(2x^2-1)$$

$$\Rightarrow \frac{du}{dx} = -\frac{4x}{\sqrt{1-(2x^2-1)^2}}$$

$$\Rightarrow \frac{du}{dx} = -\frac{4x}{\sqrt{1-4x^4+4x^2-1}}$$

$$\Rightarrow \frac{du}{dx} = -\frac{4x}{\sqrt{-4x^4+4x^2}}$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

And, $v = \cos^{-1}x$

$$\Rightarrow \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Thus, $\frac{du}{dv} = 2$

Q. 4. If $x = t^2$ and $y = t^3$ then $\frac{d^2y}{dx^2}$ is

(a) $\frac{3}{2}$ **(b)** $\frac{3}{4t}$

(c) $\frac{3}{2t}$ **(d)** $\frac{3}{4}$

[NCERT Exemp. Ex. 5.3, Q. 94, Page 115]

Ans. Correct option : (b)

Explanation : Given that,

$x = t^2$ and $y = t^3$

Then, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$

Thus,

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{3}{2}$$

Q. 5. Fill in the blanks :

Derivative of x^2 with respect to x^3 is __________

[NCERT Exemp. Ex. 5.3, Q. 98, Page 116]

Ans. $\frac{2}{3x}$

Since $u = x^2 \Rightarrow \frac{du}{dx} = 2x$ and $v = x^3 \Rightarrow \frac{dv}{dx} = 3x^2$, then

$$\frac{du}{dv} = \frac{2x}{3x^2} = \frac{2}{3x}.$$

Q. 6. Fill in the blanks :

If $f(x) = |\cos x|$ then $f'\left(\frac{\pi}{4}\right) =$ _________

[NCERT Exemp. Ex. 5.3, Q. 99, Page 116]

Ans. $-\frac{1}{\sqrt{2}}$

$$f(x) = |\cos x| = \cos x \text{ for } 0 < x < \frac{\pi}{2}$$

Thus,

$$f'(x) = \frac{d}{dx}(\cos x) = -\sin x$$

$$\Rightarrow f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

Q. 7. Fill in the blanks :

If $f(x) = |\cos x - \sin x|$ then $f'\left(\frac{\pi}{3}\right) =$ __________

[NCERT Exemp. Ex. 5.3, Q. 100, Page 116]

Ans. $\frac{\sqrt{3}+1}{2}$

$$f(x) = |\cos x - \sin x| = -(\cos x - \sin x) \text{ for } \frac{\pi}{4} < x < \frac{\pi}{2}$$

Thus,

$$f'(x) = \cos x + \sin x$$

$$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} + \sin\frac{\pi}{3}$$

$$= \frac{\sqrt{3}+1}{2}$$

Q. 8. Fill in the blanks :

For the curve $\sqrt{x} + \sqrt{y} = 1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is ________.

[NCERT Exemp. Ex. 5.3, Q. 101, Page 116]

Ans. –1

Given that,

$\sqrt{x} + \sqrt{y} = 1$

Differentiable with respect to x, we have

$$\sqrt{x} + \sqrt{y} = 1$$

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4},\frac{1}{4}\right)} = -\sqrt{\frac{\frac{1}{4}}{\frac{1}{4}}} = -1$$

Very Short Answer Type Questions **(1 or 2 marks each)**

Q. 1. Differentiate the function $2^{\cos^2 x}$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 25, Page 109]

Ans. Given that $y = 2^{\cos^2 x}$

Taking log both sides, we have

$$\log y = \cos^2 x \log 2$$

$$= \log 2 \cos^2 x$$ **[1]**

Differentiating both sides, we have

$$\frac{d}{dx}(\log y) = \log 2 \frac{d}{dx}(\cos^2 x)$$

$$\frac{1}{y}\frac{dy}{dx} = \log 2(-2\cos x \sin x)$$

$$\frac{dy}{dx} = -2^{\cos^2 x}(\sin 2x)\log 2$$ **[1]**

Q. 2. Differentiate the function $\frac{8^x}{x^8}$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 26, Page 109]

Ans. Given that $y = \frac{8^x}{x^8}$

Taking log both sides, we have

$$\log y = \log 8^x - \log x^8$$

$$= x\log 8 - 8\log x$$ **[1]**

Differentiating both sides, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x\log 8 - 8\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \log 8 - \frac{8}{x}$$

$$\frac{dy}{dx} = y\left(\log 8 - \frac{8}{x}\right)$$

$$= \frac{8^x}{x^8}\left(\log 8 - \frac{8}{x}\right)$$ **[1]**

Q. 3. Differentiate the function $\log\left(x + \sqrt{x^2 + a}\right)$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 27, Page 109]

Ans. Given that $y=\log\left(x+\sqrt{x^2+a}\right)$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(x+\sqrt{x^2+a}\right)\right]$$
$$=\frac{1}{\left(x+\sqrt{x^2+a}\right)}\times\left(1+\frac{2x}{2\sqrt{x^2+a}}\right)$$
$$=\frac{1}{\left(x+\sqrt{x^2+a}\right)}\times\left(\frac{x+\sqrt{x^2+a}}{\sqrt{x^2+a}}\right)$$
$$=\frac{1}{\sqrt{x^2+a}}$$

[2]

Q. 4. Differentiate the function $\log\left[\log\left(\log x^5\right)\right]$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 28, Page 109]

Ans. Given that,

$$y = \log [\log (\log x^5)\}]$$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left\{\log\left(\log x^5\right)\right\}\right]$$
$$=\frac{1}{\log\left(\log x^5\right)}\times\frac{1}{\log x^5}\times\frac{1}{x^5}\times 5x^4$$
$$=\frac{5}{x\left(\log x^5\right)\log\left(\log x^5\right)}$$

[2]

Q. 5. Differentiate the function $\sin\sqrt{x}+\cos^2\sqrt{x}$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 29, Page 109]

Ans. Given that,

$$y=\sin\sqrt{x}+\cos^2\sqrt{x}$$
$$=\sin\sqrt{x}+\left(\cos\sqrt{x}\right)^2$$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin\sqrt{x}+\left(\cos\sqrt{x}\right)^2\right]$$
$$=\frac{d}{dx}\left(\sin\sqrt{x}\right)+\frac{d}{dx}\left[\left(\cos\sqrt{x}\right)^2\right]$$
$$=\frac{\cos\sqrt{x}}{2\sqrt{x}}+\left[2\cos\sqrt{x}\left(-\frac{\sin\sqrt{x}}{2\sqrt{x}}\right)\right]$$
$$=\frac{1}{2\sqrt{x}}\left(\cos\sqrt{x}-\sin 2\sqrt{x}\right)$$

[2]

Q. 6. Differentiate the function $\sin^n\left(ax^2+bx+c\right)$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 30, Page 109]

Ans. Given that,

$$y=\sin^n\left(ax^2+bx+c\right)$$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin\left(ax^2+bx+c\right)\right]^n$$
$$=n\left[\sin\left(ax^2+bx+c\right)\right]^{n-1}\cos\left(ax^2+bx+c\right)\left(2ax+b\right)$$
$$=n\left(2ax+b\right)\cos\left(ax^2+bx+c\right)\left[\sin\left(ax^2+bx+c\right)\right]^{n-1}$$

[2]

Q. 7. Differentiate the function $\cos\left(\tan\sqrt{x+1}\right)$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 31, Page 109]

Ans. Given that $y=\cos\left(\tan\sqrt{x+1}\right)$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\cos\left(\tan\sqrt{x+1}\right)\right]$$
$$=\left[-\sin\left(\tan\sqrt{x+1}\right)\right]\left(\sec^2\sqrt{x+1}\right)\left(\frac{1}{2\sqrt{x+1}}\right)$$
$$=-\frac{1}{2\sqrt{x+1}}\sin\left(\tan\sqrt{x+1}\right)\sec^2\sqrt{x+1}$$

[2]

Q. 8. Differentiate the function $\sin x^2+\sin^2 x+\sin^2\left(x^2\right)$ w.r.t. x. **[NCERT Exemp. Ex. 5.3, Q. 32, Page 109]**

Ans. Given that $y=\sin x^2+\sin^2 x+\sin^2\left(x^2\right)$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin x^2+\sin^2 x+\sin^2\left(x^2\right)\right]$$
$$=\frac{d}{dx}\left(\sin x^2\right)+\frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\left[\sin^2\left(x^2\right)\right]$$
$$=2x\cos x^2+2\sin x\cos x+2\sin\left(x^2\right)\cos\left(x^2\right)\times(2x)$$
$$=2x\cos x^2+\sin 2x+2x\sin\left(2x^2\right)$$

[2]

Q. 9. Differentiate the function $\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 33, Page 109]

Ans. Given that $y=\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)$ Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}\left(\frac{1}{\sqrt{x+1}}\right)\right]$$
$$=\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x+1}}\right)^2}}\times\frac{d}{dx}\left[(x+1)^{-\frac{1}{2}}\right]$$
$$=\frac{1}{\sqrt{\frac{x+1-1}{x+1}}}\times\left[-\frac{1}{2}(x+1)^{-\frac{3}{2}}\right]$$
$$=\sqrt{\frac{x+1}{x}}\left[-\frac{1}{2(x+1)\sqrt{x+1}}\right]$$
$$=-\frac{1}{2\sqrt{x}(x+1)}$$

[2]

Q. 10. Differentiate the function $\sin^m x\cos^n x$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 35, Page 109]

Ans. Given that $y=\sin^m x\cos^n x$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin^m x\cos^n x\right)$$
$$=\sin^m x\frac{d}{dx}\left(\cos^n x\right)+\cos^n x\frac{d}{dx}\left(\sin^m x\right)$$
$$=\sin^m x\left(-n\cos^{n-1}x\sin x\right)+\cos^n x\left(m\sin^{m-1}x\cos x\right)$$
$$=\sin^m x\cos^n x\left(m\cot x-n\tan x\right)$$

[2]

Q. 11. Differentiate the function $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$, $-\frac{\pi}{4}<x<\frac{\pi}{4}$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 38, Page 110]

Ans. Given that,

$$y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$ **[1]**

Differentiating both sides, we have

$$\frac{dy}{dx} = \frac{1}{2}$$ **[1]**

Q. 12. Find $\frac{dy}{dx}$ of the function expressed in parametric form $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$.

[NCERT Exemp. Ex. 5.3, Q. 44, Page 110]

Ans. Given that $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$

$$x = t + \frac{1}{t}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(t + \frac{1}{t}\right) = 1 - \frac{1}{t^2}$$

$$y = t - \frac{1}{t}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(t - \frac{1}{t}\right) = 1 + \frac{1}{t^2}$$ **[1]**

Thus,

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$= \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}}$$

$$= \frac{t^2+1}{t^2-1}$$ **[1]**

Q. 13. Find $\frac{dy}{dx}$ of the function expressed in parametric form $\sin x = \frac{2t}{1+t^2}$, $\tan y = \frac{2t}{1-t^2}$.

[NCERT Exemp. Ex. 5.3, Q. 47, Page 110]

Ans. Given that $\sin x = \frac{2t}{1+t^2}$, $\tan y = \frac{2t}{1-t^2}$

Put $t = \tan\theta$ in $\sin x = \frac{2t}{1+t^2}$, we have

$$\sin x = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\Rightarrow \sin x = \sin 2\theta$$ **[1]**

$$\Rightarrow \quad x = 2\theta$$

$$\Rightarrow \quad \frac{dx}{d\theta} = 2$$

Put $t = \tan\theta$ in $\tan y = \frac{2t}{1-t^2}$, we have

$$\tan y = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\Rightarrow \tan y = \tan 2\theta$$

$$\Rightarrow \quad y = 2\theta$$

$$\Rightarrow \quad \frac{dy}{d\theta} = 2$$ **[1]**

Thus, $\frac{dy}{dx} = 1$

Q. 14. Find $\frac{dy}{dx}$ when x and y are connected by the relation $\tan^{-1}(x^2+y^2) = a$.

[NCERT Exemp. Ex. 5.3, Q. 56, Page 111]

Ans. Given that,

$$\tan^{-1}(x^2+y^2) = a$$

$$\Rightarrow \quad x^2 + y^2 = \tan a$$

Differentiating with respect to x, we have

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(\tan a)$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$ **[2]**

Q. 15. Find $\frac{dy}{dx}$ when x and y are connected by the relation $(x^2+y^2)^2 = xy$.

[NCERT Exemp. Ex. 5.3, Q. 57, Page 111]

Ans. Given that $(x^2+y^2)^2 = xy$

Differentiating with respect to x, we have

$$\frac{d}{dx}(x^2+y^2)^2 = \frac{d}{dx}(xy)$$

$$\Rightarrow 2(x^2+y^2)\left(2x + 2y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow 4x(x^2+y^2) + 4y(x^2+y^2)\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \left[4y(x^2+y^2) - x\right]\frac{dy}{dx} = y - 4x(x^2+y^2)$$

$$\Rightarrow (4x^2y + 4y^3 - x)\frac{dy}{dx} = y - 4x^3 - 4xy^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{y - 4x^3 - 4xy^2}{4x^2y + 4y^3 - x}$$ **[2]**

Short Answer Type Questions

(3 or 4 marks each)

Q. 1. Differentiate the function $(\sin x)\cos^x$ w.r.t. x.

Ans. Given that $y = (\sin x)^{\cos x}$

Taking log both sides, we have

$$\log y = \log(\sin x)^{\cos x}$$

$$= \cos x(\log \sin x)$$ **[1]**

Differentiating both sides, we have

$$\frac{d}{dx}(\log y)=\frac{d}{dx}\left[\cos x(\log\sin x)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\cos x\frac{d}{dx}\left[(\log\sin x)\right]+\log\sin x\frac{d}{dx}(\cos x)$$

$$\frac{dy}{dx}=y\left[\cos x\frac{d}{dx}\{(\log\sin x)\}+\log\sin x\frac{d}{dx}(\cos x)\right]$$

$$=(\sin x)^{\cos x}\left(\cos x\times\frac{\cos x}{\sin x}-\sin x\log\sin x\right)$$

$$=(\sin x)^{\cos x}\left(\cos x\cot x-\sin x\log\sin x\right)$$ **[2]**

Q. 2. Differentiate the function $(x+1)^2(x+2)^3(x+3)^4$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 36, Page 109]

Ans. Given that $y=(x+1)^2(x+2)^3(x+3)^4$

Taking log both sides, we have

$$\log y=\log\left[(x+1)^2(x+2)^3(x+3)^4\right]$$

$$=\log\left[(x+1)^2\right]+\log\left[(x+2)^3\right]+\log\left[(x+3)^4\right]$$

$$=2\log(x+1)+3\log(x+2)+4\log(x+3)$$ **[1]**

Differentiating both sides, we have

$$\frac{d}{dx}(\log y)=\frac{d}{dx}\begin{bmatrix}2\log(x+1)+3\log(x+2)\\+4\log(x+3)\end{bmatrix}$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2}{x+1}+\frac{3}{x+2}+\frac{4}{x+3}$$

$$\frac{dy}{dx}=y\left(\frac{2}{x+1}+\frac{3}{x+2}+\frac{4}{x+3}\right)$$

$$=\begin{bmatrix}(x+1)^2(x+2)^3(x+3)^4\\\left(\frac{2}{x+1}+\frac{3}{x+2}+\frac{4}{x+3}\right)\end{bmatrix}$$

$$=(x+1)(x+2)^2(x+3)^3(9x^2+34x+29)$$ **[2]**

Q. 3. Differentiate the function

$\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right),-\frac{\pi}{4}<x<\frac{\pi}{4}$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 37, Page 110]

Ans. Given that,

$$y=\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$=\cos^{-1}\left(\sin x\left(\frac{1}{\sqrt{2}}\right)+\cos x\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$=\cos^{-1}\left(\sin x\sin\frac{\pi}{4}+\cos x\cos\frac{\pi}{4}\right)$$

$$=\cos^{-1}\left(\cos\left(\frac{\pi}{4}-x\right)\right)$$

$$=\frac{\pi}{4}-x$$ **[2]**

Differentiating both sides, we have

$$\frac{dy}{dx}=-1$$ **[1]**

Q. 4. Differentiate the function

$\tan^{-1}(\sec x+\tan x),-\frac{\pi}{2}<x<\frac{\pi}{2}$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 39, Page 110]

Ans. Given that,

$$y=\tan^{-1}(\sec x+\tan x)$$

Differentiating both sides, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left[\tan\quad(\sec x+\tan x)\right]$$

$$=\frac{}{+(\sec x+\tan x)}\times(\sec x\tan x+\sec\ x)$$

$$=\frac{}{1+\sec^2 x+\tan^2 x+2\sec x\tan x}\times\sec x(\tan x+\sec x)$$

$$=\frac{}{\sec^2 x+\sec^2 x+\quad\sec x\tan x}\times\sec x(\tan x+\sec x)$$

$$=\frac{}{2\sec\ x\quad 2\sec x\tan x}\times\sec x(\tan x+\sec x)$$

$$=\frac{1}{2\sec x(\sec x+\tan x)}\times\sec x(\tan x+\sec x)$$

$$=\frac{1}{2}$$

[3]

Q. 5. Differentiate the function

$\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right),-\frac{\pi}{2}<x<\frac{\pi}{2}$ and $\frac{a}{b}\tan x>-1$

w.r.t. x. [NCERT Exemp. Ex. 5.3, Q. 40, Page 110]

Ans. Given that,

$$y=\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$$

$$=\tan^{-1}\left(\frac{\frac{a\cos x}{b\cos x}-\frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x}+\frac{a\sin x}{b\cos x}}\right)$$

$$=\tan^{-1}\left(\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right)$$

$$=\tan^{-1}\frac{a}{b}-\tan^{-1}(\tan x)$$

$$=\tan^{-1}\frac{a}{b}-x$$ **[2]**

Differentiating both sides, we have

$$\frac{dy}{dx}=-1$$ **[1]**

Q. 6. Differentiate the function

$\sec^{-1}\left(\frac{1}{4x^3-3x}\right),0<x<\frac{1}{\sqrt{2}}$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 41, Page 110]

Ans. Given that,

$$y=\sec^{-1}\left(\frac{1}{4x^3-3x}\right),\ 0<x<\frac{1}{\sqrt{2}}$$

Put $x=\cos\theta\Rightarrow\theta=\cos^{-1}x$, then

$$y = \sec^{-1}\left(\frac{1}{4\cos^3\theta - 3\cos\theta}\right)$$
$$= \sec^{-1}\left(\frac{1}{\cos 3\theta}\right)$$
$$= \sec^{-1}(\sec 3\theta)$$
$$= 3\theta$$
$$= 3\cos^{-1}x$$ **[2]**

Differentiating both sides, we have

$$\frac{dy}{dx} = 3\frac{d}{dx}\left(\cos^{-1}x\right)$$
$$= -\frac{3}{\sqrt{1-x^2}}$$ **[1]**

Q. 7. Differentiate the function

$\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right), -\frac{1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 42, Page 110]

Ans. Given that,

$$y = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right), -\frac{1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$$
$$y = \tan^{-1}\left(\frac{3\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2}\right)$$ **[1]**

Put $\frac{x}{a} = \tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$, then

$$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$
$$= \tan^{-1}(\tan 3\theta)$$
$$= 3\theta$$
$$= 3\tan^{-1}\left(\frac{x}{a}\right)$$ **[1]**

Differentiating both sides, we have

$$\frac{dy}{dx} = 3\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]$$
$$= 3 \times \frac{1}{1 + \left(\frac{x}{a}\right)^2}$$
$$= \frac{3a}{a^2 + x^2}$$ **[1]**

Q. 8. Differentiate the function

$\tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right), -1 < x < 1, x \neq 0$ w.r.t. x.

[NCERT Exemp. Ex. 5.3, Q. 43, Page 110]

Ans. Given that,

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right), -1 < x < 1, x \neq 0$$

Put $x^2 = \cos 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1}x^2$, then

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}\right)$$
$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$
$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$
$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right]$$
$$= \frac{\pi}{4} + \theta$$
$$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$$ **[2]**

Differentiating both sides, we have

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2\right)$$
$$= 0 + \frac{1}{2}\left(-\frac{2x}{\sqrt{1-x^4}}\right)$$
$$= -\frac{x}{\sqrt{1-x^4}}$$ **[1]**

Q. 9. Find $\frac{dy}{dx}$ of the function expressed in parametric form $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$.

[NCERT Exemp. Ex. 5.3, Q. 46, Page 110]

Ans. Given that,

$$x = 3\cos\theta - 2\cos^3\theta,\ y = 3\sin\theta - 2\sin^3\theta.$$
$$x = 3\cos\theta - 2\cos^3\theta$$

Differentiating both sides, we have

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(3\cos\theta - 2\cos^3\theta\right)$$
$$= -3\sin\theta + 6\cos^2\theta\sin\theta$$ **[1]**

And,

$$y = 3\sin\theta - 2\sin^3\theta$$

Differentiating both sides, we have

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(3\sin\theta - 2\sin^3\theta\right)$$
$$= 3\cos\theta - 6\sin^2\theta\cos\theta$$ **[1]**

Thus,

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)$$
$$= \frac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta + 6\cos^2\theta\sin\theta}$$
$$= \frac{3\cos\theta\left(1 - 2\sin^2\theta\right)}{3\sin\theta\left(-1 + 2\cos^2\theta\right)}$$
$$= \cot\theta \times \frac{\cos 2\theta}{\cos 2\theta}$$
$$= \cot\theta$$ **[1]**

Q. 10. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$.

[NCERT Exemp. Ex. 5.3, Q. 49, Page 110]

Ans. Given that $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

Since

$$x = e^{\cos 2t}$$

$$\log x = \cos 2t$$

$$(\log x)^2 = \cos^2 2t$$

$$y = e^{\sin 2t}$$

$$\log y = \sin 2t$$

$$(\log y)^2 = \sin^2 2t$$ **[1]**

Then

$$(\log x)^2 + (\log y)^2 = \sin^2 2t + \cos^2 2t$$

$$= 1$$ **[1]**

Thus,

$$\Rightarrow \frac{2(\log x)}{x} + \frac{2(\log y)}{y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y \log x}{x \log y}$$ **[1]**

Q. 11. If $x = 3\sin t - \sin 3t$ and $y = 3\cos t - \cos 3t$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.

[NCERT Exemp. Ex. 5.3, Q. 51, Page 110]

Ans. Given that,

$x = 3\sin t - \sin 3t$ and $y = 3\cos t - \cos 3t$

$$x = 3\sin t - \sin 3t$$

Differentiating both sides, we have

$$\frac{dx}{dt} = 3\cos t - 3\cos 3t$$

And,

$$y = 3\cos t - \cos 3t$$

Differentiating both sides, we have

$$\frac{dy}{dt} = -3\sin t + 3\sin 3t$$ **[1]**

Thus,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{-3\sin t + 3\sin 3t}{3\cos t - 3\cos 3t}$$

$$= \frac{-\sin t + \sin 3t}{\cos t - \cos 3t}$$ **[1]**

Therefore, $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$ is

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{3}} = \frac{-\sin\frac{\pi}{3} + \sin \pi}{\cos\frac{\pi}{3} - \cos \pi}$$

$$= \frac{-\frac{\sqrt{3}}{2} + 0}{\frac{1}{2} + 1}$$

$$= -\frac{1}{\sqrt{3}}$$ **[1]**

Q. 12. Differentiate $\frac{x}{\sin x}$ with respect to $\sin x$.

[NCERT Exemp. Ex. 5.3, Q. 52, Page 111]

Ans. Let $u = \frac{x}{\sin x}$ and $v = \sin x$.

Then,

$$u = \frac{x}{\sin x}$$

Differentiating both sides, we have

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{x}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}(x) - x\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x}$$ **[1]**

And,

$$v = \sin x$$

Differentiating both sides, we have

$$\frac{dv}{dx} = \frac{d}{dx}(\sin x)$$

$$= \cos x$$ **[1]**

Thus,

$$\frac{du}{dv} = \frac{\frac{\sin x - x\cos x}{\sin^2 x}}{\cos x}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x \cos x}$$

$$= \frac{\tan x - x}{\sin^2 x}$$ **[1]**

Q. 13. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}x$ when $x \neq 0$.

[NCERT Exemp. Ex. 5.3, Q. 53, Page 111]

Ans. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1} x$.

Put $x = \tan\theta$ in $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, we have

$$u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\tan^{-1} x$$ **[2]**

Thus, $\frac{du}{dx}=\frac{1}{2(1+x^2)}$

And,

$v=\tan^{-1}x\Rightarrow\frac{dv}{dx}=\frac{1}{1+x^2}$

Thus,

$$\frac{du}{dv}=\frac{\frac{1}{2(1+x^2)}}{\frac{1}{(1+x^2)}}=\frac{1}{2}$$

[1]

Q. 14. Find $\frac{dy}{dx}$ when x and y are connected by the relation $\sin(xy)+\frac{x}{y}=x^2-y$.

[NCERT Exemp. Ex. 5.3, Q. 54, Page 111]

Ans. Given that $\sin(xy)+\frac{x}{y}=x^2-y$.

Differentiating with respect to x, we have

$$\frac{d}{dx}\left[\sin(xy)+\frac{x}{y}\right]=\frac{d}{dx}(x^2-y)$$

$$\Rightarrow\frac{d}{dx}[\sin(xy)]+\frac{d}{dx}\left(\frac{x}{y}\right)=\frac{d}{dx}(x^2)-\frac{d}{dx}(y)$$

$$\Rightarrow\cos(xy)\left(x\frac{dy}{dx}+y\right)+\left(\frac{y-x\frac{dy}{dx}}{y^2}\right)=2x-\frac{dy}{dx}$$

$$\Rightarrow\cos(xy)\left(xy^2\frac{dy}{dx}+y^3\right)+y-x\frac{dy}{dx}=2xy^2-\frac{dy}{dx}y^2$$

$$\Rightarrow xy^2\cos(xy)\frac{dy}{dx}+y^3\cos(xy)+y-x\frac{dy}{dx}=2xy^2-y^2\frac{dy}{dx}$$

$$\Rightarrow xy^2\cos(xy)\frac{dy}{dx}-x\frac{dy}{dx}+y^2\frac{dy}{dx}=2xy^2-y^3\cos(xy)-y$$

$$\Rightarrow\frac{dy}{dx}\left[xy^2\cos(xy)-x+y^2\right]=2xy^2-y^3\cos(xy)-y$$

$$\Rightarrow\frac{dy}{dx}=\frac{2xy^2-y^3\cos(xy)-y}{xy^2\cos(xy)-x+y^2}$$

[3]

Q. 15. Find $\frac{dy}{dx}$ when x and y are connected by the relation $\sec(x+y)=xy$.

[NCERT Exemp. Ex. 5.3, Q. 55, Page 111]

Ans. Given that, $\sec(x+y)=xy$.

Differentiating with respect to x, we have

$$\frac{d}{dx}[\sec(x+y)]=\frac{d}{dx}(xy)$$

$$\Rightarrow\sec(x+y)\tan(x+y)\left(1+\frac{dy}{dx}\right)=x\frac{dy}{dx}+y$$

$$\Rightarrow\sec(x+y)\tan(x+y)+\sec(x+y)\tan(x+y)\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$\Rightarrow\sec(x+y)\tan(x+y)\frac{dy}{dx}-x\frac{dy}{dx}=y-\sec(x+y)\tan(x+y)$$

$$\Rightarrow\left[\sec(x+y)\tan(x+y)-x\right]\frac{dy}{dx}=y-\sec(x+y)\tan(x+y)$$

$$\Rightarrow\left[\sec(x+y)\tan(x+y)-x\right]\frac{dy}{dx}=y-\sec(x+y)\tan(x+y)$$

$$\therefore\quad\frac{dy}{dx}=\frac{y-\sec(x+y)\tan(x+y)}{\sec(x+y)\tan(x+y)-x}$$

[3]

Q. 16. If $ax^2+2hxy+by^2+2gx+2fy+c=0$, then show that $\frac{dy}{dx}\times\frac{dx}{dy}=1$.

[NCERT Exemp. Ex. 5.3, Q. 58, Page 111]

Ans. Given that, $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Differentiating with respect to x, we have

$$2ax+2h\left(x\frac{dy}{dx}+y\right)+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}+0=0$$

$$\Rightarrow 2ax+2hx\frac{dy}{dx}+2hy+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$\Rightarrow(2hx+2by+2f)\frac{dy}{dx}=-(2ax+2hy+2g)$$

$$\Rightarrow\frac{dy}{dx}=-\frac{2ax+2hy+2g}{2hx+2by+2f}=-\frac{ax+hy+g}{hx+by+f}$$

$$\Rightarrow\frac{dx}{dy}=-\frac{hx+by+f}{ax+hy+g}$$

[2]

Thus,

$$\Rightarrow\frac{dy}{dx}\times\frac{dx}{dy}=\left(-\frac{ax+hy+g}{hx+by+f}\right)\left(-\frac{hx+by+f}{ax+hy+g}\right)=1$$

[1]

Q. 17. If $x=e^{\frac{x}{y}}$, then prove that $\frac{dy}{dx}=\frac{x-y}{x\log x}$.

[NCERT Exemp. Ex. 5.3, Q. 59, Page 111]

Ans. Given that, $x=e^{\frac{x}{y}}$.

Differentiating with respect to x, we have

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(e^{\frac{x}{y}}\right)$$

$$1=e^{\frac{x}{y}}\left[\frac{y\frac{d}{dx}(x)-x\frac{dy}{dx}}{y^2}\right]$$

$$y^2=e^{\frac{x}{y}}\left(y-x\frac{dy}{dx}\right)$$

$$=ye^{\frac{x}{y}}-xe^{\frac{x}{y}}\frac{dy}{dx}$$

$$xe^{\frac{x}{y}}\frac{dy}{dx}=ye^{\frac{x}{y}}-y^2$$

$$\frac{dy}{dx}=\frac{y\left(e^{\frac{x}{y}}-y\right)}{xe^{\frac{x}{y}}}$$

$$=\frac{\left(e^{\frac{x}{y}}-y\right)}{\frac{x}{y}e^{\frac{x}{y}}}$$

$$=\frac{x-y}{x\log x}$$

[3]

Q. 18. If $y^x = e^{y-x}$, then prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

[NCERT Exemp. Ex. 5.3, Q. 60, Page 111]

Ans. Given that, $y^x = e^{y-x}$

Taking log both sides, we have

$$x\log y = y - x$$

$$x(1 + \log y) = y$$

$$x = \frac{y}{1+\log y}$$

Differentiating w.r.t. y both sides, we have

$$\frac{dy}{dx} = \frac{(1+\log y)\frac{d}{dy}(y) - y\frac{d}{dy}(1+\log y)}{(1+\log y)^2}$$

$$= \frac{(1+\log y) - y.\frac{1}{y}}{(1+\log y)^2}$$

$$\frac{dy}{dx} = \frac{\log y}{(1+\log y)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$$ **[2]**

Q. 19. If $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$, then show that $\frac{dy}{dx} = \frac{y^2 \tan x}{y \log \cos x - 1}$.

[NCERT Exemp. Ex. 5.3, Q. 61, Page 111]

Ans. Given that,

$$y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$$

$$\Rightarrow y = (\cos x)^y$$

Taking log both sides, we have

$$\log y = y \log\cos x$$ **[1]**

Differentiating with respect to x, we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(y\log\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = y\frac{d}{dx}(\log\cos x) + \log\cos x\frac{dy}{dx}$$

$$\left(\frac{1}{y} - \log\cos x\right)\frac{dy}{dx} = y\left(\frac{-\sin x}{\cos x}\right)$$

$$\left(\frac{1-y\log\cos x}{y}\right)\frac{dy}{dx} = -y\tan x$$

$$\frac{dy}{dx} = -\frac{y^2\tan x}{1-y\log\cos x}$$

$$= \frac{y^2\tan x}{y\log\cos x - 1}$$ **[2]**

Q. 20. If $x\sin(a+y) + \sin a\cos(a+y) = 0$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

[NCERT Exemp. Ex. 5.3, Q. 62, Page 111]

Ans. Given that,

$$x\sin(a+y) + \sin a\cos(a+y) = 0$$

$$x = \frac{-\sin a\cos(a+y)}{\sin(a+y)}$$

$$= -\sin a\cot(a+y)$$ **[1]**

Differentiating with respect to y, we have

$$\frac{dx}{dy} = -\sin a\frac{d}{dy}\left[\cot(a+y)\right]$$

$$= -\sin a\left[-\mathrm{cosec}^2(a+y)\right]$$

$$\frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$ **[2]**

Q. 21. If $y = \tan^{-1}x$, then $\frac{d^2y}{dx^2}$ in terms of y alone.

[NCERT Exemp. Ex. 5.3, Q. 64, Page 111]

Ans. Given that, $y = \tan^{-1}x \Rightarrow x = \tan y$

Differentiating with respect to x, we have

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$

$$\Rightarrow \quad 1 = \sec^2 y\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \cos^2 y$$ **[1½]**

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cos^2 y)$$

$$= 2\cos y(-\sin y)\frac{dy}{dx}$$

$$= 2\cos y(-\sin y)\cos^2 y$$

$$= -\sin 2y\cos^2 y$$ **[1½]**

Q. 22. If $x^m y^n = (x+y)^{m+n}$, then prove that $\frac{d^2y}{dx^2} = 0$.

[NCERT Exemp. Ex. 5.3, Q. 80, Page 113]

Ans. Given that, $x^m y^n = (x+y)^{m+n}$

Taking log both sides, we have

$$\log(x^m y^n) = \log(x+y)^{m+n}$$

$$\Rightarrow m\log x + n\log y = (m+n)\log(x+y)$$ **[1]**

Differentiating with respect to x, we have

$$\Rightarrow \frac{d}{dx}(m\log x + n\log y) = \frac{d}{dx}\left[(m+n)\log(x+y)\right]$$

$$\Rightarrow \quad m\times\frac{1}{x} + n\times\frac{1}{y}\frac{dy}{dx} = (m+n)\times\frac{1}{x+y}\times\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{my-nx}{x(x+y)} = \frac{my-nx}{y(x+y)}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$ **[1]**

Again, differentiating with respect to x, we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{x\times\frac{y}{x}-y}{x^2}=0$$ **[1]**

Q. 23. If $x^m y^n=(x+y)^{m+n}$, then prove that $\frac{dy}{dx}=\frac{y}{x}$.

[NCERT Exemp. Ex. 5.3, Q. 80(i), Page 113]

Ans. Given that, $x^m y^n=(x+y)^{m+n}$

Taking log both sides, we have

$$\log(x^m y^n)=\log(x+y)^{m+n}$$

$$\Rightarrow m\log x+n\log y=(m+n)\log(x+y)$$ **[1]**

Differentiating with respect to x, we have

$$\Rightarrow \frac{d}{dx}(m\log x+n\log y)=\frac{d}{dx}\left[(m+n)\log(x+y)\right]$$

$$\Rightarrow \quad m\times\frac{1}{x}+n\times\frac{1}{y}\frac{dy}{dx}=(m+n)\times\frac{1}{x+y}\times\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{my-nx}{x(x+y)}=\frac{my-nx}{y(x+y)}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y}{x}$$

[2]

Long Answer Type Questions

(5 or 6 marks each)

Q. 1. Find $\frac{dy}{dx}$ of the function expressed in parametric form $x=e^{\theta}\left(\theta+\frac{1}{\theta}\right)$, $y=e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$.

[NCERT Exemp. Ex. 5.3, Q. 45, Page 110]

Ans. Given that,

$$x=e^{\theta}\left(\theta+\frac{1}{\theta}\right),\ y=e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$$

$$x=e^{\theta}\left(\theta+\frac{1}{\theta}\right)$$

$$\Rightarrow \frac{dx}{d\theta}=\frac{d}{d\theta}\left[e^{\theta}\left(\theta+\frac{1}{\theta}\right)\right]$$

$$=\left[e^{\theta}\frac{d}{d\theta}\left(\theta+\frac{1}{\theta}\right)+\left(\theta+\frac{1}{\theta}\right)\frac{d}{d\theta}\left(e^{\theta}\right)\right]$$

$$=e^{\theta}\left(1-\frac{1}{\theta^2}\right)+\left(\theta+\frac{1}{\theta}\right)\left(e^{\theta}\right)$$

$$=e^{\theta}\left(1-\frac{1}{\theta^2}+\theta+\frac{1}{\theta}\right)$$

$$=e^{\theta}\left(\frac{\theta^3+\theta^2+\theta-1}{\theta^2}\right)$$

[2]

And,

$$y=e^{-\theta}\left(\theta-\frac{1}{\theta}\right)$$

$$\Rightarrow \frac{dy}{d\theta}=\frac{d}{d\theta}\left[e^{-\theta}\left(\theta-\frac{1}{\theta}\right)\right]$$

$$=\left[e^{-\theta}\frac{d}{d\theta}\left(\theta-\frac{1}{\theta}\right)+\left(\theta-\frac{1}{\theta}\right)\frac{d}{d\theta}\left(e^{-\theta}\right)\right]$$

$$=e^{-\theta}\left(1+\frac{1}{\theta^2}\right)-\left(\theta-\frac{1}{\theta}\right)\left(e^{-\theta}\right)$$

$$=e^{-\theta}\left(1+\frac{1}{\theta^2}-\theta+\frac{1}{\theta}\right)$$

$$=e^{-\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^2}\right)$$

[2]

Thus,

$$\frac{dy}{dx}=e^{-2\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^3+\theta^2+\theta+1}\right)$$

[1]

Q. 2. Find $\frac{dy}{dx}$ of the function expressed in parametric form $x=\frac{1+\log t}{t^2}$, $y=\frac{3+2\log t}{t}$.

[NCERT Exemp. Ex. 5.3, Q. 48, Page 110]

Ans. Given that,

$$x=\frac{1+\log t}{t^2},\ y=\frac{3+2\log t}{t}$$

$$x=\frac{1+\log t}{t^2}$$

Differentiating both sides with respect to t, we have

$$\frac{dx}{dt}=\frac{t^2\frac{d}{dt}(1+\log t)-(1+\log t)\frac{d}{dt}(t^2)}{t^4}$$

$$=\frac{t^2\times\frac{1}{t}-2t(1+\log t)}{t^4}$$

$$=\frac{t-2t(1+\log t)}{t^4}$$

$$=\frac{1-2(1+\log t)}{t^3}$$

$$=\frac{-1-2\log t}{t^3}$$

[2]

And,

$$y=\frac{3+2\log t}{t}$$

Differentiating both sides with respect to t, we have

$$\frac{dy}{dt}=\frac{t\frac{d}{dt}(3+2\log t)-(3+2\log t)\frac{d}{dt}(t)}{t^2}$$

$$=\frac{t\times\frac{2}{t}-(3+2\log t)}{t^2}$$

$$=\frac{2-3-2\log t}{t^2}$$

$$=\frac{-1-2\log t}{t^2}$$

[2]

Thus,

$$\frac{dy}{dx}=\frac{\frac{-1-2\log t}{t^2}}{\frac{-1-2\log t}{t^3}}=t$$

[1]

Q. 3. If $x = a\sin 2t(1+\cos 2t)$ **and** $y = b\cos 2t(1-\cos 2t)$, **show that** $\left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}} = \frac{b}{a}$.

[NCERT Exemp. Ex. 5.3, Q. 50, Page 110]

Ans. Given that,

$x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$.

$x = a\sin 2t(1+\cos 2t)$

Differentiating both sides with respect to t, we have

$$\frac{dx}{dt} = a\left[\sin 2t\frac{d}{dt}(1+\cos 2t) + (1+\cos 2t)\frac{d}{dt}(\sin 2t)\right]$$
$$= a\left[\sin 2t(-2\sin 2t) + (1+\cos 2t)(2\cos 2t)\right]$$
$$= a\left[-2\sin^2 2t + 2\cos 2t + 2\cos^2 2t\right]$$
$$= -2a\sin^2 2t + 2a\cos 2t + 2a\cos^2 2t$$

[2]

And,

$$\frac{dy}{dt} = b\frac{d}{dt}\left[\cos 2t(1-\cos 2t)\right]$$
$$= b\left[\cos 2t\frac{d}{dt}(1-\cos 2t) + (1-\cos 2t)\frac{d}{dt}(\cos 2t)\right]$$
$$= b\left[2\sin 2t\cos 2t - 2\sin 2t(1-\cos 2t)\right]$$
$$= 2b\sin 2t\cos 2t - 2b\sin 2t(1-\cos 2t)$$

[1]

Thus,

$$\frac{dy}{dx} = \frac{2b\sin 2t\cos 2t - 2b\sin 2t(1-\cos 2t)}{-2a\sin^2 2t + 2a\cos 2t + 2a\cos^2 2t}$$
$$= \frac{b\left[\sin 4t - 2\sin 2t(1-\cos 2t)\right]}{-2a\left[\sin^2 2t - \cos 2t - \cos^2 2t\right]}$$

[1]

Therefore,

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{b\left[\sin(\pi) - 2\sin\left(\frac{\pi}{2}\right)\left(1-\cos\left(\frac{\pi}{2}\right)\right)\right]}{-2a\left[\sin^2\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) - \cos^2\left(\frac{\pi}{2}\right)\right]}$$
$$= \frac{b\left[0-2(1-0)\right]}{-2a\left[1-0-0\right]}$$
$$= \frac{b}{a}$$

[1]

Q. 4. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, **then prove that** $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

[NCERT Exemp. Ex. 5.3, Q. 63, Page 111]

Ans. Given that, $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

Put $x = \sin\alpha$ and $y = \sin\beta$, we have

$$\sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a(\sin\alpha - \sin\beta)$$
$$\Rightarrow \sqrt{\cos^2\alpha} + \sqrt{\cos^2\beta} = a(\sin\alpha - \sin\beta)$$
$$\Rightarrow \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$$
$$\Rightarrow 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2a\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\Rightarrow \cos\frac{\alpha-\beta}{2} = a\sin\frac{\alpha-\beta}{2}$$
$$\Rightarrow \cot\frac{\alpha-\beta}{2} = a$$
$$\Rightarrow \frac{\alpha-\beta}{2} = \cot^{-1}a$$
$$\Rightarrow \alpha-\beta = 2\cot^{-1}a$$
$$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$$

[3]

Differentiating with respect to x, we have

$$\Rightarrow \frac{d}{dx}\left(\sin^{-1}x - \sin^{-1}y\right) = \frac{d}{dx}\left(2\cot^{-1}a\right)$$
$$\Rightarrow \frac{d}{dx}\left(\sin^{-1}x\right) - \frac{d}{dx}\left(\sin^{-1}y\right) = \frac{d}{dx}\left(2\cot^{-1}a\right)$$
$$\Rightarrow \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$
$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

[2]

Q. 5. If $x = \sin t$ **and** $y = \sin pt$, **prove that** $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

[NCERT Exemp. Ex. 5.3, Q. 81, Page 113]

Ans. Given that,

$x = \sin t$ and $y = \sin pt$.

Thus,

$x = \sin t$

$$\Rightarrow \frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

[1]

And,

$y = \sin pt$

$$\Rightarrow \frac{dy}{dt} = \frac{d}{dt}(\sin pt) = p\cos pt$$

[1]

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{p\cos pt}{\cos t}$$
$$\Rightarrow \cos t\frac{dy}{dx} = p\cos pt$$
$$\Rightarrow \cos^2 t\left(\frac{dy}{dx}\right)^2 = p^2\cos^2 pt$$
$$\Rightarrow \left(1-\sin^2 t\right)\left(\frac{dy}{dx}\right)^2 = p^2\left(1-\sin^2 pt\right)$$
$$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = p^2\left(1-y^2\right)$$

[1½]

Differentiating with respect to x, we have

$$\Rightarrow \left(1-x^2\right)\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] + \left(\frac{dy}{dx}\right)^2\frac{d}{dx}\left[\left(1-x^2\right)\right]$$
$$= \frac{d}{dx}\left[p^2\left(1-y^2\right)\right]$$

$$\Rightarrow (1-x^2)\times 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2(-2x)$$

$$= p^2\left(-2y\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2}(1-x^2)-x\frac{dy}{dx}=-p^2y$$

$$\Rightarrow \frac{d^2y}{dx^2}(1-x^2)-x\frac{dy}{dx}+p^2y=0$$

[1½]

Q. 6. Find $\frac{dy}{dx}$, if $y = x^{\tan x}+\sqrt{\frac{x^2+1}{2}}$.

[NCERT Exemp. Ex. 5.3, Q. 82, Page 113]

Ans. Let $u = x^{\tan x}$ and $v=\sqrt{\frac{x^2+1}{2}}$

Now,

$$u = x^{\tan x}$$

$$\Rightarrow \log u = \tan x \log x$$

[1]

Differentiating both sides, we have

$$\frac{d}{dx}(\log u)=\frac{d}{dx}(\tan x\log x)$$

$$\frac{1}{u}\frac{du}{dx}=\frac{\tan x}{x}+\log x\sec^2 x$$

$$\frac{du}{dx}=u\left[\frac{\tan x}{x}+\log x\sec^2 x\right]$$

$$= x^{\tan x}\left[\frac{\tan x}{x}+\log x\sec^2 x\right]$$

[2]

Now,

$$v=\sqrt{\frac{x^2+1}{2}}$$

Differentiating both sides, we have

$$\frac{dv}{dx}=\frac{d}{dx}\left(\sqrt{\frac{x^2+1}{2}}\right)$$

$$=\frac{x}{\sqrt{2(x^2+1)}}$$

[1]

Thus,

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$=x^{\tan x}\left[\frac{\tan x}{x}+\log x\sec^2 x\right]+\frac{x}{\sqrt{2(x^2+1)}}$$

[1]

TOPIC-3
Rolle's Theorem and MVT

TIPS...

Rolle's theorem as the slope of the tangent at any point on the graph of y = f(x) is nothing but the derivative of f(x) at that point.

Quick Review

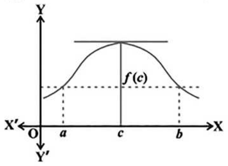
- Rolle's Theorem : For a function $f(x)$ to be applicable Rolle's Theorem, there are three conditions which is to be satisfied :

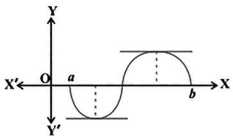
  (xxxvi) $f(x)$ should be continuous on $[a, b]$

  (xxxvii) $f(x)$ should be differentiable on (a, b)

  (xxxviii) $f(a) = f(b)$, where a and b are some real numbers.

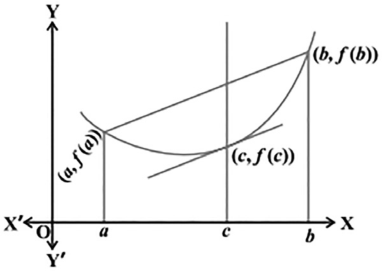
- If all the above conditions are satisfied, then there exists some c in (a, b) such that $f'(c) = 0$. Geometrically Rolle's theorem ensures that there is at least one point on the curve $y = f(x)$ at which tangent is parallel to x-axis [abscissa of the point lying in (a, b)].





- MVT (Mean Valued Theorem)

  Let f : $[a, b]$ R be a continuous function on [a, b] and differentiable on (a, b). Then there exists at least one point c in (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$ Geometrically, Mean Value Theorem states that there exists at least one point c in (a, b) such that the tangent at the point (c, f(c)) is parallel to the secant joining the points $(a, f(a))$ and $(b, f(b))$.



Multiple Choice Questions (1 mark each)

Q. 1. The value of c in the Rolle's Theorem for the function $f(x)=x^3-3x$ in the interval $\left[0,\sqrt{3}\right]$ is

(a) 1 **(b)** –1

(c) $\frac{3}{2}$ **(d)** $\frac{1}{3}$

[NCERT Exemp. Ex. 5.3, Q. 95, Page 115]

Ans. Correct option : (a)

Explanation : Given that $f(x)=x^3-3x$. It is continuous and differentiable. And, $f(0)=f\left(\sqrt{3}\right)=0$.

Thus, by Rolle's Theorem, there exists c for which

$$f'(c)=0$$

$$\Rightarrow 3c^2-3=0$$

$$\Rightarrow c^2=1$$

$$\Rightarrow c=\pm1$$

$$\Rightarrow c=1\in\left[0,\sqrt{3}\right]$$

Q. 2. For the function $f(x)=x+\frac{1}{x}$, $x\in[1,3]$, the value of c for the mean value theorem is

(a) 1 **(b)** $\sqrt{3}$

(c) 2 **(d)** None of these

[NCERT Exemp. Ex. 5.3, Q. 96, Page 116]

Ans. Correct option : (b)

Explanation : Given that $f(x)=x+\frac{1}{x}$, $x\in[1,3]$. It is continuous and differentiable.

Thus, by mean value theorem, there exists c for which

$$f(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow 1-\frac{1}{c^2}=\frac{\frac{10}{3}-2}{3-1}$$

$$\Rightarrow c^2=3$$

$$\Rightarrow c=\sqrt{3}\in(1,3)$$

Q. 3. State True or False for the statements :

Rolle's theorem is applicable for the function $f(x)=|x-1|$ in [0, 2].

[NCERT Exemp. Ex. 5.3, Q. 102, Page 116]

Ans. False

Explanation : Since the given function $f(x)=|x-1|$ is continuous function but not differentiable at $x=1$ in [0, 2], then the Rolle's theorem is not applicable to the given function.

Short Answer Type Questions (3 or 4 marks each)

Q. 1. Verify the Rolle's theorem for the function $f(x)=x(x-1)^2$ in [0, 1].

[NCERT Exemp. Ex. 5.3, Q. 65, Page 112]

Ans. Given that $f(x)=x(x-1)^2$ in $[0,1]$

It is continuous and differentiable in [0, 1].

And, $f(0)=f(1)=0$ **[1]**

Thus, Rolle's theorem is applicable to the given function.

Then, there exists a value c such that

$$f'(c)=0$$

$$\Rightarrow 3c^2-4c+1=0$$

$$\Rightarrow (3c-1)(c-1)=0$$

$$\Rightarrow c=\frac{1}{3}\in(0,1)$$ **[2]**

Therefore, the Rolle's theorem is verified for the given function.

Q. 2. Verify the Rolle's theorem for the function $f(x)=\log\left(x^2+2\right)-\log 3$ in $[-1,1]$.

[NCERT Exemp. Ex. 5.3, Q. 67, Page 112]

Ans. Given that $f(x)=\log\left(x^2+2\right)-\log 3$ in $[-1,1]$

It is continuous in [–1, 1] and differentiable in (–1, 1). **[1]**

And, $f(-1)=f(1)=0$ **[1]**

Thus, Rolle's theorem is applicable to the given function.

Then, there exists a value c such that

$$f'(c)=0$$

$$\Rightarrow \frac{2c}{c^2+2}=0$$

$$\Rightarrow c=0\in(-1,1)$$ **[1]**

Therefore, the Rolle's theorem is verified for the given function.

Q. 3. Verify the Rolle's theorem for the function $f(x)=x(x+3)e^{-\frac{x}{2}}$ in [–3, 0].

[NCERT Exemp. Ex. 5.3, Q. 68, Page 112]

Ans. Given that $f(x)=x(x+3)e^{-\frac{x}{2}}$ in $[-3,0]$

It is continuous in [–3, 0] and differentiable in (–3, 0). [1]

And, $f(-3)=f(0)=0$ **[1]**

Thus, Rolle's theorem is applicable to the given function.

Then, there exists *a* value *c* such that

$$f'(c)=0$$

$$\Rightarrow (2c+3)e^{-\frac{c}{2}}-\frac{1}{2}c(c+3)e^{-\frac{c}{2}}=0$$

$$\Rightarrow -\frac{1}{2}e^{-\frac{c}{2}}\left(c^2-c-6\right)=0$$

$\Rightarrow \quad -\frac{1}{2}e^{-\frac{c}{2}}(c+2)(c-3)=0$

$\Rightarrow \quad c=-2\in(-3,0)$ **[1]**

Therefore, the Rolle's theorem is verified for the given function.

Q. 4. Verify the Rolle's theorem for the function $f(x)=\sqrt{4-x^2}$ in $[-2,2]$.

[NCERT Exemp. Ex. 5.3, Q. 69, Page 112]

Ans. Given that $f(x)=\sqrt{4-x^2}$ in $[-2,2]$

It is continuous in [–2, 2] and differentiable in (–2, 2). **[1]**

And, $f(-2)=f(2)=0$ **[1]**

Thus, Rolle's theorem is applicable to the given function.

Then, there exists a value c such that

$f'(c)=0$

$\Rightarrow \frac{-2c}{2\sqrt{4-c^2}}=0$

$\Rightarrow \quad c=0\in(-2,2)$ **[1]**

Therefore, the Rolle's theorem is verified for the given function.

Q. 5. Find the points on the curve y = (cos x – 1) in [0, 2π], where the tangent is parallel to *x*-axis.

[NCERT Exemp. Ex. 5.3, Q. 71, Page 112]

Ans. Since the tangent of the given curve is parallel to *x*-axis, then

$\frac{dy}{dx}=0$

$\Rightarrow -\sin x-0=0$

$\Rightarrow \quad \sin x=0$ **[1]**

Thus,

$\Rightarrow \quad x=\pi\in(0,2\pi)$ **[1]**

Put $x=\pi$ in $y=\cos x-1$, we have

$y=\cos\pi-1=-1-1=-2$

Thus, the required point is $(\pi,-2)$. **[1]**

Q. 6. Using Rolle's theorem, find the point on the curve $y=x(x-4)$, $x\in[0,4]$, where the tangent is parallel to *x*-axis. [NCERT Exemp. Ex. 5.3, Q. 72, Page 112]

Ans. Since the tangent of the given curve is parallel to *x*-axis, then

$\frac{dy}{dx}=0$

$\Rightarrow \frac{d}{dx}\left[x^2-4x\right]=0$

$\Rightarrow \quad 2x-4=0$

$\Rightarrow \quad x=2$ **[1]**

Thus,

$\Rightarrow \quad x=2\in(0,4)$ **[1]**

Put $x=2$ in $y=x(x-4)$, we have

$y=2(2-4)=-4$

Thus, the required point is $(2,-4)$. **[1]**

Q. 7. Verify mean value theorem for the function $f(x)=\sqrt{25-x^2}$ in $[1,5]$.

[NCERT Exemp. Ex. 5.3, Q. 76, Page 112]

Ans. Given that $f(x)=\sqrt{25-x^2}$ in $[1,5]$

It is continuous in $[1,5]$. **[1]**

Now, $f'(x)=\frac{-2x}{2\sqrt{25-x^2}}=-\frac{x}{\sqrt{25-x^2}}$, it is defined in $(1,5)$ and hence it is differentiable in $(1,5)$. **[1]**

Thus, the mean value theorem is applicable to the given function.

Then, there exists a value $c\in(1,5)$ such that

$f'(c)=\frac{f(5)-f(1)}{5-1}$

$\Rightarrow -\frac{c}{\sqrt{25-c^2}}=\frac{0-\sqrt{24}}{4}$

$\Rightarrow \quad 16c^2=24(25-c^2)$

$\Rightarrow \quad 40c^2=600$

$\Rightarrow \quad c^2=15$

$\Rightarrow \quad c=\sqrt{15}\in(1,5)$ **[1]**

Thus, the mean value verified for the given function.

Q. 8. Find a point on the curve $y=(x-3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

[NCERT Exemp. Ex. 5.3, Q. 77, Page 112]

Ans. Since the tangent of the given curve is parallel to the chord joining the points (3, 0) and (4, 1), then

$\frac{dy}{dx}=\frac{0-1}{3-4}$

$\Rightarrow \frac{d}{dx}\left[(x-3)^2\right]=1$

$\Rightarrow \quad 2(x-3)=1$

$\Rightarrow \quad x=\frac{7}{2}$ **[1]**

Thus,

$\Rightarrow x=\frac{7}{2}\in(3,4)$ **[1]**

Put x = $\frac{7}{2}$ in $y=(x-3)^2$, we have

$y=\left(\frac{7}{2}-3\right)^2=\frac{1}{4}$

Thus, the required point is $\left(\frac{7}{2},\frac{1}{4}\right)$. **[1]**

Long Answer Type Questions

(5 or 6 marks each)

Q. 1. Verify the Rolle's theorem for the function $f(x)=\sin^4x+\cos^4x$ in $\left[0,\frac{\pi}{2}\right]$.

[NCERT Exemp. Ex. 5.3, Q. 66, Page 112]

Ans. Given that,

$f(x)=\sin^4x+\cos^4x$ in $\left[0,\frac{\pi}{2}\right]$

It is continuous and differentiable in $\left[0, \frac{\pi}{2}\right]$ **[1]**

And, $f(0) = f\left(\frac{\pi}{2}\right) = 1$ **[1]**

Thus, Rolle's theorem is applicable to the given function.

Then, there exists a value c such that

$$f'(c) = 0$$

$$\Rightarrow 4\sin^3 c\cos c - 4\cos^3 c\sin c = 0$$

$$\Rightarrow 4\sin c\cos c\left(\sin^2 c - \cos^2 c\right) = 0$$

$$\Rightarrow 4\sin c\cos c\left(-\cos 2c\right) = 0$$

$$\Rightarrow -2\left(2\sin c\cos c\right)\cos 2c = 0$$

$$\Rightarrow -2\sin 2c\cos 2c = 0$$

$$\Rightarrow \sin 4c = 0$$

$$\Rightarrow 4c = \pi$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

[3]

Therefore, the Rolle's theorem is verified for the given function.

Q. 2. Discuss the applicability of Rolle's theorem on the function given

$$f(x) = \begin{cases} x^2 + 1, \text{ if } 0 \le x \le 1 \\ 3 - x, \text{ if } 1 \le x \le 2 \end{cases}.$$

[NCERT Exemp. Ex. 5.3, Q. 70, Page 112]

Ans. Given that,

$$f(x) = \begin{cases} x^2 + 1, \text{ if } 0 \le x \le 1 \\ 3 - x, \text{ if } 1 \le x \le 2 \end{cases}$$

At $x = 1$,

$$\text{LHL} = \lim_{x \to 1^-}\left(x^2 + 1\right) = 1 + 1 = 2$$ **[1]**

$$\text{RHL} = \lim_{x \to 1^+}\left(3 - x\right) = 3 - 1 = 2$$ **[1]**

And, $f(1) = 3 - 1 = 2$

Since LHL = RHL = $f(1) = 2$, the function is continuous function at $x = 1$.

$$f(x) = \begin{cases} x^2 + 1, \text{ if } 0 \le x \le 1 \\ 3 - x, \text{ if } 1 \le x \le 2 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x, \text{ if } 0 < x < 1 \\ -1, \text{ if } 1 < x < 2 \end{cases}$$ **[2]**

Now,

$$\text{LHD} = 2(1) = 2$$

$$\text{RHD} = -1$$

Since $\text{LHD} \ne \text{RHD}$, the given function is not differentiable at $x = 1$.

Thus, Rolle's theorem is not applicable to the given function. **[1]**

Q. 3. Verify mean value theorem for the function $f(x) = \frac{1}{4x - 4}$ in $[1, 4]$.

[NCERT Exemp. Ex. 5.3, Q. 73, Page 112]

Ans. Given that $f(x) = \frac{1}{4x - 4}$ in $[1, 4]$

It is continuous in [1, 4]. **[1]**

Now, $f'(x) = -\frac{4}{(4x - 4)^2}$, it is defined in (1, 4) and hence it is differentiable in (1, 4).

Thus, the mean value theorem is applicable to the given function. **[1]**

Then, there exists a value $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\Rightarrow -\frac{4}{(4c - 1)^2} = \frac{\frac{1}{16 - 1} - \frac{1}{4 - 1}}{3}$$

$$\Rightarrow -\frac{4}{(4c - 1)^2} = \frac{\frac{1}{15} - \frac{1}{3}}{3}$$

$$\Rightarrow -\frac{4}{(4c - 1)^2} = \frac{-4}{45}$$

$$\Rightarrow (4c - 1)^2 = 45$$

$$\Rightarrow 4c - 1 = \pm 3\sqrt{5}$$

$$\Rightarrow c = \frac{3\sqrt{5} + 1}{4} \in (1, 4)$$

[3]

Thus, the mean value verified for the given function.

Q. 4. Verify mean value theorem for the function $f(x) = x^3 - 2x^2 - x + 3$ in $[0, 1]$.

[NCERT Exemp. Ex. 5.3, Q. 74, Page 112]

Ans. Given that $f(x) = x^3 - 2x^2 - x + 3$ in $[0, 1]$

It is continuous in [0, 1]. **[1]**

Now, $f'(x) = 3x^2 - 4x - 1$, it is defined in (0, 1) and hence it is differentiable in (0, 1).

Thus, the mean value theorem is applicable to the given function. **[1]**

Then, there exists a value $c \in (0, 1)$ such that

$$f'(c) = \frac{f(0) - f(1)}{0 - 1}$$

$$\Rightarrow 3c^2 - 4c - 1 = \frac{[0 + 3] - [1 - 2 - 1 + 3]}{0 - 1}$$

$$\Rightarrow 3c^2 - 4c - 1 = -2$$

$$\Rightarrow 3c^2 - 4c + 1 = 0$$

$$\Rightarrow (3c - 1)(c - 1) = 0$$

$$\Rightarrow c = 1, \frac{1}{3}$$

$$\Rightarrow c = \frac{1}{3} \in (0, 1)$$

[3]

Thus, the mean value verified for the given function.

Q. 5. Verify mean value theorem for the function $f(x) = \sin x - \sin 2x$ in $[0, \pi]$.

[NCERT Exemp. Ex. 5.3, Q. 75, Page 112]

Ans. Given that $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

It is continuous in $[0, \pi]$. **[1]**

Now, $f'(x) = \cos x - 2\cos 2x$, it is defined in $(0, \pi)$ and hence it is differentiable in $(0, \pi)$. **[1]**

Thus, the mean value theorem is applicable to the given function.

Then, there exists a value $c \in (0, \pi)$ such that

$$f'(c)=\frac{f(\pi)-f(0)}{\pi-0}$$

$\Rightarrow \quad \cos c-2\cos 2c=\dfrac{\sin\pi-\sin 2\pi-\sin 0+\sin 0}{\pi-0}$

$\Rightarrow \quad 2\cos 2c-\cos c=0$

$\Rightarrow 2\left(2\cos^2 c-1\right)-\cos c=0$

$\Rightarrow \quad 4\cos^2 c-\cos c-2=0$

$\Rightarrow \quad \cos c=\dfrac{1\pm\sqrt{1+32}}{8}=\dfrac{1\pm\sqrt{33}}{8}$

$\Rightarrow \quad c=\cos^{-1}\left(\dfrac{1\pm\sqrt{33}}{8}\right)\in(0,\pi)$

[3]

Thus, the mean value verified for the given function.

Q. 6. Using mean value theorem, prove that there is a point on the curve $y=2x^2-5x+3$ between the points A (1, 0) and B (2, 1), where tangent is parallel to the chord AB. Also, find that point.

[NCERT Exemp. Ex. 5.3, Q. 78, Page 112]

Ans. Given that $y=2x^2-5x+3$ in $[1, 2]$

It is continuous in [1, 2]. **[1]**

Now, $f'(x)=4x-5$, it is defined in $(1,2)$ and hence it is differentiable in $(1,2)$. **[1]**

Thus, the mean value theorem is applicable to the given function.

Then, there exists a value $c\in(1,2)$ such that **[1]**

$$f'(c)=\frac{f(2)-f(1)}{2-1}$$

$\Rightarrow 4c-5=\dfrac{1-0}{1}$

$\Rightarrow \quad c=\dfrac{3}{2}\in(1,2)$

[1]

Thus, the mean value verified for the given function.

Now, put $x=\dfrac{3}{2}$ in $y=2x^2-5x+3$, we get

$$y=2\left(\frac{3}{2}\right)^2-5\left(\frac{3}{2}\right)+3=0$$

Thus, the required point is $\left(\dfrac{3}{2},0\right)$. **[1]**