Complex Numbers and Quadratic Equations

Chapter 2

Syllabus

Complex numbers as ordered pairs of reals. Representation of complex numbers in the form $a+ib$ and their representation in a plane, Argand diagram, algebra of complex number, modulus and argument (or amplitude) of a complex number, Quadratic equations in real and complex number system and their solutions. Relation between roots and coefficients, nature of roots, the formation of quadratic equations with given roots.

Topic-1
Complex Numbers

JEE (Main) Previous Year Questions

Multiple Choice Questions

1. Let a, b be two real numbers such that $ab < 0$. If the complex number $\frac{1+ai}{b+i}$ is of unit modulus and $a + ib$ lies on the circle $|z-1| = |2z|$, then a possible value of $\frac{1+[a]}{4b}$, where $[t]$ is greatest integer function, is :

(1) $-\frac{1}{2}$ **(2)** -1

(3) 1 **(4)** $\frac{-(1+\sqrt{7})}{4}$

[JEE (Main) – 1st February 2023 - Shift-2]

2. For two non-zero complex numbers z_1 and z_2, if $\text{Re}(z_1z_2) = 0$ and $\text{Re}(z_1 + z_2) = 0$, then which of the following are possible?

A. $\text{Im}(z_1) > 0$ and $\text{Im}(z_2) > 0$
B. $\text{Im}(z_1) < 0$ and $\text{Im}(z_2) > 0$
C. $\text{Im}(z_1) > 0$ and $\text{Im}(z_2) < 0$
D. $\text{Im}(z_1) < 0$ and $\text{Im}(z_2) < 0$

Choose the correct answer from the options given below:

(1) B and D **(2)** A and B

(3) B and C **(4)** A and C

[JEE (Main) – 29th January 2023 - Shift-1]

3. The sum of the coefficients of three consecutive terms in the binomial expansion of $(1 + x)^{n+2}$, which are in the ratio 1 : 3 : 5, is equal to:

(1) 63 **(2)** 92 **(3)** 25 **(4)** 41

[JEE (Main) – 11th April 2023 - Shift-2]

4. If the coefficients of x and x^2 in $(1 + x)^p (1- x)^q$ are 4 and – 5 respectively, then $2p + 3q$ is equal to

(1) 60 **(2)** 63

(3) 66 **(4)** 69

[JEE (Main) – 10th April 2023 - Shift-2]

5. Let the complex number $z = x + iy$ be such that $\frac{2z-3i}{2z+i}$ is purely imaginary. If $x + y^2 = 0$, then $y^4 + y^2 - y$ is equal to:

(1) $\frac{3}{2}$ **(2)** $\frac{2}{3}$ **(3)** $\frac{4}{3}$ **(4)** $\frac{3}{4}$

[JEE (Main) – 10th April 2023 - Shift-1]

6. If for $z = \alpha + i\beta$, $|z + 2| = z + 4(1 + i)$, then $\alpha + \beta$ and $\alpha\beta$ are the roots of the equation

(1) $x^2 + 3x - 4 = 0$ (2) $x^2 + 7x + 12 = 0$

(3) $x^2 + x - 12 = 0$ (4) $x^2 + 2x - 3 = 0$

[JEE (Main) – 8th April 2023 - Shift-1]

7. If $z = 2 + 3i$, then $z^5 + (\bar{z})^5$ is equal to:

(1) 244 (2) 224 (3) 245 **(4)** 265

[JEE (Main) – 29th July 2022 - Shift-1]

8. If $z \neq 0$ is a complex number such that $\left|z - \frac{1}{z}\right| = 2$, then the maximum value of $|z|$ is:

(1) $\sqrt{2}$ **(2)** 1 **(3)** $\sqrt{2} - 1$ **(4)** $\sqrt{2} + 1$

[JEE (Main) – 29th July 2022 - Shift-2]

9. Let S = $\{z = x + iy : |z - 1 + i| \geq |z|, |z| < 2, |z + i| = |z - 1|\}$. Then, the set of all values of x, for which $w = 2x + iy \in S$ for some $y \in R$ is:

(1) $\left(-\sqrt{2}, \frac{1}{2\sqrt{2}}\right]$ **(2)** $\left(-\frac{1}{\sqrt{2}}, \frac{1}{4}\right]$

(3) $\left(-\sqrt{2}, \frac{1}{2}\right)$ **(4)** $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

[JEE (Main) – 29th July 2022 - Shift-2]

10. Let $S_1 = \left\{z_1 \in C : |z_1 - 3| = \frac{1}{2}\right\}$ and $S_2 = \{z_2 \in C : |z_2 - |z_2 + 1|| = |z_2 + |z_2 - 1||\}$. then for $z_1 \in S_1$ and $z_2 \in S_2$, the least value of $|z_2 - z_1|$ is :

(1) 0 **(2)** $\frac{1}{2}$ **(3)** $\frac{3}{2}$ **(4)** $\frac{5}{2}$

[JEE (Main) – 28th July 2022 - Shift-1]

11. Let the minimum value v_0 of $v = |z|^2 + |z - 3|^2 + |z - 6i|^2$, $z \in C$ is attained at $z = z_0$. Then, $\left|2z_0^2 - \bar{z}_0^3 + 3\right|^2 + v_0{}^2$ is equal to:

(1) 1000 **(2)** 1024 **(3)** 1105 **(4)** 1196

[JEE (Main) – 27th July 2022 - Shift-1]

12. Let S be the set of all $(\alpha, \beta), \pi < \alpha, \beta < 2\pi$, for which the complex number $\frac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$ is purely imaginary and $\frac{1 + i\cos\beta}{1 - 2i\cos\beta}$ is purely real. Let $z_{\alpha\beta} = \sin 2\alpha + i \cos 2\beta$ $(\alpha, \beta) \in \Sigma$. Then $\sum_{(\alpha,\beta)\in s}\left(iz_{\alpha\beta} + \frac{1}{i\bar{z}_{\alpha\beta}}\right)$ is equal to :

(1) 3 **(2)** $3i$ **(3)** 1 **(4)** $2 - i$

[JEE (Main) – 27th July 2022 - Shift-2]

13. Let O be the origin and A be the point $z_1 = 1 + 2i$. If B is the point z_2, $Re(z_2) < 0$, such that OAB is a right angled isosceles triangle with OB as hypotenuse, then which of the following is NOT true ?

(1) $\arg z_2 = \pi - \tan^{-1}3$

(2) $\arg(z_1 - 2z_2) = -\tan^{-1}\frac{4}{3}$

(3) $|z_2| = \sqrt{10}$

(4) $|2z_1 - z_2| = 5$

[JEE (Main) – 26th July 2022 - Shift-1]

14. If $z = x + iy$ satisfies $|z| - 2 = 0$ and $|z - i| - |z + 5i| = 0$, then

(1) $x + 2y - 4 = 0$ **(2)** $x^2 + y - 4 = 0$

(3) $x + 2y + 4 = 0$ **(4)** $x^2 - y + 3 = 0$

[JEE (Main) – 26th July 2022 - Shift-2]

15. For $n \in N$, let $S_n = \left\{z \in C; |z - 3 + 2i| = \frac{n}{4}\right\}$. and $T_n = \left\{z \in C; |z - 2 + 3i| = \frac{1}{n}\right\}$. Then the number of elements in the set $\{n \in N ; S_n \cap T_n = \phi\}$ is :

(1) 0 **(2)** 2 **(3)** 3 **(4)** 4

[JEE (Main) – 25th July 2022 - Shift-1]

16. The area of the polygon, whose vertices are the non – real roots of the equation $\bar{z} = iz^2$ is

(1) $\frac{3\sqrt{3}}{4}$ **(2)** $\frac{3\sqrt{3}}{2}$ **(3)** $\frac{3}{2}$ **(4)** $\frac{3}{4}$

[JEE (Main) – 27th June 2022 - Shift-1]

17. The number of points of intersection of $|z - (4 + 3i)| = 2$ and $|z| + |z - 4| = 6$, $z \in C$, is

(1) 0 **(2)** 1 **(3)** 2 **(4)** 3

[JEE (Main) – 27th June 2022 - Shift-2]

18. Let for some real numbers α and β, $a = \alpha - i\beta$. If the system of equation $4ix + (1 + i)y = 0$ and $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)x + \bar{a}y = 0$ has more than one solutions. Then $\frac{\alpha}{\beta}$ is equal to

(1) $-2 + \sqrt{3}$ **(2)** $2 - \sqrt{3}$

(3) $2 + \sqrt{3}$ **(4)** $-2 - \sqrt{3}$

[JEE (Main) – 27th June 2022 - Shift-2]

19. Let a circle C in complex plane pass through the point $z_1 = 3 + 4i$, $z_2 = 4 + 3i$ and $z_3 = 5i$. If $z (\neq z_1)$ is a point on C such that the line through z and z_1 is perpendicular to the line through z_2 and z_3, then $\arg(z)$ is equal to:

(1) $\tan^{-1}\left(\frac{2}{\sqrt{5}}\right) - \pi$ **(2)** $\tan^{-1}\left(\frac{24}{7}\right) - \pi$

(3) $\tan^{-1}(3) - \pi$ **(4)** $\tan^{-1}\left(\frac{3}{4}\right) - \pi$

[JEE (Main) – 25th June 2022 - Shift-1]

20. Let z_1 and z_2 be two complex numbers such that $\bar{z}_1 = i\bar{z}_2$ and $\arg\left(\frac{z_1}{\bar{z}_2}\right) = \pi$. Then

(1) $\arg z_2 = \frac{\pi}{4}$ **(2)** $\arg z_2 = -\frac{3\pi}{4}$

(3) $\arg z_1 = \frac{\pi}{4}$ **(4)** $\arg z_1 = -\frac{3\pi}{4}$

[JEE (Main) – 25th June 2022 - Shift-2]

21. If $S = \left\{z \varepsilon C : \frac{z-i}{z+2i} \in R\right\}$, then:

(1) S is a circle in the complex plane
(2) S contains only one element
(3) S is a straight line in the complex plane
(4) S contains exactly two elements

[JEE (Main) – 27th Aug. 2021 - Shift-1]

22. The equation $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ represents a circle with:

(1) centre at (0, –1) and radius $\sqrt{2}$
(2) centre at (0, 1) and radius 2
(3) centre at (0, 1) and radius $\sqrt{2}$
(4) centre at (0, 0) and radius $\sqrt{2}$

[JEE (Main) – 26th Aug. 2021 - Shift-1]

23. If $(\sqrt{3}+i)^{100} = 2^{99}$ $(p + iq)$, then p and q are roots of the equation:

(1) $x^2 + (\sqrt{3}+1)x + \sqrt{3} = 0$

(2) $x^2 + (\sqrt{3}-1)x - \sqrt{3} = 0$

(3) $x^2 - (\sqrt{3}+1)x + \sqrt{3} = 0$

(4) $x^2 - (\sqrt{3}-1)x - \sqrt{3} = 0$

[JEE (Main) – 26th Aug. 2021 - Shift-2]

24. Let n denote the number of solutions of the equation $z^2 + 3\bar{z} = 0$, where z is a complex number. Then, the value of $\sum_{k=0}^{\infty} \frac{1}{n^k}$ is equal to:

(1) 1 **(2)** 2 **(3)** $\frac{4}{3}$ **(4)** $\frac{3}{2}$

[JEE (Main) – 22nd July 2021 - Shift-2]

25. If z and ω are two complex numbers such that $|z\omega| = 1$ and $arg(z) - arg(\omega) = \frac{3\pi}{2}$, then $arg\left(\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega}\right)$ is:

(Here $arg(z)$ denotes the principal argument of complex number z)

(1) $\frac{3\pi}{4}$ **(2)** $-\frac{\pi}{4}$ **(3)** $-\frac{3\pi}{4}$ **(4)** $\frac{\pi}{4}$

[JEE (Main) – 20th July 2021 - Shift-1]

26. If the equation $a|z|^2 + \overline{\overline{\alpha}z + \alpha\bar{z}} + d = 0$ represents a circle where a, d are real constants, then which of the following conditions is correct?

(1) $|\alpha|^2 - ad \neq 0$

(2) $|\alpha|^2 - ad > 0$ and $a \in R - \{0\}$

(3) $\alpha = 0, a, d \in R^+$

(4) $|\alpha|^2 - ad \geq 0$ and $a \in R$

[JEE (Main) – 18th March 2021 - Shift-1]

27. Let a complex number be $\omega = 1 - i\sqrt{3}$. Let another complex number z be such that $|z\omega| = 1$ and arg(z) – arg(ω) = $\frac{\pi}{2}$. Then the area of the triangle with vertices origin, z and ω is equal to:

(1) $\frac{1}{2}$ **(2)** 4 **(3)** 2 **(4)** $\frac{1}{4}$

[JEE (Main) – 18th March 2021 - Shift-2]

28. The area of the triangle with vertices A(z), B(iz) and C($z + iz$) is:

(1) $\frac{1}{2}|z+iz|^2$ **(2)** 1

(3) $\frac{1}{2}$ **(4)** $\frac{1}{2}|z|^2$

[JEE (Main) – 17th March 2021 - Shift-1]

29. If $\alpha, \beta \in R$ are such that $1 - 2i$ (here $i^2 = -1$) is a root of $z^2 + \alpha z + \beta = 0$, then $(\alpha - \beta)$ is equal to:

(1) 7 **(2)** –3 **(3)** 3 **(4)** –7

[JEE (Main) – 25th Feb. 2021 - Shift-2]

30. All the points in the set $S = \left\{\frac{\alpha+i}{\alpha-i}, \alpha \in R\right\}$, $(i = \sqrt{-1})$ lie on a :

(1) straight line whose slope is 1
(2) circle whose radius is $\sqrt{2}$
(3) straight line whose slope is –1
(4) circle whose radius is 1

[JEE (Main) – 9th April 2019 - Shift-1]

ANSWER – KEY

1. (4)	2. (3)	3. (1)	4. (2)
5. (4)	6. (2)	7. (1)	8. (4)
9. (2)	10. (3)	11. (1)	12. (3)
13. (4)	14. (3)	15. (4)	16. (1)
17. (3)	18. (2)	19. (2)	20. (3)
21. (3)	22. (3)	23. (4)	24. (3)
25. (4)	26. (2)	27. (1)	28. (4)
29. (4)	30. (4)		

ANSWERS WITH EXPLANATIONS

1. Option (4) is correct.

Since, $|1 + ai| = |b + i|$

$\Rightarrow a^2 + 1 = b^2 + 1 \Rightarrow a^2 = b^2$

And $|a + ib - 1| = |2a + 2bi||$

$\Rightarrow a^2 + 1 - 2a + b^2 = 4a^2 + 4b^2$

$\Rightarrow 3a^2 + 3b^2 + 2a - 1 = 0$

$\Rightarrow 6a^2 + 2a - 1$ $\quad (\because a^2 = b^2)$

$$\Rightarrow a = \frac{-2 \pm \sqrt{4+24}}{12}$$

$$= \frac{-2 \pm 2\sqrt{7}}{12} = \frac{-1 \pm \sqrt{7}}{6}$$

$$\therefore (a,b) \equiv \left(\frac{-1+\sqrt{7}}{6}, \frac{1-\sqrt{7}}{6}\right) \text{ or } \left(\frac{-1-\sqrt{7}}{6}, \frac{1+\sqrt{7}}{6}\right)$$

$\Rightarrow [a] = 0$

$$\frac{1+[a]}{4b} = 0 \text{ or } \frac{-(1+\sqrt{7})}{4}$$

2. Option (3) is correct.

$\Rightarrow$ Let $z_1 = p_1 + iq_1$ & $z_2 = p_2 + iq_2$

$z_1 + z_2 = (p_1 + p_2) + i(q_1 + q_2)$

Since, Re $(z_1 + z_2) = 0$

$\Rightarrow p_1 + p_2 = 0$ ...(1)

Now, $z_1z_2 = (p_1 + iq_1)(p_2 + iq_2)$

$\Rightarrow z_1z_2 = p_1p_2 + ip_1q_2 + ip_2q_1 + i^2q_1q_2$

$\Rightarrow z_1z_2 = (p_1p_2 - q_1q_2) + i(p_1q_2 + p_2q_1)$

Since, Re $(z_1z_2) = 0$

$\Rightarrow p_1p_2 - q_1q_2 = 0$

From (i) $p_2 = -p_1$

$\Rightarrow -p_1^2 - q_1q_2 = 0$

$\Rightarrow q_1q_2 = -p_1^2 < 0$, as $p_1 \in$ R & $q_1, q_2 \in$ R

Since, $q_1q_2 < 0$

$\Rightarrow$ Im (z_1) & Im (z_2) are of opposite signs.

HINT:
(1) Use $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$
(2) If $x \in R_1$ then $x^2 \geq 0$

3. Option (1) is correct.

Given that 3 consecutive terms are in the ratio 1 : 3 : 5. So

${}^{n+2}C_{r-1} : {}^{n+2}C_r : {}^{n+2}C_{r+1} = 1 : 3 : 5$

$$\Rightarrow \frac{{}^{n+2}C_{r-1}}{{}^{n+2}C_r} = \frac{1}{3} \text{ and } \frac{{}^{n+2}C_r}{{}^{n+2}C_{r+1}} = \frac{3}{5}$$

$$\Rightarrow \frac{r}{n+2-r+1} = \frac{1}{3} \text{ and } \frac{r+1}{n+2-(r+1)+1} = \frac{3}{5}$$

$\Rightarrow 3r = n - r + 3$ and $5r + 5 = 3n - 3r + 6$

or $4r = n + 3$ ...(i)

and $8r = 3n + 1$ ...(ii)

On solving eqn. (i) and (ii), we get

$r = 2$ and $n = 5$

Hence, sum of terms $= {}^7C_1 + {}^7C_2 + {}^7C_3$

$= 7 + 21 + 35 = 63$

4. Option (2) is correct.

We have $(1 + x)^p (1 - x)^q$

$$= \left(1 + px + \frac{p(p-1)}{2}x^2 + ...\right)\left(1 - qx + \frac{q(q-1)}{2}x^2 - ...\right)$$

Coefficient of x is 4

$\Rightarrow p - q = 4$ ...(i)

and coefficient of x^2 is -5

$$\Rightarrow \frac{p(p-1)}{2} - pq + \frac{q(q-1)}{2} = -5$$

or $p^2 - p - 2pq + q^2 - q + 10 = 0$ ...(ii)

on solving equation (i) and (ii), we get

$p = 15$ and $q = 11$

so $2p + 3q = 30 + 33 = 63$

5. Option (4) is correct.

If $z = x + iy$, then

$$\frac{2z-3i}{2z+i} = \frac{2x+2iy-3i}{2x+2iy+i}$$

$$\frac{2x+i(2y+3)}{2x+i(2y+1)} \times \frac{2x-i(2y+1)}{2x-i(2y+1)}$$

$$\frac{2z-3i}{2z+i} = \left(\frac{4x^2+(2y-3)(2y+1)}{4x^2+(2y+1)^2}\right) + i\left(\frac{2x(2y-3)-2x(2y+1)}{4x^2+(2y+1)^2}\right)$$

Now $\dfrac{2z-3i}{2z+i}$ is purely imaginary if

$$\frac{4x^2+(2y-3)(2y+1)}{4x^2+(2y+1)^2} = 0$$

$\Rightarrow 4x^2 + 4y^2 - 4y - 3 = 0$

but $x + y^2 = 0$ is given

so $x = -y^2$ and $x^2 = y^4$

$\Rightarrow 4y^4 + 4y^2 - 4y - 3 = 0$

or $y^4 + y^2 - y = \dfrac{3}{4}$

6. Option (2) is correct.

Given: $|z + 2| = z + 4(1 + i)$

Also, $z = \alpha + i\beta$

$\therefore |z + 2| = |\alpha + i\beta + 2| = (\alpha + i\beta) + 4 + 4i$

$\Rightarrow |(\alpha + 2) + i\beta| = (\alpha + 4) + i(\beta + 4)$

$$\Rightarrow \sqrt{(\alpha+2)^2+\beta^2} = (\alpha+4)+i(\beta+4)$$

$\Rightarrow \beta + 4 = 0 \Rightarrow \beta = -4$

Now, $(\alpha + 2)^2 + \beta^2 = (\alpha + 4)^2$

$\Rightarrow \alpha^2 + 4 + 4\alpha + \beta^2 = \alpha^2 + 16 + 8\alpha$

$\Rightarrow 4 + 4\alpha + 16 = 16 + 8\alpha$

$\Rightarrow 4\alpha = 4 \Rightarrow \alpha = 1$

So, $\alpha + \beta = -3$ and $\alpha\beta = -4$

$\therefore$ Required equation is

$x^2 - (-3 - 4)x + (-3)(-4) = 0$

$\Rightarrow x^2 + 7x + 12 = 0$

7. Option (1) is correct.

Given $z = 2 + 3i$ and $\bar{z} = 2 - 3i$

$\Rightarrow z^5 + \left(\bar{z}\right)^5 = (2 + 3i)^5 + (2 - 3i)^5$

By Binomial Expression

$(x + a)^n = {}^nC_o x^n a^0 + {}^nC_1 x^n a + \ldots\ldots\ldots.$

$\Rightarrow z^5 + (\bar{z})^5$

$= {}^5C_0 2^5 (3i)^0 + {}^5C_1 2^4 (3i)^1 + {}^5C_2 2^3 (3i)^2 + {}^5C_3 2^2 (3i)^3$

$+ \ldots\ldots + {}^5C_0 2^5 (3i)^0 + {}^5C_1 2^4 (-3i)^1 + {}^5C_2 2^3 (-3i)^2 -$

$+ {}^5C_3 2^2 (-3i)^3 + \ldots\ldots$

$= 2\left({}^5C_0 2^5 (3i)^0 + {}^5C_2 2^3 (3i)^2 + \ldots\ldots\right)$

$= 2\left({}^5C_0 2^5 (3i)^0 + {}^5C_2 2^3 (3i)^2 + {}^5C_4 2.(3i)^4\right)$

$= 2(32 - 10 \times 8 \times 9 + 5 \times 2 \times 27 \times 3)$

$= 2(32 - 80 \times 9 + 10 \times 81) = 244$

8. Option (4) is correct.

Given, $\left|z - \frac{1}{z}\right| = 2$

Let $|z| = r$ represent circle of centre origin and radius r.

$\therefore \left||z| - \frac{1}{|z|}\right| \le \left|z - \frac{1}{z}\right| \le |z| + \frac{1}{|z|}$

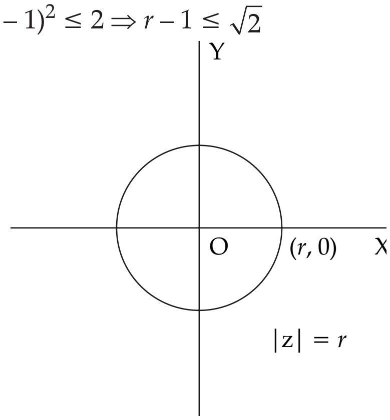
$\Rightarrow \left|r - \frac{1}{r}\right| \le 2 \le r + \frac{1}{r}$

$\Rightarrow \left|r - \frac{1}{r}\right| \le 2$ and $r + \frac{1}{r} \ge 2$ always true.

$\Rightarrow \left|r - \frac{1}{r}\right| \le 2$ and $r - \frac{1}{r} \le 2$

$\Rightarrow r^2 - 1 \le 2r \Rightarrow r^2 - 2r \le 1$

$\Rightarrow (r - 1)^2 \le 2 \Rightarrow r - 1 \le \sqrt{2}$

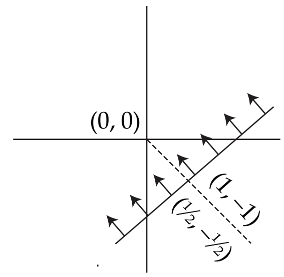


$\therefore \quad |Z|_{max} = 1 + \sqrt{2}$

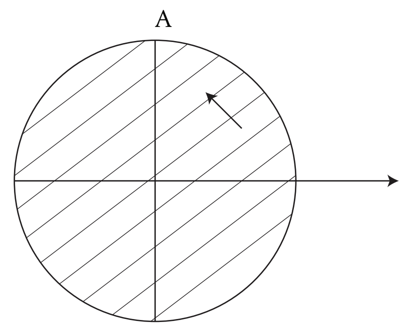
9. Option (2) is correct.

$|z - 1 + i| \ge |z|,\ |z| < 2,\ |z + i| = |z - 1|$

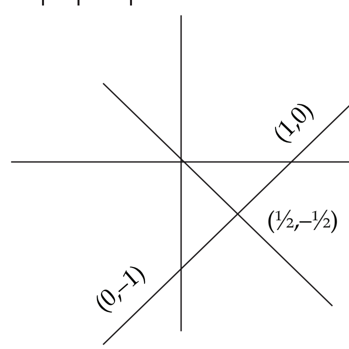
(i) $|z-1+i| \ge |z|$

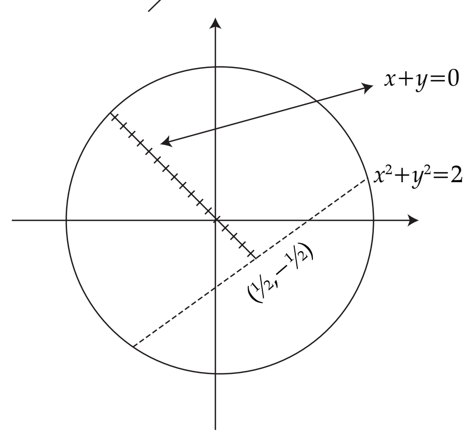


(ii) $|z| < 2$



(iii) $|z+i| = |z-1|$





$\therefore w = 2x + iy \in S$

$\Rightarrow 2x \le \frac{1}{2} \quad \therefore x \le \frac{1}{4}$

And $(2x)^2 + (-2x)^2 = 4$

$\therefore x^2 < \frac{1}{2} \Rightarrow x \in \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\therefore \ x \in \left(\frac{-1}{\sqrt{2}}, \frac{1}{4}\right]$

10. Option (3) is correct.

Given $\left|z_2-|z_2+1|\right|=\left|z_2+|z_2-1|\right|$

$\Rightarrow \left|z_2-|z_2+1|\right|^2=\left|z_2+|z_2-1|\right|^2$

$\Rightarrow \left(z_2-|z_2+1|\right)\left(\bar{z}_2-|z_2+1|\right)$

$=\left(z_2+|z_2-1|\right)\left(\bar{z}_2+|z_2-1|\right)$

$\Rightarrow z_2\bar{z}_2-z_2|z_2+1|-\bar{z}_2|z_2+1|-z_2\bar{z}_2-z_2|z_2-1|$
$-\bar{z}_2|z_2-1|$

$=|z_2-1|^2-|z_2+1|^2$

$\Rightarrow z_2\left(|z_2+1|+|z_2-1|\right)+\bar{z}_2\left(|z_2+1|+|z_2-1|\right)$

$=|z_2+1|^2-|z_2-1|^2$

$\Rightarrow \left(z_2+\bar{z}_2\right)\left(|z_2+1|+|z_2-1|\right)=|z_2+1|^2-|z_2-1|^2$

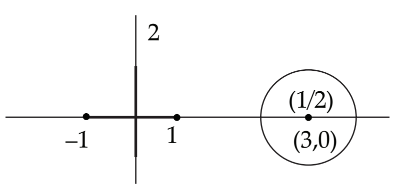
$\Rightarrow \left(z_2+\bar{z}_2\right)\left(|z_2+1|+|z_2-1|\right)=2\left(z_2+\bar{z}_2\right)$

$\Rightarrow \left(z_2+\bar{z}_2\right)\left(|z_2+1|+|z_2-1|-2\right)=0$

$\Rightarrow z_2+\bar{z}_2=0$ or $|z_2+1|+|z_2-1|=2$

Hence, z_2 lies on imaginary axis or on real axis under [–1,2].

Also $|z-3|=\dfrac{1}{2}$, so z, is circle with centre (3, 0) and radius $=\dfrac{1}{2}$



Clearly, min $|z_1-z_2|=\dfrac{3}{2}$

11. Option (1) is correct.

Let $z=x+iy$

$\therefore v=|z|^2+|z-3|^2+|z-6i|^2$

$=x^2+y^2+(x-3)^2+y^2+x^2+(y-6)^2$

$=3x^2+3y^2-6x-12y+45$

$=3[x^2-2x+y^2-4y+15]$

$=3[x^2-2x+1+y^2-4y+4+10]$

$=3[(x-1)^2+(y-2)^2+10]$

$\therefore v$ is minimum when $x=1$ $y=2$

$\therefore z_0=1+2i$ and $v_0=30$

Now, $|2z_0^2-z_0^3+3|^2+v_0^2$

$=|2(1+2i)^2-(1-2i)^3+3|^2+(30)^2$

$=|-6+8i-(1-6i-12+8i)+3|^2+900$

$=|-3+8i+11-2i|^2+900$

$=|8+6i|^2+900=64+36+900=1000$

12. Option (3) is correct.

$\dfrac{1-i\sin\alpha}{1+2i\sin\alpha}=\dfrac{1-i\sin\alpha}{1+2i\sin\alpha}\times\dfrac{1-2i\sin\alpha}{1-2i\sin\alpha}$

$=\dfrac{1-2\sin^2\alpha}{1+4\sin^2\alpha}-\dfrac{3i\sin\alpha}{1+4\sin^2\alpha}$ is purely imaginary

$\Rightarrow \dfrac{1-2\sin^2\alpha}{1+4\sin^2\alpha}=0\Rightarrow 1-2\sin^2\alpha=0$

$\Rightarrow \sin\alpha=\pm\dfrac{1}{\sqrt{2}}$

$\therefore \alpha=\dfrac{5\pi}{4},\dfrac{7\pi}{4}\ (\because \pi<\alpha<2\pi)$

And $\dfrac{1+i\cos\beta}{1-2i\cos\beta}=\dfrac{1+i\cos\beta}{1-2i\cos\beta}\times\dfrac{1+2i\cos\beta}{1+2i\cos\beta}$

$=\dfrac{1-2\cos^2\beta}{1+4\cos^2\beta}+\dfrac{3i\cos\beta}{1+4\cos^2\beta}$ is purely real

$\Rightarrow \dfrac{3\cos\beta}{1+4\cos^2\beta}=0$

$\Rightarrow \cos\beta=0$

$\therefore \beta=\dfrac{3\pi}{2}\ [\because \pi<\beta<2\pi]$

$\therefore S=\left\{\left(\dfrac{5\pi}{4},\dfrac{3\pi}{2}\right),\left(\dfrac{7\pi}{4},\dfrac{3\pi}{2}\right)\right\}$

$Z_{\alpha\beta}=\sin2\alpha+i\cos2\beta=1-i$ or $Z_{\alpha\beta}=-1-i$

Now, $\Sigma_{(\alpha,\beta)\in s}\left(iz_{\alpha\beta}+\dfrac{1}{i\bar{z}_{\alpha\beta}}\right)$

$=i+1+\dfrac{1}{-1+i}+1-i+\dfrac{1}{-1-i}$

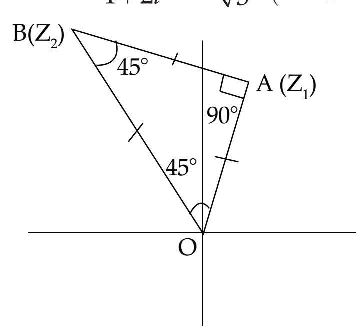
$=2-\left(\dfrac{1}{1-i}+\dfrac{1}{1+i}\right)=2-\left(\dfrac{2}{1+1}\right)=2-1=1$

13. Option (4) is correct.

$|OA|=\sqrt{5}=|AB|$ and $|OB|=\sqrt{5+5}=\sqrt{10}$

Now, $\dfrac{z_2-0}{z_1-0}=\dfrac{|OB|}{|OA|}e^{\frac{i\pi}{4}}$

$\Rightarrow \dfrac{z_2}{1+2i}=\dfrac{\sqrt{10}}{\sqrt{5}}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$



$\Rightarrow \dfrac{z_2}{1+2i}=\sqrt{2}\left(\dfrac{1}{\sqrt{2}}+i\dfrac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad z_2 = (1 + 2i)(1 + i) = -1 + 3i$

$\Rightarrow \quad \arg z_2 = \pi - \tan^{-1}3$

Option (1) is correct

Now, $z_1 - 2z_2 = (1 + 2i) - 2(-1 + 3j)$

$= 3 - 4i$

$\Rightarrow \arg (z_1 - 2z_2) = -\tan^{-1}\frac{4}{3}$

$\because$ Option (2) is correct.

$\therefore |z_2| = |OB| = \sqrt{10}$

$\therefore$ Option (3) is true.

Now, $2z_1 - z_2 = 2(1 + 2i) - (-1 + 3i)$

$= 3 + i$

$\therefore \quad |2z_1 - z_2| = \sqrt{9+1} = \sqrt{10}$

$\therefore$ Option (4) is not true.

14. Option (3) is correct.

Given that $|z| - 2 = 0$

$|z| = 2 \Rightarrow x^2 + y^2 = 4$ ...(1)

And $|z-i| - |z+5i| = 0$

$|z-i|^2 = |z+5i|^2$

$\Rightarrow x^2 + (y-1)^2 = x^2 + (y+5)^2$

$\Rightarrow y^2 - 2y + 1 = y^2 + 10y + 25$

$\Rightarrow -12y = 24$

$\Rightarrow y = -2$

Putting in (i), we get

$x = 0$, which satisfy the equation $x + 2y + 4 = 0$

15. Option (4) is correct.

Given, $S_n = |z - 3 + 2i| = \frac{n}{4}$

And $T_n : |z - 2 + 3i| = \frac{1}{n}$

$\Rightarrow \quad S_n : (x-3)^2 + (y+2)^2 = \left(\frac{n}{4}\right)^2$

and $T_n : (x-2)^2 + (y+3)^2 = \left(\frac{1}{n}\right)^2$

For $S_1 \cap S_2 = 5$

$c_1c_2 < |r_1 - r_2|$

or $c_1c_2 > r_1 + r_2$

$\Rightarrow \quad \sqrt{2} > \frac{n}{4} + \frac{1}{n}$

or $\sqrt{2} < \left|\frac{n}{4} - \frac{1}{n}\right|$

$\Rightarrow \quad n \in \{1, 2, 3, 4\}$

16. Option (1) is correct.

Let $z = x + iy, x, y \in R$

Given that $\bar{z} = iz^2$

$\Rightarrow \quad x - iy = i(x + iy)^2$

$\Rightarrow \quad x - iy = i(x^2 - y^2 + 2ixy)$

$= -2xy + i(x^2 - y^2)$

On comparing both sides we get,

$x = -2xy$ ...(i)

and $y = (y^2 - x^2)$ ...(ii)

From (i), $x + 2xy = 0$

$\Rightarrow \quad x(1 + 2y) = 0$

$\Rightarrow \quad x = 0$

or $1 + 2y = 0$

$\Rightarrow \quad y = -\frac{1}{2}$

If $x = 0$ then from (ii) we get,

$y - y^2 = 0$

$\Rightarrow \quad y = 0$ or 1

$\therefore \quad z = 0 + 0\,i$ and $0 + i$

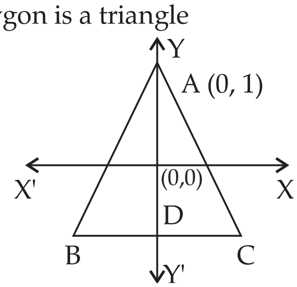
When $y = -\frac{1}{2}$, then from (ii)

$-\frac{1}{2} = \frac{1}{4} - x^2$

$\Rightarrow \quad x = \pm\frac{\sqrt{3}}{2}$

$\therefore \quad z = \pm\frac{\sqrt{3}}{2} - \frac{1}{2}i$

The polygon is a triangle



So, vertices of triangle are

A (0, 1), B$\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$ and C$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

$\therefore \quad$ Area of $\Delta ABC = \frac{1}{2} BC \times AD$

$= \frac{1}{2} \times \sqrt{3} \times \frac{3}{2} = \frac{3\sqrt{3}}{4}$

17. Option (3) is correct.

Let $z = x + iy$

Given, $|z - (4 + 3i)| = 2$

$\Rightarrow \quad (x-4)^2 + (y-3)^2 = 4$

It represent equation of circle

Since, $|z| + |z - 4| = 6$ (Given)

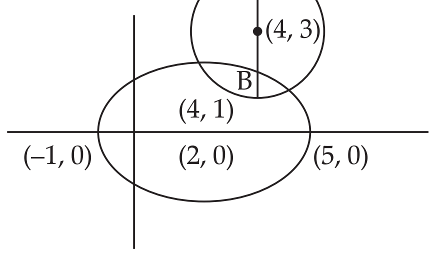
$\Rightarrow \sqrt{x^2 + y^2} + \sqrt{(x-4)^2 + y^2} = 6$

By simplification, we get

$5x^2 - 20x + 9y^2 = 25$

$\Rightarrow \quad \frac{(x-2)^2}{9} + \frac{y^2}{5} = 1$

It represent equation of ellipse



AB is vertical diameter of circle where vertex of B is (4, 1)

Puting the coordinates of B in equation of ellipse, we get

$$\frac{(4-2)^2}{9}+\frac{1}{5}-1 = \frac{4}{9}+\frac{1}{5}-1 = \frac{-16}{45}<0$$

$\therefore$ B (4, 1) lies inside the ellipse. So circle intersect the ellipse at two points.

18. Option (2) is correct.

$$4ix + (1+i)\,y = 0 \quad \text{...(i)}$$

$$8\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)+\bar{a}y = 0$$

$$8\left(\frac{-1}{2}+\frac{i\sqrt{3}}{2}\right)+\bar{a}y = 0 \quad \text{...(ii)}$$

Given that equation (i) and (ii) has more than one solution

$$\therefore \begin{vmatrix} 4i & 1+i \\ 8\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) & \bar{a} \end{vmatrix} = 0$$

$$\Rightarrow 4i\bar{a}-8(1+i)\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) = 0$$

$$[\because a=\alpha-i\beta \Rightarrow \bar{a}=\alpha+i\beta]$$

$$\Rightarrow \quad i(\alpha+i\beta)-(1+i)(-1+i\sqrt{3}) = 0$$

$$\Rightarrow \quad \alpha i-\beta+1+i-i\sqrt{3}+\sqrt{3} = 0$$

$$\Rightarrow (-\beta+1+\sqrt{3})+i\,(\alpha+1-\sqrt{3}) = 0$$

$$\Rightarrow -\beta+1+\sqrt{3} = 0$$

$$\Rightarrow \beta = \sqrt{3}+1$$

And $\alpha+1-\sqrt{3} = 0$

$$\Rightarrow \alpha = \sqrt{3}-1$$

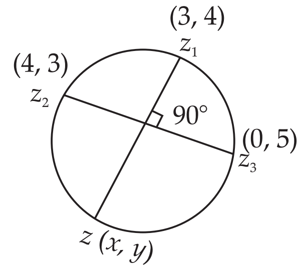
Now, $\dfrac{\alpha}{\beta} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$

19. Option (2) is correct.

Let C be a circle in complex plane passes through the points $z_1 = 3 + 4i$ i.e., (3, 4)

$z_2 = 4 + 3i$ i.e., (4, 3) and $z_3 = 5i$ i.e., (0, 5)

$z = x + iy$ be any complex number



$$C : |z| = 5, x^2 + y^2 = 25 \quad \text{...(i)}$$

$$m_1 \times m_2 = -1 \quad \text{...(ii)}$$

$$\Rightarrow \left(\frac{4-y}{3-x}\right)\left(\frac{5-3}{0-4}\right) = -1$$

$$\Rightarrow \left(\frac{4-y}{3-x}\right)\left(\frac{2}{-4}\right) = -1 \Rightarrow \frac{4-y}{3-x}\left(\frac{-1}{2}\right) = -1$$

$$\Rightarrow (4-y) = 2(3-x)$$

$$\Rightarrow 4-y = 6-2x$$

$$L : 2x-y = 2$$

Now, z is intersection of C and L

$x^2 + y^2 = 25 \qquad y = 2x-2$

$\Rightarrow x^2 + (2a-2)^2 = 25$

$\Rightarrow x^2 + 4(x^2 + 1-2x) = 25$

$\Rightarrow x^2 + 4x^2 + 4-8x = 25$

$\Rightarrow 5x^2-8x = 21$

$\Rightarrow 5x^2-8x-21 = 0$

$\Rightarrow 5x^2-15x + 7x-21 = 0$

$\Rightarrow 5x(x-3) + 7(x-3) = 0$

$\Rightarrow (x-3)(5x + 7) = 0$

$$\Rightarrow x = \frac{-7}{5} \quad x = 3$$

At $x = 3, y = 4, z(3, 4)$ and $x = \dfrac{-2}{5}, y = \dfrac{2\times7-2}{5}$,

$z\left(\dfrac{-7}{5}, \dfrac{-24}{5}\right)$ which lies in IV quadrant.

Then $\quad \text{Arg}(z) = -\pi+\tan^{-1}\left(\dfrac{\frac{+24}{5}}{\frac{7}{5}}\right)$

$$= -\pi+\tan^{-1}\left(\frac{24}{7}\right)$$

20. Option (3) is correct.

z_1 and z_2 be two complex numbers

S.t

$\Rightarrow z_1 = -iz_2$ ...(i)

Given $\text{Arg}\left(\dfrac{z_1}{\bar{z}_2}\right) = \pi$

$$\Rightarrow \quad \text{Arg}\left(\frac{-iz_2}{\bar{z}_2}\right) = \pi$$

We have, $\text{Arg}(z_1 \cdot z_2) = \text{Arg } z_1 + \text{Arg } z_2$

$$\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg } z_1 - \text{Arg } z_2$$

$$\text{Arg}(-i) + \text{Arg}\left(\frac{z_1}{\bar{z}_2}\right) = \pi$$

$$-\frac{\pi}{2} + \text{Arg } z_1 - \text{Arg}\left(\bar{z}_2\right) = \pi$$

$$-\frac{\pi}{2} + \theta + \theta = \pi$$

Let $\text{Arg } z_1 = \theta$

$\Rightarrow \text{Arg } \bar{z}_2 = -\theta$

$$\Rightarrow \quad 2\theta = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{3\pi}{4}$$

$$\text{Arg } z_2 = \frac{3\pi}{4}$$

$$\text{Arg}\left(\frac{z_1}{\bar{z}_2}\right) = \text{Arg } z_1 - \text{Arg } \bar{z}_2 = \pi$$

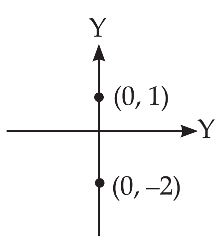
$$\Rightarrow \quad \text{Arg } z_1 + \frac{3\pi}{4} = \pi$$

$$\therefore \quad \text{Arg } z_1 = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$$

$$\therefore \quad \text{Arg } z_1 = \frac{\pi}{4}$$

21. Option (3) is correct.

$$\because \frac{z-i}{z+2i} \in R \;\&\; z \in C$$



$$\therefore \text{ arg}\left(\frac{z-2}{z+2i}\right) = 0 \text{ or } \pi$$

$\Rightarrow z$ lies on y-axis

$\Rightarrow$ S is straight line in complex plane

22. Option (3) is correct.

Let $z = x + iy$

$$\text{So, } \frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1}$$

$$\Rightarrow \frac{z-1}{z+1} = \left(\frac{x+iy-1}{x+iy+1}\right) \times \left(\frac{(x+1)-iy}{(x+1)-iy}\right)$$

$$= \frac{(x+1)(x-1) - iy(x-1) + iy(x+1) - i^2y^2}{(x+1)^2 + y^2}$$

$$\Rightarrow \frac{z-1}{z+1} = \frac{(x^2+y^2-1) + i(xy+y-xy+y)}{(x+1)^2+y^2}$$

$$\Rightarrow \frac{z-1}{z+1} = \frac{(x^2+y^2-1) + 2iy}{(x+1)^2+y^2}$$

$$\text{Given, arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{2y}{x^2+y^2-1} = 1$$

$\Rightarrow x^2 + y^2 - 1 = 2y$

$\Rightarrow x^2 + y^2 - 2y - 1 = 0$

$\Rightarrow (x-0)^2 + (y-1)^2 = (\sqrt{2})^2$

So, centre (0, 1) & radius = $\sqrt{2}$ units

23. Option (4) is correct.

We have, $(\sqrt{3}+i)^{100} = 2^{99}(p + iq)$

$$\Rightarrow 2^{100}\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]^{100} = 2^{99}(p+iq)$$

$$\Rightarrow 2^{100} \cdot e^{i\frac{50\pi}{3}} = 2^{99}(p+iq)$$

$$\Rightarrow 2\left[\cos\frac{50\pi}{3} + i\sin\frac{50\pi}{3}\right) = p + iq$$

$$\Rightarrow 2\left(\frac{-1}{2} + \frac{i\sqrt{3}}{2}\right) = p + iq$$

$\Rightarrow p = -1 \;\&\; q = \sqrt{3}$

$\therefore p$ and q are roots of the equation

$$x^2 - (\sqrt{3} - 1)x - \sqrt{3} = 0$$

24. Option (3) is correct.

Let $z = x + iy$

So, $(x + iy)^2 + 3(x - iy) = 0$

$\Rightarrow x^2 - y^2 + 2ixy + 3x - 3iy = 0$

$\Rightarrow x^2 - y^2 + 3x = 0$ and $2xy - 3y = 0$

Case 1: $y = 0$

$\Rightarrow x^2 - y^2 + 3x = 0$

$\Rightarrow x = 0$ or $x = -3$

Solutions are $z = 0$ and $z = -3$

Case 2: $x = \frac{3}{2}$

$\Rightarrow \quad x^2 - y^2 + 3x = 0$

$\Rightarrow \quad y = \dfrac{3\sqrt{3}}{2}$ or $y = \dfrac{-3\sqrt{3}}{2}$

Solutions are $z = \dfrac{3}{2} + i\dfrac{3\sqrt{3}}{2}$ and

$z = \dfrac{3}{2} - i\dfrac{3\sqrt{3}}{2}$

Total number of solutions $= n = 4$

So, $\sum_{k=0}^{\infty} \dfrac{1}{4^k} = \dfrac{1}{1-\frac{1}{4}} = \dfrac{4}{3}$

25. Option (4) is correct.

Given, $|z\omega| = 1$ and $\arg(z) - \arg(\omega) = \dfrac{3\pi}{2}$

$\Rightarrow |z||\omega| = 1$ and $\arg(z) = \arg(\omega) + \dfrac{3\pi}{2}$

Let $\omega = re^{i\theta} \Rightarrow z = \dfrac{1}{r}e^{i\left(\frac{3\pi}{2}+\theta\right)}$

$$\Rightarrow \quad \frac{1-2\bar{z}\omega}{1+3\bar{z}\omega} = \frac{1-2re^{i\theta}\frac{1}{r}e^{i\left(-\frac{3\pi}{2}-\theta\right)}}{1+3re^{i\theta}\frac{1}{r}e^{i\left(-\frac{3\pi}{2}-\theta\right)}}$$

$$= \frac{1-2e^{-i3\pi/2}}{1+3e^{-i3\pi/2}} = \frac{1-2i}{1+3i}$$

So, $\arg\left(\dfrac{1-2\bar{z}\omega}{1+3\bar{z}\omega}\right) = \arg\left(\dfrac{1-2i}{1+3i}\right)$

$= \tan^{-1}(-2) - \tan^{-1}(3) = \dfrac{\pi}{4}$

26. Option (2) is correct.

We have to identify the condition where $a|z|^2 + \overline{\overline{\alpha z + \alpha \bar{z}}} + d = 0$ represents a circle where a, d are real constants.

We can rewrite the given equation of circle,

$$a|z|^2 + \overline{\overline{\alpha z} + \alpha\bar{z}} + d = 0$$

$$a|z|^2 + \bar{\bar{\alpha}}\bar{z} + \bar{\alpha}\bar{\bar{z}} + d = 0$$

$$a|z|^2 + \alpha\bar{z} + \bar{\alpha}z + d = 0$$

Dividing both sides by 'a' we get

$$|z|^2 + \left(\frac{\alpha}{a}\right)\bar{z} + \left(\frac{\bar{\alpha}}{a}\right)z + \frac{d}{a} = 0$$

We also know that

$$z\bar{z} = |z|^2$$

$$\bar{a} = a$$

$$\bar{d} = d$$

Thus, simplifying above equation further,

$$z\bar{z} + \left(\frac{\alpha}{a}\right)\bar{z} + \left(\frac{\bar{\alpha}}{a}\right)z + \frac{d}{a} = 0$$

We know that if we have equation of circle in complex plane $z\bar{z} + b\bar{z} + \bar{b}z + c = 0$ then its center and radius are given as:

Centre $= -b$

Radius, $r = \sqrt{|b|^2 - c}$

Comparing with given equation

Centre $= \dfrac{-\alpha}{a}$

Radius, $r = \sqrt{\left|\dfrac{\alpha}{a}\right|^2 - \dfrac{d}{a}}$

We know the condition for square root,

$$\left|\frac{\alpha}{a}\right|^2 - \frac{d}{a} \geq 0$$

If $r = 0$ then circle will not exist therefore,

$$\left|\frac{\alpha}{a}\right|^2 - \frac{d}{a} > 0$$

$$\Rightarrow \left|\frac{\alpha}{a}\right|^2 > \frac{d}{a} \Rightarrow \frac{|\alpha|^2}{a^2} \times a^2 > \frac{d}{a} \times a^2$$

$\Rightarrow \quad |\alpha|^2 > ad$

$\Rightarrow \quad |\alpha|^2 - ad > 0$

Since in $\dfrac{\alpha}{a}$, a cannot be zero

therefore $a \in R - \{0\}$

Hint:
Use the property of conjugate and terminologies of equation of circle in complex plane.

Shortcut Method:

$$a|z|^2 + \alpha\bar{z} + \bar{\alpha}z + d = 0$$

$$z\bar{z} + \left(\frac{\alpha}{a}\right)\bar{z} + \left(\frac{\bar{\alpha}}{a}\right)z + \frac{d}{a} = 0$$

Centre $= -\dfrac{\alpha}{a}$, $a \in R - \{0\}$

$$r = \sqrt{\left|\frac{\alpha}{a}\right|^2 - \frac{d}{a}}$$

$$\left|\frac{\alpha}{a}\right|^2 - \frac{d}{a} > 0$$

$$|\alpha|^2 - ad > 0$$

27. Option (1) is correct.

Given

Complex number $\omega = 1 - \sqrt{3}i, |z\omega| = 1$ and arg(z) − arg(ω) $= \frac{\pi}{2}$.

If complex number $a = x + iy$ then

$$|a| = \sqrt{x^2 + y^2}$$

$$\therefore \quad |\omega| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3}$$

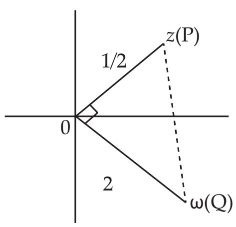
$$\Rightarrow \quad |\omega| = 2 \quad \ldots\text{(i)}$$

$$|z\omega| = 1$$

$$\Rightarrow \quad |z||\omega| = 1 \quad (\because |xy| = |x||y|)$$

$$\Rightarrow \quad |z| = \frac{1}{2} \quad \ldots\text{(ii)}$$

$$\arg(z) - \arg(\omega) = \frac{\pi}{2}$$



Area of ΔOPQ where O is origin and P, Q are z and ω respectively.

$$\therefore \quad \text{Area} = \frac{1}{2}|z||\omega| = \frac{1}{2} \times \frac{1}{2} \times 2 = \frac{1}{2}$$

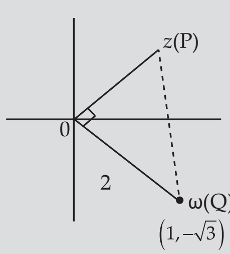
Hint:

(i) Find the modulus of complex number ω using $\sqrt{(\text{Real})^2 + (\text{Imag.})^2} = |\omega|$

(ii) Find $|z|$ using given relation $|z\omega| = 1$

(iii) Use area of right angled triangle to find required area as angle between OP and OQ is given as $\frac{\pi}{2}$.

Shortcut Method:



$$|z||\omega| = 1$$

$$\Rightarrow \quad |z| = \frac{1}{2}$$

$$\text{Area of } \Delta OPQ = \frac{1}{2} \times 2 \times \frac{1}{2} = \frac{1}{2}$$

28. Option (4) is correct.

Given

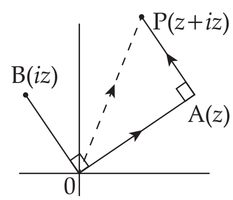
Triangle with vertices A(z), B(iz) and C($z + iz$)

Take origin as O

$$\overrightarrow{OA} = z - O$$

$$\overrightarrow{OB} = iz - O$$

$\overrightarrow{OB}$ is formed by rotating $\overrightarrow{OA}$ about origin by a right angle.



Draw $\overrightarrow{AP}$ parallel to $\overrightarrow{OB}$

Resultant of $\overrightarrow{OA}$ and $\overrightarrow{AP}$ is $z + iz$.

So, we need to find area of right angled triangle OAP.

$$\text{Area} = \frac{1}{2} \times \left|\overrightarrow{OA}\right| \times \left|\overrightarrow{AP}\right| = \frac{1}{2}|z||iz|$$

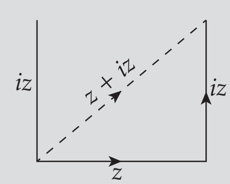
$$= \frac{1}{2}|z||i||z| = \frac{1}{2}|z|^2 \quad (\because |i| = 1)$$

Hint:

(i) Draw vectorial diagram.

(ii) Find area of right angle formed.

Shortcut Method:



$$\text{Area} = \frac{1}{2}|z|^2$$

29. Option (4) is correct.

Given: $\alpha, \beta \in R$, $1 - 2i$ is a root of

$$z^2 + \alpha z + \beta = 0$$

$$(1 - 2i) \text{ is a root of } z^2 + \alpha z + \beta = 0$$

$$\therefore \quad (1 - 2i)^2 + \alpha(1 - 2i) + \beta = 0$$

$$\Rightarrow \quad 1 + 4i^2 - 4i + \alpha - 2i\alpha + \beta = 0$$

$\Rightarrow \quad 1 - 4 - 2i(\alpha + 2) + \alpha + \beta = 0$

$\Rightarrow \quad \alpha + \beta - 3 - 2i(\alpha + 2) = 0$

Compare real and imaginary parts

$\alpha + \beta - 3 = 0$ and $\alpha + 2 = 0$

$\Rightarrow \alpha = -2$

$\therefore -2 + \beta - 3 = 0$

$\Rightarrow \beta = 5$

$\therefore \quad \alpha - \beta = -2 - 5$

$\Rightarrow \quad \alpha - \beta = -7$

Hint:

(i) $i^2 = -1$

(ii) $(a + b)^2 = a^2 + b^2 + 2ab$

Shortcut Method:

$\because \ \alpha, \beta \in R$

$\Rightarrow$ other root must be $1 + 2i$

$\alpha = -$ (sum of roots)

$= -(1 - 2i + 1 + 2i) = -2$

$\beta =$ product of roots

$= (1 - 2i)(1 + 2i) = 1 + 4 = 5$

$\therefore \quad \alpha - \beta = -7$

30. Option (4) is correct.

Let $\quad z = \dfrac{\alpha + i}{\alpha - i}$

$\Rightarrow \quad |z| = \left|\dfrac{\alpha + i}{\alpha - i}\right|$

$\Rightarrow \quad |z| = \dfrac{|\alpha + i|}{|\alpha - i|}$

$\Rightarrow \quad |z| = \dfrac{\alpha^2 + 1}{\alpha^2 + 1}$

$\Rightarrow \quad |z| = 1$

Circle of radius 1.

Topic-2
Quadratic Equations

JEE (Main) Previous Year Questions

Multiple Choice Questions

1. Let $\lambda \neq 0$ be a real number. Let α, β be the roots of the equation $14x^2 - 31x + 3\lambda = 0$ and α, γ be the roots of the equation $35x^2 - 53x + 4\lambda = 0$. Then $\dfrac{3\alpha}{\beta}$ and $\dfrac{4\alpha}{\gamma}$ are the roots of the equation

(1) $49x^2 - 245x + 250 = 0$

(2) $7x^2 + 245x - 250 = 0$

(3) $7x^2 - 245x + 250 = 0$

(4) $49x^2 + 245x + 250 = 0$

[JEE (Main) – 29th January 2023 - Shift-1]

2. The sum of all the roots of the equation $|x^2 - 8x + 15| - 2x + 7 = 0$ is :

(1) $11 - \sqrt{3}$ **(2)** $9 - \sqrt{3}$ **(3)** $9 + \sqrt{3}$ **(4)** $11 + \sqrt{3}$

[JEE (Main) – 6th April 2023 - Shift-1]

3. Let α, β be the roots of the equation $x^2 - \sqrt{2}x + \sqrt{6} = 0$ and $\dfrac{1}{\alpha^2} + 1, \dfrac{1}{\beta^2} + 1$ be the roots of the equation $x^2 + ax + b = 0$. Then the roots of the equation $x^2 - (a + b - 2)x + (a + b + 2) = 0$ are :

(1) non − real complex numbers

(2) real and both negative

(3) real and both positive

(4) real and exactly one of them is positive

[JEE (Main) – 28th July 2022 - Shift-2]

4. If α, β are the roots of the equation

$$x^2 - \left(5 + 3\sqrt{\log_3^5} - 5\sqrt{\log_5^3}\right)x + 3\left(3(\log_3^5)^{\frac{1}{3}} - 5(\log_5^3)^{\frac{2}{3}-1} - 1\right) = 0,$$

then the equation, whose roots are $\alpha + \dfrac{1}{\beta}$ and $\beta + \dfrac{1}{\alpha}$ is

(1) $3x^2 - 20x - 12 = 0$ **(2)** $3x^2 - 10x - 4 = 0$

(3) $3x^2 - 10x + 2 = 0$ **(4)** $3x^2 - 20x + 16 = 0$

[JEE (Main) – 27th July 2022 - Shift-2]

5. The minimum value of the sum of the squares of the roots of $x^2 + (3 - a)x + 1 = 2a$ is :

(1) 4 **(2)** 5 **(3)** 6 **(4)** 8

[JEE (Main) – 26th July 2022 - Shift-2]

6. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + x^3 + x^2 + x + 1 = 0$, then $\alpha^{2021} + \beta^{2021} + \gamma^{2021} + \delta^{2021}$ is equal to :

(1) –4 **(2)** –1 **(3)** 1 **(4)** 4

[JEE (Main) – 25th July 2022 - Shift-1]

7. Let $f(x)$ be a quadratic polynomial such that $f(-2) + f(3) = 0$. If one of the roots of $f(x) = 0$ is -1, then the sum of the roots $f(x) = 0$ is equal to:

(1) $\frac{11}{3}$ **(2)** $\frac{7}{3}$ **(3)** $\frac{13}{3}$ **(4)** $\frac{14}{3}$

[JEE (Main) – 28th June 2022 - Shift-2]

8. Let $a, b \in$ R be such that the equation $ax^2 - 2bx + 15 = 0$ has a repeated root α . If α and β are the roots of the equation $x^2 - 2bx + 21 = 0$, then $\alpha^2 + \beta^2$ is equal to:

(1) 37 **(2)** 58 **(3)** 68 **(4)** 92

[JEE (Main) – 25th June 2022 - Shift-2]

9. If the sum of the squares of the reciprocal of the roots α and β of the equation $3x^2 + \lambda x - 1 = 0$ is 15, then $6(\alpha^3 + \beta^3)$ is equal to:

(1) 18 **(2)** 24 **(3)** 36 **(4)** 96

[JEE (Main) – 24th June 2022 - Shift-1]

10. The sum of all the real roots of the equation $(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$ is

(1) $\log_e 3$ **(2)** $-\log_e 3$ **(3)** $\log_e 6$ **(4)** $-\log_e 6$

[JEE (Main) – 24th June 2022 - Shift-2]

11. The number of pairs (a, b) of real numbers, such that whenever α is a root of the equation $x^2 + ax + b - 0$, $\alpha^2 - 2$ is also a root of this equation is:

(1) 6 **(2)** 4 **(3)** 8 **(4)** 2

[JEE (Main) – 1st Sep. 2021 - Shift-2]

12. The sum of the roots of the equation, $x + 1 - 2\log_2(3 + 2^x) + 2\log_4(10 - 2^{-x}) = 0$, is:

(1) $\log_2 12$ **(2)** $\log_2 14$ **(3)** $\log_2 11$ **(4)** $\log_2 13$

[JEE (Main) – 31st Aug. 2021 - Shift-2]

13. The set of all value of $k > -1$, for which the equation $(3x^2 + 4x + 3)^2 - (k + 1)(3x^2 + 4x + 3)(3x^2 + 4x + 2) + k(3x^2 + 4x + 2)^2 = 0$ has real roots, is:

(1) $\left[-\frac{1}{2}, 1\right)$ **(2)** $[2, 3)$

(3) $\left(1, \frac{5}{2}\right]$ **(4)** $\left(\frac{1}{2}, \frac{3}{2}\right] - \{1\}$

[JEE (Main) – 27th Aug. 2021 - Shift-2]

14. Let α, β be two roots of the equation $x^2 + (20)^{\frac{1}{4}}x + (5)^{\frac{1}{2}} = 0$. Then $\alpha^8 + \beta^8$ is equal to:

(1) 10 **(2)** 50 **(3)** 160 **(4)** 100

[JEE (Main) – 27th July 2021 - Shift-1]

15. Let $a = \max\limits_{x \in R}\{8^{2\sin 3x}\ 4^{4\cos 3x}\}$ and

$\beta = \min\limits_{x \in R}\{8^{2\sin 3x}\ 4^{4\cos 3x}\}$

If $8x^2 + bx + c = 0$ is a quadratic equation whose roots are $\alpha^{1/5}$ and $\beta^{1/5}$, then the value of $(c - b)$ is equal to:

(1) 43 **(2)** 42 **(3)** 50 **(4)** 47

[JEE (Main) – 27th July 2021 - Shift-2]

16. The number of real solutions of the equation, $x^2 - |x| - 12 = 0$ is:

(1) 3 **(2)** 1 **(3)** 2 **(4)** 4

[JEE (Main) – 25th July 2021 - Shift-2]

17. Let $[x]$ denote the greatest integer less than or equal to x. Then, the values of $x \in R$ satisfying the equation $[e^x]^2 + [e^x + 1] - 3 = 0$ lie in the interval:

(1) $[\log_e 2, \log_e 3)$ **(2)** $[0, 1/e)$

(3) $[0, \log_e 2)$ **(4)** $[1, e)$

[JEE (Main) – 22nd July 2021 - Shift-2]

18. If α and β are the distinct roots of the equation $x^2 + (3)^{1/4}x + 3^{1/2} = 0$, then the value of $\alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1)$ is equal to:

(1) 56×3^{25} **(2)** 52×3^{24} **(3)** 56×3^{24} **(4)** 28×3^{25}

[JEE (Main) – 20th July 2021 - Shift-1]

19. The value of $3 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{3 + \ldots \infty}}}}$

is equal to:

(1) $3 + 2\sqrt{3}$ **(2)** $4 + \sqrt{3}$ **(3)** $2 + \sqrt{3}$ **(4)** $1.5 + \sqrt{3}$

[JEE (Main) – 18th March 2021 - Shift-1]

20. The value of $4 + \cfrac{1}{5 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{4 + \ldots\ldots \infty}}}}$ is :

(1) $2 + \frac{4}{\sqrt{5}}\sqrt{30}$ **(2)** $4 + \frac{4}{\sqrt{5}}\sqrt{30}$

(3) $2 + \frac{2}{5}\sqrt{30}$ **(4)** $5 + \frac{2}{5}\sqrt{30}$

[JEE (Main) – 17th March 2021 - Shift-1]

21. The number of elements in the set $\{x \in R : (|x| - 3)|x + 4| = 6\}$ is equal to:

(1) 2 **(2)** 1 **(3)** 3 **(4)** 4

[JEE (Main) – 16th March 2021 - Shift-1]

22. The integer 'k', for which the inequality $x^2 - 2(3k - 1)x + 8k^2 - 7 > 0$ is valid for every x in R is

(1) 3 **(2)** 2 **(3)** 4 **(4)** 0

[JEE (Main) – 25th Feb. 2021 - Shift-1]

23. Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{3a_9}$ is :

(1) 4 **(2)** 1 **(3)** 2 **(4)** 3

[JEE (Main) – 25th Feb 2021 - Shift-2]

24. Let p and q be two positive numbers such that $p + q = 2$ and $p^4 + q^4 = 272$. Then p and q are roots of the equation :

(1) $x^2 - 2x + 2 = 0$ **(2)** $x^2 - 2x + 8 = 0$

(3) $x^2 - 2x + 136 = 0$ **(4)** $x^2 - 2x + 16 = 0$

[JEE (Main) – 24th Feb. 2021 - Shift-1]

ANSWER – KEY

1. (1)	2. (3)	3. (2)	4. (2)
5. (3)	6. (2)	7. (1)	8. (2)
9. (2)	10. (2)	11. (1)	12. (3)
13. (3)	14. (2)	15. (2)	16. (3)
17. (3)	18. (2)	19. (4)	20. (3)
21. (1)	22. (1)	23. (3)	24. (4)

ANSWERS WITH EXPLANATIONS

1. Option (1) is correct.

Roots of equation $14x^2 - 31x + 3\lambda = 0$ are α and β

Roots of equation $35x^2 - 53x + 4\lambda = 0$ are α and γ

Here α is a common root.

So, $\frac{\alpha^2}{-124\lambda + 159\lambda} = \frac{-\alpha}{56\lambda - 105\lambda} = \frac{1}{-742 + 1085}$

$\Rightarrow \frac{\alpha^2}{35\lambda} = \frac{\alpha}{49\lambda} = \frac{1}{343}$

Here, $\frac{\alpha}{49\lambda} = \frac{1}{343} \Rightarrow \alpha = \frac{\lambda}{7}$

Also, $\frac{\alpha^2}{35\lambda} = \frac{1}{343} \Rightarrow \alpha^2 = \frac{35\lambda}{343}$

So, $\frac{35\lambda}{343} = \left(\frac{\lambda}{7}\right)^2 \Rightarrow \frac{35\lambda}{343} = \frac{\lambda^2}{49} \Rightarrow \lambda^2 - 5\lambda = 0 \Rightarrow \lambda = 0, 5$

But $\lambda \neq 0 \Rightarrow \lambda = 5$

$\therefore \alpha = \frac{\lambda}{7} = \frac{5}{7}$

Also, $\alpha + \beta = \frac{31}{14} \Rightarrow \beta = \frac{31}{14} - \frac{5}{7} = \frac{21}{14} = \frac{3}{2}$

Again, $\alpha + \gamma = \frac{53}{35} \Rightarrow \gamma = \frac{53}{35} - \frac{5}{7} = \frac{28}{35} = \frac{4}{5}$

Now, $S = \frac{3\alpha}{\beta} + \frac{4\alpha}{\gamma} = \frac{10}{7} + \frac{25}{7} = \frac{35}{7}$

$P = \frac{3\alpha}{\beta} \cdot \frac{4\alpha}{\gamma} = \frac{10}{7} \cdot \frac{25}{7} = \frac{250}{49}$

Equation whose roots are

$\frac{3\alpha}{\beta}$ & $\frac{4\alpha}{\gamma}$ is

$x^2 - Sx + P = 0$

$\Rightarrow x^2 - \left(\frac{35}{7}\right)x + \frac{250}{49} = 0$

$\Rightarrow 49x^2 - 245x + 250 = 0$

2. Option (3) is correct.

$|x^2 - 8x + 15| = 2x - 7$

take '+'

$x^2 - 8x + 15 = 2x - 7$

$\Rightarrow x^2 - 10x + 22 = 0$

$x_1 = 5 + \sqrt{3}, x_2 = 5 - \sqrt{3}$ (reject)

take '–'

$x^2 - 8x + 15 = 7 - 2x$

$x^2 - 6x + 8 = 0$

$x_3 = 4, x_4 = 2$ (reject)

Sum of roots $= 5 + \sqrt{3} + 4 = 9 + \sqrt{3}$

3. Option (2) is correct.

Given that α and β are roots of quadratic equation $x^2 - \sqrt{2}x + \sqrt{6} = 0$,

$\therefore \alpha + \beta = \sqrt{2}$ and $\alpha.\beta = \sqrt{6}$

$\therefore$ Sum of roots $= \frac{1}{\alpha^2} + 1 + \frac{1}{\beta^2} + 1$

$= \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} + 2 = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2} + 2$

$= \frac{(\sqrt{2})^2 - 2\sqrt{6}}{6} + 2 = \frac{14 - 2\sqrt{6}}{6} = -a$

Now, product of roots

$= \left(\frac{1}{\alpha^2} + 1\right)\left(\frac{1}{\beta^2} + 1\right) = \frac{1}{\alpha^2\beta^2} + \frac{1}{\alpha^2} + \frac{1}{\beta^2} + 1$

$= \frac{1}{\alpha^2\beta^2} + \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2} + 1$

$= \frac{1}{6} + \frac{2 - 2\sqrt{6}}{6} + 1 = \frac{9 - 2\sqrt{6}}{6} = b$

$a + b = \frac{-14 + 2\sqrt{6} + 9 - 2\sqrt{6}}{6} = \frac{-5}{6}$

So, equation is $x^2 + \frac{17}{6}x + \frac{7}{6} = 0$

$\Rightarrow 6x^2 + 17x + 7 = 0$

$\therefore x = \frac{-7}{3}, \frac{-1}{2}$

4. Option (2) is correct.

$3^{\sqrt{\log_3 5}} - 5^{\sqrt{\log_5 3}} = 3^{\sqrt{\log_3 5}} - \left(3^{\log_3 5}\right)^{\sqrt{\log_5 3}}$

$= 3^{\sqrt{\log_3 5}} - 3^{\frac{\log_3 5 \,.\, \log_5 3}{\sqrt{\log_5 3}}} = 3^{\sqrt{\log_3 5}} - \frac{1}{3^{\sqrt{\log_5 3}}}$

$= 3^{\sqrt{\log_3 5}} - 3^{\sqrt{\log_3 5}} = 0$

$$3^{(\log_3 5)^{\frac{1}{3}}} - 5^{(\log_5 3)^{\frac{2}{3}}} = 5^{(\log_5 3)^{\frac{2}{3}}} - 5^{(\log_5 3)^{\frac{2}{3}}} = 0$$

$\therefore x^2-5x-3=0$ has two roots, α and β

$\therefore \alpha + \beta = 5$ and $\alpha . \beta = -3$

Now, sum of root $\alpha + \frac{1}{\beta} + \beta + \frac{1}{\alpha} = \alpha + \beta + \frac{\alpha+\beta}{\alpha\beta}$

$$= 5 - \frac{5}{3} = \frac{10}{3}$$

Product of roots $= \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$

$$= \alpha\beta + \frac{\alpha}{\alpha} + \frac{\beta}{\beta} + \frac{1}{\alpha\beta} = -3 + 2 - \frac{1}{3} = \frac{-4}{3}$$

$\therefore$ Equation is

$$x^2 - \left(\frac{10}{3}\right)x - \frac{4}{3} = 0 \Rightarrow 3x^2 - 10x - 4 = 0$$

5. **Option (3) is correct.**

Let α, β are roots of

$x^2 + (3 - a)x + 1 = 2a$

$\Rightarrow x^2 + (3 - a)x + (1 - 2a) = 0$

$\therefore \alpha + \beta = -(3 - a) = a - 3$

and $\alpha . \beta = 1 - 2a$

Sum of squares of the roots

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (a - 3)^2 - 2(1 - 2a)$

$= a^2 - 6a + 9 - 2 + 4a$

$= a^2 - 2a + 7$

$= a^2 - 2a + 1 + 6$

$= (a - 1)^2 + 6$

$\because (a - 1)^2 \geq 0 \Rightarrow (a - 1)^2 + 6 \geq 6$

$\therefore \alpha^2 + \beta^2 \geq 6$

Hence, minimum value = 6

6. **Option (2) is correct.**

Given equation, $x^4 + x^3 + x^2 + x + 1 = 0$

Multiplying $(x - 1)$ both sides, we get

$(x - 1)(x^4 + x^3 + x^2 + x + 1) = 0\,(x - 1)$

$\Rightarrow \quad (x^5 - 1) = 0$

$\Rightarrow \quad x^5 = 1$

$x = 1$ and $\alpha^5 = \beta^5 = \gamma^5 = \delta^5 = 1$,

Now, $S = \alpha^{2021} + \beta^{2021} + \gamma^{2021} + \delta^{2021}$

$\Rightarrow S = \Sigma\alpha.(\alpha^5)^{400} = \Sigma\alpha.(1)^{400} = \Sigma\alpha$

$\Rightarrow S = -1$

7. **Option (1) is correct.**

Let $f(x) = ax^2 + bx + c$ ...(i)

be the quadratic polynomial.

$\because x = -1$ is one of the root of this equation

$\Rightarrow \quad a - b + c = 0$ ...(ii)

$\because \quad f(-2) + f(3) = 0$

$\Rightarrow 4a - 2b + c + 9a + 3b + c = 0$

$\Rightarrow \quad 13a + b + 2c = 0$ ...(iii)

From (ii) and (iii) by cross multiplication method

$a - b + c = 0$

$13a + b + 2c = 0$

$$\Rightarrow \frac{a}{-2-1} = \frac{-b}{2-13} = \frac{c}{1+13}$$

$$\Rightarrow \frac{a}{-3} = \frac{b}{11} = \frac{c}{14} = (k \text{ say})$$

$\Rightarrow a = -3k,\ b = 11k,\ c = 14k$

$\therefore f(x) = -3kx^2 + 11kx + 14k$ (k be any no.)

Now, sum of the roots $= \frac{-11k}{-3k} = \frac{11}{3}$

8. **Option (2) is correct.**

Equation $ax^2 - 2bx + 15 = 0$ ...(i) has repeated root α.

Then $\alpha + \alpha = \frac{2b}{a}$

$\Rightarrow \quad 2\alpha = \frac{2b}{a} \quad \therefore \alpha = \frac{b}{a}$ ....(i)

$(\alpha.\alpha) = \frac{15}{a}$ ...(ii)

$\therefore \quad \alpha^2 = \frac{15}{a}$ ...(iii)

From (ii) and (iii)

$$\frac{b^2}{a^2} = \frac{15}{a}$$

$\therefore \quad b^2 = 15a$ ...(iv)

If α, β are the roots of the equation

$x^2 - 2bx + 21 = 0$ ...(v)

$\alpha + \beta = 2b,\ \alpha.\beta = 21$

Then $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= 4b^2 - 2 \times 21 = 4b^2 - 42$ ...(vi)

From (iv), $b^2 = 15a$

$\alpha^2 + \beta^2 = 4 \times 15a - 42$

$= 60a - 42$ ...(vii)

Putting $\alpha = \frac{b}{a}$ in equation (v)

$x^2 - 2bx + 21 = 0$

$\Rightarrow \quad \frac{b^2}{a^2} - 2b \times \frac{b}{a} + 21 = 0$

$\Rightarrow \quad b^2 - 2ab^2 + 21a^2 = 0$ Put $b^2 = 15a$

$\Rightarrow 15a - 2a \times 15a + 21a^2 = 0$

$\Rightarrow \quad 15a - 30a^2 + 21a^2 = 0 \Rightarrow 15a - 9a^2 = 0$

$\therefore a(15 - 9a) = 0$

$\Rightarrow a = 0$

$\Rightarrow a = \frac{15}{9}$

$a = 0$ not possible

From (vii) $\alpha^2 + \beta^2 = 60a - 42$

$= 60 \times \frac{15}{9} - 42$

$= 100 - 42 = 58$

9. Option (2) is correct.

Given, α and β are the roots of $3x^2 + \lambda x - 1 = 0$

$\Rightarrow \alpha + \beta = -\frac{\lambda}{3}$ and $\alpha\beta = -\frac{1}{3}$

Also given, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 15$

$\Rightarrow \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = 15$

$\Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\frac{1}{9}} = 15 \Rightarrow \frac{\lambda^2}{9} + \frac{2}{3} = \frac{15}{9}$

$\Rightarrow \lambda^2 = 9$

Now, $6(\alpha^3 + \beta^3)^2 = 6\{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\}^2$

$= 6\{(\alpha+\beta)((\alpha+\beta)^2 - 3\alpha\beta)\}^2$

$= 6\left\{\left(\frac{-\lambda}{3}\right)\left(\frac{\lambda^2}{9} + 1\right)\right\}^2$

$= 6\left\{\frac{-\lambda}{3}(1+1)\right\}^2$

$= 6(4)\left(\frac{\lambda^2}{9}\right) = 24$

Hints:
Sum of the roots of quadratic equation $ax^2 + bx + c = 0$ is $\frac{-b}{a}$ and product of the roots is $\frac{c}{a}$.

Shortcut method
Given, α and β are the roots of $3x^2 + \lambda x - 1 = 0$

$\Rightarrow \alpha + \beta = \frac{-\lambda}{3}$ and $\alpha\beta = \frac{-1}{3}$

Also, given $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 15$

$\Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2} = 15$

$\Rightarrow \lambda^2 = 9$

Now, $6(\alpha^3 + \beta^3)^2 = 6\{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\}^2$

$= 6\left\{\left(\frac{-\lambda}{3}\right)\left(\frac{\lambda^2}{9} + 1\right)\right\}^2 = (6)(4) = 24$

10. Option (2) is correct.

Given equation

$(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$ ...(i)

let $e^x = t$

$(t^2 - 4)(6t^2 - 5t + 1) = 0$

$\Rightarrow (t-2)(t+2)(6t^2 - 3t - 2t + 1) = 0$

$\Rightarrow (t-2)(t+2)[3t(2t-1) - 1(2t-1)] = 0$

$\Rightarrow (t-2)(t+2)(2t-1)(3t-1) = 0$

$\Rightarrow t = 2, -2, 1/2, 1/3 \because t = e^x$

$e^x = 2$

$\therefore x = \log 2e^x = 1/2$

$\therefore x = \log\left(\frac{1}{2}\right) \Rightarrow x = -\log 2$

$\therefore e^x = 1/3$

$\therefore x = \log 1/3 = -\log 3$

Sum of all real roots $= \log 2 - \log 2 - \log 3$

$= -\log 3$

11. Option (1) is correct.

Consider the equation $x^2 + ax + b = 0$

It has two roots (not necessarily real, α & β)

Either $\alpha = \beta$ or $\alpha \neq \beta$

Case-I: If $\alpha = \beta$, so $\alpha = \alpha^2 - 2 \Rightarrow \alpha = -1, 2$

When $\alpha = -1$, then $(a, b) = (2, 1)$

$\alpha = 2$, then $(a, b) = (-4, 4)$

Case-II: If $\alpha \neq \beta$ then

(i) $\alpha = \alpha^2 - 2$ & $\beta = \beta^2 - 2$

Here, $(\alpha, \beta) = (2, -1)$ or $(-1, 2)$

Hence, $(a, b) = (-(\alpha+\beta), \alpha\beta) = (-1, -2)$

(ii) $\alpha = \beta^2 - 2$ & $\beta = \alpha^2 - 2$

Then, $\alpha - \beta = \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha)$

Since, $\alpha \neq \beta, \alpha + \beta + 1 = 0$

Also we get $\alpha + \beta = \beta^2 + \alpha^2 - 4$

$\alpha + \beta = (\beta + \alpha)^2 - 2\alpha\beta - 4$

Thus, $-1 = 1 - 2\alpha\beta - 4$ which implies

$\alpha\beta = -1$,

therefore $(a, b) = (-(\alpha+\beta), \alpha\beta) = (1, -1)$

(iii) $\alpha = \alpha^2 - 2 = \beta^2 - 2$ & $\alpha \neq \beta$

Subtracting both sides, we get

$(\alpha^2 - 2) - (\beta^2 - 2) = 0$

$\Rightarrow \alpha^2 - \beta^2 = 0 \Rightarrow (\alpha+\beta)(\alpha-\beta) = 0$

$\Rightarrow \alpha = -\beta$

Thus, $\alpha = 2, \beta = -2$

or $\alpha = -1, \beta = 1$

So, $(a, b) = (0, -4)$ & $(0, -1)$

(iv) $\beta = \alpha^2 - 2 = \beta^2 - 2$ & $\alpha \neq \beta$ not same as (iii), therefore we get 6 pairs of (a, b) which are $(2, 1), (-4, 4), (-1, -2), (1, -1), (0, -4), (0, -1)$.

12. Option (3) is correct.

$x + 1 - 2\log_2(3 + 2^x) + 2\log_4(10 - 2^{-x}) = 0$

$\Rightarrow x + 1 + \log_2\left[\frac{10 \cdot 2^x - 1}{(3+2^x)^2}\right] - x = 0$

$\Rightarrow 1 + \log_2\left[\frac{10 \cdot 2^x - 1}{(3+2^x)^2}\right] = 0$

$\Rightarrow \frac{10 \cdot 2^x - 1}{9 + (2^x)^2 + 6.2^x} = \frac{1}{2}$

$\Rightarrow (2^x)^2 - 14.2^x + 11 = 0$

Let $2^x = y$

$\Rightarrow y^2 - 14y + 11 = 0$

$\Rightarrow y = \dfrac{14 \pm \sqrt{152}}{2} = 7 \pm \dfrac{\sqrt{152}}{2}$

$\Rightarrow y_1 = 7 + \dfrac{\sqrt{152}}{2}, y_2 = 7 - \dfrac{\sqrt{152}}{2}$

$\Rightarrow 2^{x_1} = 7 + \dfrac{\sqrt{152}}{2}, 2^{x_2} = 7 - \dfrac{\sqrt{152}}{2}$

$\therefore x_1 = \log_2\left(7 + \dfrac{\sqrt{152}}{2}\right),$

$x_2 = \log_2\left(7 - \dfrac{\sqrt{152}}{2}\right)$

$\therefore x_1 + x_2 = \log_2\left(49 - \dfrac{152}{4}\right)$

$\Rightarrow x_1 + x_2 = \log_2 11$

13. Option (3) is correct.

Let $t = (3x^2 + 4x + 2)$

$\Rightarrow (t + 1)^2 - (k + 1)(t)(t + 1) + k(t)^2 = 0$

$\Rightarrow t(1 - k) = -1$

$\therefore 3x^2 + 4x + 2 = t = \dfrac{1}{k-1}$

For real roots, $D \geq 0$

$\Rightarrow \dfrac{3}{k-1} \geq 2$

$\Rightarrow \dfrac{(2k-5)}{(k-1)} \leq 0 \Rightarrow k \in \left(1, \dfrac{5}{2}\right]$

14. Option (2) is correct.

Given that α & β are roots of the equation

$x^2 + (20)^{1/4}x + (5)^{1/2} = 0$

So, $\alpha^8 + \beta^8 = ?$

$\Rightarrow \left(x^2 + \sqrt{5}\right)^2 = \sqrt{20}x^2$

$\Rightarrow x^4 + 5 + 2\sqrt{5}x^2 = \sqrt{20}\,x^2 = 2\sqrt{5}x^2$

$\Rightarrow x^4 = -5$ and $x^8 = 25$

So, $\alpha^8 = 25, \beta^8 = 25$

$\therefore \alpha^8 + \beta^8 = 50$

15. Option (2) is correct.

$\alpha = \max(2^{6\sin 3x + 8\cos 3x}) = 2^{10}$

$\beta = \min(2^{6\sin 3x + 8\cos 3x}) = 2^{-10}$

$\alpha^{1/5} = 2^2$ & $\beta^{1/5} = 2^{-2}$

Now, $8x^2 + bx + c = 0 \;\rightarrow\; \alpha^{\frac{1}{5}},\ \beta^{\frac{1}{5}}$

$\Rightarrow \alpha^{\frac{1}{5}} + \beta^{\frac{1}{5}} = 4 + \dfrac{1}{4} = \dfrac{17}{4} = \dfrac{-b}{8}$

$\Rightarrow b = -34$

Also, $\alpha^{\frac{1}{5}} \times \beta^{\frac{1}{5}} = 1 \Rightarrow \dfrac{c}{8} = 1 \Rightarrow c = 8$

$\therefore c - b = 42$

16. Option (3) is correct.

$|x^2| - |x| - 12 = 0$

$\therefore |x| = 4, -3$ (not possible)

$\Rightarrow |x| = 4 \Rightarrow x = \pm 4$

$\therefore$ Number of real solutions = 2

17. Option (3) is correct.

$[e^x]^2 + [e^x + 1] - 3 = 0$

$\Rightarrow [e^x]^2 + [e^x] + 1 - 3 = 0,$

$\ldots [[x + I] = [x] + I$, for I – integers]

$\Rightarrow [e^x]^2 + [e^x] - 2 = 0$

Let $[e^x] = t$

$\Rightarrow t^2 + t - 2 = 0$

$\Rightarrow (t + 2)(t - 1) = 0$

$\Rightarrow t = 1, -2$

$\Rightarrow [e^x] = 1, [e^x] = -2$

Here, $[e^x] = -2$, not possible

$\Rightarrow [e^x] = 1$

$1 \leq e^x < 2$

$\Rightarrow x \in [0, \ln 2)$

18. Option (2) is correct.

Given $x^2 + (3)^{1/4} x + 3^{1/2} = 0$

$\Rightarrow x^2 + \sqrt{3} = -3^{1/4} x$

Squaring both sides,

$\Rightarrow x^4 + 2\sqrt{3}x^2 + 3 = \sqrt{3}\, x^2$

$\Rightarrow x^4 + \sqrt{3}x^2 + 3 = 0$

$\Rightarrow x^4 + 3 = -\sqrt{3}x^2$

Now squaring both the sides again,

$\Rightarrow x^8 + 6x^4 + 9 = 3x^4$

$\Rightarrow x^8 + 3x^4 + 9 = 0$

Put $x = \alpha$, $\alpha^8 = -9 - 3\alpha^4$

$\therefore \alpha^{12} = -9\alpha^4 - 3\alpha^8 = -9\alpha^4 - 3(-9 - 3\alpha^4) = 27$

Similarly $\beta^{12} = 27$

$\Rightarrow \alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1)$

$= (27)^8 \times 26 + (27)^8 \times 26 = 52 \times (27)^8$

$= 52 \times 3^{24}$

19. Option (4) is correct.

Let, $y = 3 + \left[\dfrac{1}{4 + \dfrac{1}{y}}\right]$

$y = 3 + \left[\dfrac{y}{4y+1}\right]$

$$y = \frac{3(4y+1)+y}{4y+1}$$

$$y\,(4y + 1) = 12y + 3 + y$$

$$4y^2 - 12y - 3 = 0$$

We know, solution of quadratic equation of the form, $ax^2 + bx + c = 0$ $(a \neq 0)$ is given by quadratic formula as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$4y^2 - 12y - 3 = 0$$

Here, $a = 4, b = -12, c = -3$

$$y = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(-3)}}{2(4)}$$

$$y = \frac{12 \pm \sqrt{144+48}}{8} = \frac{12 \pm 8\sqrt{3}}{8}$$

$$y = \frac{3 \pm 2\sqrt{3}}{2}$$

$$y = \frac{3 + 2\sqrt{3}}{2} = 1.5 + \sqrt{3} \text{ as } y > 0$$

Hint:

(i) For these types of sums, we assume a variable till that part after which that part (assumed part) repeats.

(ii) We will get a quadratic equation in some variable which we can solve further using factorization or quadratic formula.

Shortcut Method:

Let, $y = 3 + \cfrac{1}{4 + \cfrac{1}{y}}$

$\Rightarrow y(4y + 1) = 3(4y + 1) + y$

$\Rightarrow 4y^2 - 12y - 3 = 0$

$\Rightarrow y = \dfrac{12 \pm \sqrt{144+48}}{8} \Rightarrow y = \dfrac{12 \pm 8\sqrt{3}}{8}$

$\Rightarrow y = \dfrac{3 + 2\sqrt{3}}{2}$, as $y > 0$

20. Option (3) is correct.

Given: $4 + \cfrac{1}{5 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{4 + \ldots}}}}$

Let $x = 4 + \cfrac{1}{5 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{4 + \ldots}}}}$

$$\Rightarrow \quad x = 4 + \cfrac{1}{5 + \cfrac{1}{x}}$$

$$\Rightarrow \quad x - 4 = \frac{x}{5x+1}$$

$$\Rightarrow \quad (5x + 1)(x - 4) = x$$

$$\Rightarrow \quad 5x^2 - 20x + x - 4 = x$$

$$\Rightarrow \quad 5x^2 - 20x - 4 = 0$$

Solution of quadratic equation $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here, $a = 5, b = -20, c = -4$

$$\therefore \quad x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \times 5 \times (-4)}}{10}$$

$$\Rightarrow \quad x = \frac{20 \pm \sqrt{480}}{10} \Rightarrow x = \frac{20 \pm 4\sqrt{30}}{10}$$

$$\Rightarrow \quad x = 2 + \frac{2}{5}\sqrt{30} \text{ or } x = 2 - \frac{2}{5}\sqrt{30}$$

$x = 2 - \dfrac{2}{5}\sqrt{30}$ must be neglected as such operations on positive numbers cannot give a negative value.

Hence, $x = 2 + \dfrac{2}{5}\sqrt{30}$

Hint :

(i) Consider the given value as x.

(ii) Find the terms that are repeated infinite times and that is also equal to x.

(iii) Solve the quadratic formed.

(iv) Negative value of x must be neglected as x in this case will always be positive.

Shortcut Method:

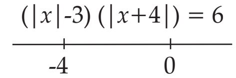
Let $x = 4 + \cfrac{1}{5 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{4 + \ldots}}}}$

$$x = \frac{20 + \sqrt{20^2 + 4 \times 5 \times 4}}{2 \times 5}$$

$$x = 2 + \frac{2}{5}\sqrt{30}$$

21. Option (1) is correct.

Given, $(|x|-3)\,(|x+4|) = 6$



Case I: When $x < -4$

Since, $|x| = \begin{cases} x; & x > 0 \\ -x; & x < 0 \end{cases}$

$\Rightarrow \quad (-x-3)(-x-4) = 6$

$\Rightarrow \quad (x+3)(x+4) = 6$

$\Rightarrow \quad x^2 + 7x + 6 = 0$

$\Rightarrow \quad (x+6)(x+1) = 0$

$\Rightarrow \quad x = -1 \text{ and } -6$

As $x < -4$ so for this case $x = \{-6\}$

Case II: When $x \in [-4, 0)$

Using $|x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$

$$(-x-3)(x+4) = 6$$

$\Rightarrow x^2 + 7x + 18 = 0$

$\Rightarrow$ No solution (D < 0)

So, for this case $x \in \phi$

Case III:

When $x \geq 0$

$$(x-3)(x+4) = 6$$

$\Rightarrow \quad x^2 + x - 18 = 0$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{1+72}}{2}$

As $x \geq 0$ so for this case $x = \left\{\dfrac{\sqrt{73}-1}{2}\right\}$

So, final solution of the given equation are

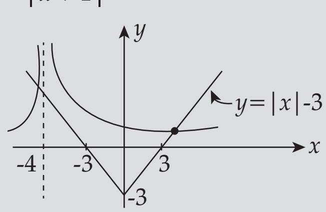
$x \in \left\{-6, \dfrac{\sqrt{73}-1}{2}\right\}$

Hence number of solution will be 2.

Shortcut Method:

$(|x| - 3)(|x+4|) = 6$

$\Rightarrow |x| - 3 = \dfrac{6}{|x+4|}$



From the above figure, we see that here 2 intersection are present so number of solution will be two.

22. Option (1) is correct.

The inequality will be greater than zero, if D is less than zero.

D < 0

$\Rightarrow (2(3k-1))^2 - 4(8k^2 - 7) < 0$

$\Rightarrow 4(9k^2 - 6k + 1) - 4(8k^2 - 7) < 0$

Simplifying in this we get

$\Rightarrow k^2 - 6k + 8 < 0$

$\Rightarrow (k-4)(k-2) < 0$

$\therefore \quad k$ lies in (2, 4)

In this interval only one integer is present i.e. 3

$\therefore k = 3$

23. Option (3) is correct.

Given: α and β are roots of $x^2 - 6x - 2 = 0$

$$a_n = \alpha^n - \beta^n \text{ for } n \geq 1$$

$\therefore \quad \alpha$ and β must satisfy $x^2 - 6x - 2 = 0$

$\therefore \quad \alpha^2 - 6\alpha - 2 = 0 \Rightarrow \alpha^2 - 2 = 6\alpha \quad$ ...(i)

$\quad \beta^2 - 6\beta - 2 = 0 \Rightarrow \beta^2 - 2 = 6\beta \quad$ ...(ii)

Given expression,

$$\frac{a_{10} - 2a_8}{3a_9} = \frac{(\alpha^{10} - \beta^{10}) - 2(\alpha^8 - \beta^8)}{3(\alpha^9 - \beta^9)}$$

$$= \frac{\alpha^{10} - 2\alpha^8 - \beta^{10} + 2\beta^8}{3(\alpha^9 - \beta^9)}$$

$$= \frac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)}{3(\alpha^9 - \beta^9)}$$

Now, from (i) and (ii)

$$\frac{a_{10} - 2a_8}{3a_9} = \frac{\alpha^8.6\alpha - \beta^8.6\beta}{3(\alpha^9 - \beta^9)} = \frac{6(\alpha^9 - \beta^9)}{3(\alpha^9 - \beta^9)}$$

$\Rightarrow \quad \dfrac{a_{10} - 2a_8}{3a_9} = 2$

Hint:

(i) α and β are roots of $x^2 - 6x - 2 = 0$

$\therefore \quad \alpha^2 - 2 = 6\alpha$ and $\beta^2 - 2 = 6\beta$

(ii) Manipulate the given expression such that they can directly be used.

Shortcut Method:

$\alpha^2 - 6\alpha - 2 = 0$

$\alpha^{10} - 6\alpha^9 - 2\alpha^8 = 0$

Similarly, $\quad \beta^{10} - 6\beta^9 - 2\beta^8 = 0$

$(\alpha^{10} - \beta^{10}) - 6(\alpha^9 - \beta^9) - 2(\alpha^8 - \beta^8) = 0$

$\Rightarrow \quad a_{10} - 6a_9 - 2a_8 = 0$

$\Rightarrow \quad a_{10} - 2a_8 = 6a_9$

$\Rightarrow \quad \dfrac{a_{10} - 2a_8}{3a_9} = 2$

24. Option (4) is correct.

Given: p and q are two positive numbers such that $p + q = 2$ and $p^4 + q^4 = 272$

$\Rightarrow (p+q)^2 = 2^2$

$\Rightarrow p^2 + q^2 + 2pq = 4$

$\Rightarrow p^2 + q^2 = 4 - 2pq$

Squaring both sides,

$$(p^2+q^2)^2 = (4-2pq)^2$$

$\Rightarrow \quad p^4 + q^4 + 2p^2q^2 = 16 + 4p^2q^2 - 16pq$

$\Rightarrow \quad 272 - 2p^2q^2 = 16 - 16pq$

$(\because p^4 + q^4 = 272)$

$\Rightarrow \quad p^2q^2 - 8pq - 128 = 0$

$\Rightarrow \quad (pq)^2 - 8pq - 128 = 0$

$$pq = \frac{8 \pm 24}{2} = 16 \text{ or } -8$$

$\Rightarrow \quad pq = 16$

Now, quadratic equation whose roots are p, q is

$x^2 - (p+q)x + pq = 0$

$\Rightarrow \quad x^2 - 2x + 16 = 0$

Hint:

(i) $(p+q)^2 = p^2 + q^2 + 2pq$

(ii) $p^4 + q^4 = (p^2+q^2)^2 - 2p^2q^2$

$= ((p+q)^2 - 2pq)^2 - 2p^2q^2$

Shortcut Method:

$p^4 + q^4 = (p^2+q^2)^2 - 2p^2q^2$

$= ((p+q)^2 - 2pq)^2 - 2p^2q^2$

$\Rightarrow (pq)^2 - 8pq - 128 = 0$

$\Rightarrow pq = 16 \quad (\because p, q$ are positive)

Now, quadratic eq^n whose roots are p, q is

$x^2 - (p+q)x + pq = 0$

$\Rightarrow x^2 - 2x + 16 = 0$

Integer Type Questions (Chapter Based)

1. Let $a \in \mathbb{R}$ and let α, β be the roots of the equation

$$x^2 + 60^{\frac{1}{4}}x + a = 0$$

If $\alpha^4 + \beta^4 = -30$, then the product of all possible values of a is

[JEE (Main) – 25th January 2023 - Shift-2]

Sol. The correct answer is (45).

Given: $a \in R, \alpha, \beta$ are roots of $x^2 + 60^{1/4}x + a = 0$

$\Rightarrow$ Sum of roots $= \alpha + \beta = -60^{1/4}$

and product of roots $= \alpha\beta = a$

[If α, β are roots of $ax^2 + bx + c = 0$ then, $\alpha + \beta = -\frac{b}{a}$ and $\frac{c}{a}$ }

Now, $(\alpha + \beta)^2 = (-60^{1/4})^2$

$\Rightarrow \alpha^2 + \beta^2 + 2\alpha\beta = 60^{1/2}$

$\Rightarrow \alpha^2 + \beta^2 = 60^{1/2} - 2a$

Squaring both sides of the above equation, we get

$(\alpha^2 + \beta^2)^2 = (60^{1/2} - 2a)^2$

$\Rightarrow \alpha^4 + \beta^4 + 2\alpha^2\beta^2 = 60 + 4a^2 - 4(60)^{1/2}a$

$\Rightarrow \alpha^4 + \beta^4 = 60 + 4a^2 - 4a(60)^{1/2} - 2a^2$

$\Rightarrow \alpha^4 + \beta^4 = 60 + 2a^2 - 4a(60)^{1/2}$

$\Rightarrow -30 = 60 + 2a^2 - 4a(60)^{1/2}$

$\Rightarrow 2a^2 - 4a(60)^{1/2} + 90 = 0$

$\Rightarrow a^2 - 2a(60)^{1/2} + 45 = 0$

The above equation is also a quadratic equation of the form $ax^2 + bx + c = 0$ whose product of roots $= \frac{c}{a}$

$\Rightarrow$ Product of all possible values of $a = 45$

2. Let $\left\{S = \{z \in C - \{i, 2i\} : \frac{z^2 + 8iz - 15}{z^2 - 3iz - 2} \in R\right\}$. If $\alpha - \frac{13}{11}i \in S, a \in R - \{0\}$, then $242\alpha^2$ is equal to _______. **[JEE (Main) – 11th April 2023 - Shift-2]**

Sol. The correct answer is (1680).

Given that $\frac{z^2 + 8iz - 15}{z^2 - 3iz - 2} \in R$

$\Rightarrow \frac{(z^2 - 3iz - 2) + (11iz - 13)}{z^2 - 3iz - 2} \in R$

$\Rightarrow 1 + \frac{11iz - 13}{z^2 - 3iz - 2} \in R \quad$ ...(i)

but $\alpha - \frac{13}{11}i \in S$

or $z = \alpha - \frac{13i}{11} = \alpha + \frac{13}{11i}$

$\Rightarrow 11iz - 13 = i\alpha$

So eqn. (i) becomes

$1 + \frac{i\alpha}{z^2 - 3iz - 2} \in R$

$\Rightarrow z^2 - 3iz - 2 \in$ Img.

Let $z = x + iy$

$\Rightarrow x^2 - y^2 - 2ixy - 3ix + 3y - 2 \in$ Img

$\Rightarrow$ Re $(x^2 - y^2 + 3y - 2 - i(3x + 2xy)) = 0$

$\Rightarrow x^2 - y^2 + 3y - 2 = 0$

$\Rightarrow x^2 = y^2 - 3y + 2$

$\Rightarrow x^2 = (y-1)(y-2)$

$\therefore z = \alpha - \frac{13}{11}i$

Putting $x = \alpha$, $y = \frac{-13}{11}$

$$\alpha^2 = \left(\frac{-13}{11} - 11\right)\left(\frac{-13}{11} - 2\right)$$

$$\alpha^2 = \frac{24 \times 35}{121}$$

$\Rightarrow 242\alpha^2 = 48 \times 35 = 1680$

3. If a and b are the roots of the equation $x^2 - 7x - 1 = 0$, then the value of $\frac{a^{21}+b^{21}+a^{17}+b^{17}}{a^{19}+b^{19}}$ is equal to ________.

[JEE (Main) – 11th April 2023 - Shift-1]

Sol. The correct answer is (51).

Given that a and b are the roots of $x^2 - 7x - 1 = 0$

So, by using Newton's theorem, we get

$S_{n+2} - 7\,S_{n+1} - S_n = 0$

On putting $n = 19$, 18 and 17 we get,

$S_{21} - 7\,S_{20} - S_{19} = 0$

$S_{20} - 7\,S_{19} - S_{18} = 0$

$S_{19} - 7\,S_{18} - S_{17} = 0$

Now

$$\frac{a^{21}+b^{21}+a^{17}+b^{17}}{a^{19}+b^{19}} = \frac{S_{21}+S_{17}}{S_{19}}$$

$$= \frac{S_{21}+(S_{19}-7S_{18})}{S_{19}} = \frac{50S_{19}+(S_{21}-7S_{20})}{S_{19}}$$

$$= 51 \times \frac{S_{19}}{S_{19}} = 51$$

4. Let m and n be the numbers of real roots of the quadratic equations $x^2 - 12x + [x] + 31 = 0$ and $x^2 - 5\,[x + 2] - 4 = 0$ respectively, where $[x]$ denotes the greatest integer $\le x$. Then $m^2 + mn + n^2$ is equal to ________.

[JEE (Main) – 8th April 2023 - Shift-2]

Sol. The correct answer is (9).

The given eqn is : $x^2 - 12x + [x] + 31 = 0$

$\Rightarrow \{x\} - x = x^2 - 12x + 31$

$\Rightarrow \{x\} = x^2 - 11x + 31$

So, $0 \le x^2 - 11x + 31 < 1$

$\Rightarrow x^2 - 11x + 30 \le 0$

$\Rightarrow x \in (5, 6)$

$\therefore [x] = 5$

$\therefore x^2 - 12x + 5 + 31 = 0$

$\Rightarrow (x - 6)^2 = 0 \Rightarrow x = 6$

Hence, $x \in \phi$ $\quad (\because x \in (5, 6))$

$\therefore m = 0$

Another equation is $x^2 - 5[x + 2] - 4 = 0$

Case I: $x \ge -2$

$x^2 - 5x - 14 = 0 \Rightarrow x = 7, -2$

Case II: $x < -2$

$x^2 + 5x + 6 = 0 \Rightarrow x = -3 - 2$

$\therefore x \in \{-3, -2, 7\}$

$\therefore n = 3$

Hence, $m^2 + mx + n^2 = 0 + 0 + 9 = 9$

HINT:
The relation between the greatest integer function and fractional part is : $[x] = x - \{x\}$

5. Let α, β $(\alpha > \beta)$ be the roots of the quadratic equation $x^2 - x - 4 = 0$. If $P_n = \alpha^n - \beta^n, n \in N$, then $\frac{P_{15}P_{16} - P_{14}P_{16} - P_{15}^2 + P_{14}\,P_{15}}{P_{13}\,P_{14}}$ is equal to ___.

[JEE (Main) – 29th July 2022 - Shift-2]

Sol. The correct answer is (16).

Let α,β $(\alpha>\beta)$ be the root of quadratic equation

$x^2 - x - 4 = 0$ …(1)

$\therefore \alpha + \beta = 1$ and $\alpha\beta = -4$

And $\alpha^2 - \alpha - 4 = 0$ & $\beta^2 - \beta - 4 = 0$

Now,

$$\frac{P_{15}P_{16} - P_{14}\,P_{16} - P_{15}^{\,2} + P_{14}P_{15}}{P_{13}\,P_{14}}$$

$$= \frac{P_{16}(P_{15} - P_{14}) + P_{15}(P_{14} - P_{15})}{P_{13}\,P_{14}}$$

$$= \frac{(P_{15} - P_{14})(P_{16} - P_{15})}{P_{13}\,P_{14}}$$

Now, $P_n - P_{n-1} = (\alpha^n - \beta^n) - (\alpha^{n-1} - \beta^{n-1})$

$= \alpha^{n-2}(\alpha^2 - \alpha) - \beta^{n-2}(\beta^2 - \beta)$

$= 4(\alpha^{n-2} - \beta^{n-2}) = 4\,p_{n-2}$

$$\therefore \frac{P_{15}\;P_{16} - P_{14}\;P_{16} - P_{15}^2 + P_{14}P_{15}}{P_{13}\;P_{14}} = \frac{16\;P_{13}\;P_{14}}{P_{13}\;P_{14}} = 16$$

6. The sum of all real values of x for which $\frac{3x^2 - 9x + 17}{x^2 + 3x + 10} = \frac{5x^2 - 7x + 19}{3x^2 + 5x + 12}$ is equal to :

[JEE (Main) – 28th July 2022 - Shift-1]

Sol. The correct answer is (6).

$$\frac{3x^2 - 9x + 17}{x^2 + 3x + 10} = \frac{5x^2 - 7x + 19}{3x^2 + 5x + 12}$$

$$\Rightarrow \frac{3x^2 - 9x + 17}{5x^2 - 7x + 19} - 1 = \frac{x^2 + 3x + 10}{3x^2 + 5x + 12} - 1$$

$$\Rightarrow \frac{-2x^2 - 2x - 2}{5x^2 - 7x + 19} = \frac{-2x^2 - 2x - 2}{3x^2 + 5x + 12}$$

Either $x^2+x+1=0$ or $5x^2-7x+19=3x^2+5x+12$

$\therefore x^2+x+1>0$

$\therefore 5x^2-7x+19 = 3x^2+5x+12$

$\Rightarrow 2x^2-12x+7 = 0$

So, sum of roots = 6

7. Let $z = a + ib$, $b \ne 0$ be complex numbers satisfying $z^2 = \bar{z}.2^{1-|z|}$. then the least value of $n \in N$, such that $z^n = (z + 1)^n$, is equal to :

[JEE (Main) – 28th July 2022 - Shift-2]

Sol. The correct answer is (6).

Given that $z^2 = \bar{z}.2^{1-|z|}$

$\Rightarrow |z^2| = |\bar{z}| \cdot 2^{1-|z|}$

$\Rightarrow |z|^2 = |\bar{z}| \cdot 2^{1-|z|}, \qquad [\because |z| = |\bar{z}|]$

$\Rightarrow |z| = 2^{1-|z|}$

$\Rightarrow |z| = 1$

$\therefore z^2 = \bar{z} \Rightarrow z^3 = z\bar{z} = 1 \Rightarrow z = 1, \omega, \omega^2$

Now, $z^n = (z+1)^n \Rightarrow \omega^n = (1+\omega)^n = (-\omega^2)$

$\Rightarrow \omega^n = (-1)^n(\omega^n)^2 \Rightarrow \omega^n - (-1)^n(\omega^n)^2 = 0$

$\therefore n = 6$

8. Let $S = \{z \in C : z^2 + \bar{z} = 0\}$. Then $\sum_{z \in s}(\text{Re}(z) + im(z))$ is equal to __________.

[JEE (Main) – 27th July 2022 - Shift-1]

Sol. The correct answer is (0).

Given that

$S = \{z \in C : z^2 + \bar{z} = 0\}$

Let $z^2 + \bar{z} = 0$

$\Rightarrow (x + iy)^2 + (x - iy) = 0$

$\Rightarrow x^2 - y^2 + 2xyi + x - iy = 0$

$\Rightarrow (x^2 - y^2 + x) + (2xy - y)i = 0$

On comparing both sides we get

$x^2 - y^2 + x = 0$ ...(i)

$2xy - y = 0$

$y(2x - 1) = 0 \Rightarrow y = 0$ or $x = \frac{1}{2}$

When $y = 0$ from equation (i)

$x = 0, -1$

When $x = \frac{1}{2}$ from equation (i)

$y = \pm\frac{\sqrt{3}}{2}$

$$\therefore \sum_{Z \in s}(\text{Re}(2) + \text{Im}(z)) = \left(0 - 1 + \frac{1}{2} + \frac{1}{2}\right) + \left(0 + 0 + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = 0$$

9. If for some $p, q, r \in R$, not all have same sign, one of the roots of the equation $(p^2 + q^2)x^2 - 2q(p + r)x + q^2 + r^2 = 0$ is also a root of the equation $x^2 + 2x - 8 = 0$, then $\frac{q^2 + r^2}{p^2}$ is equal to ________.

[JEE (Main) – 26th July 2022 - Shift-1]

Sol. The correct answer is (272).

Given that

$(p^2 + q^2)x^2 - 2q(p + r)x + q^2 + r^2 = 0$

$\Rightarrow p^2x^2 - 2pqx + q^2 + q^2x^2 - 2qrx + r^2 = 0$

$\Rightarrow (px - q)^2 + (qx - r)^2 = 0$

$\Rightarrow px - q = 0$ and $qx - r = 0$

$\Rightarrow x = \frac{q}{p} = \frac{r}{q}$

Now, $x^2 + 2x - 8 = 0$

$\Rightarrow x^2 + 4x - 2x - 8 = 0 \Rightarrow (x + 4)(x - 2) = 0$

$\Rightarrow x = -4, 2$

Since both quadratic equations has one root common and $p, q, r \in R$ not all have same sign

$\therefore \frac{q}{p} = \frac{r}{q} = -4 \Rightarrow q = -4p$ and $r = 16p$

Now $\frac{q^2 + r^2}{p^2} = \frac{16p^2 + 16^2p^2}{p^2}$

$= 16(16 + 1) = 272$

10. Let a, b be two non-zero real numbers. If p and r are the roots of the equation $x^2 - 8ax + 2a = 0$ and q and s are the roots of the equation $x^2 + 12bx + 6b = 0$, such that $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}$ are in A.P., then $a^{-1} - b^{-1}$ equal to _______.

[JEE (Main) – 25th July 2022 - Shift-1]

Sol. The correct answer is (38).

p, r, are the roots of the equation

$x^2 - 8ax + 2a = 0$ ...(i)

$p + r = 8a$

$pr = 2a$ ...(ii)

q, s are the roots of the equation

$x^2 + 12bx + 6b = 0$ ...(iii)

$q + s = -12b, q.s = 6b$ ...(iv)

From (ii), $\frac{p + r}{pr} = \frac{8a}{2a} = 4$

$\Rightarrow \frac{1}{r} + \frac{1}{p} = 4$ ...(v)

From (iv), $\frac{q + s}{qs} = \frac{-12b}{6b}$

$\Rightarrow \frac{1}{q} + \frac{1}{s} = -2$ ...(v)

Given $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}$ are in AP with common difference d and first term α

$\therefore \alpha + \alpha + 2d = 4$

and $\alpha + d + \alpha + 3d = -2$

$\Rightarrow \alpha = 5$ and $d = -3$

So, $\frac{1}{p} = \alpha \Rightarrow \frac{1}{p} = 5$

$\Rightarrow p = \frac{1}{5}$

Now, $\frac{1}{q} = \alpha + d = 5 - 3 = 2$

$\therefore q = \frac{1}{2}$

Now, $\frac{1}{r} = \alpha + 2d = 5 - 6$

$\Rightarrow \quad \frac{1}{r} = -1$

$\Rightarrow \quad r = -1$

Now, $\frac{1}{s} = \alpha + 3d = 5 - 9 = -4$

$\therefore \quad s = \frac{-1}{4}$

$\because \quad p + r = 8a$

$\Rightarrow \quad \frac{1}{5} - 1 = 8a \Rightarrow \frac{-4}{5} = 8a$

$\Rightarrow \quad a = \frac{-1}{10}$

$\because \quad q + s = -12b$

$\Rightarrow \quad \frac{1}{2} - \frac{1}{4} = -12b \Rightarrow \frac{1}{4} = -12b$

$\therefore \quad b = \frac{-1}{48}$

Hence $\quad a^{-1} - b^{-1} = -10 + 48 = 38$

11. Sum of squares of modulus of all the complex numbers z satisfying $\bar{z} = iz^2 + z^2 - z$ is equal to _______. **[JEE (Main) – 28th June 2022 - Shift-2]**

Sol. The correct answer is (2).

Let $z = x + iy$ be any complex number

Satisfying $\bar{z} = iz^2 + z^2 - z$

$(x - iy) = i(x + iy)^2 + (x + iy)^2 - (x + iy)$

$= i(x^2 - y^2 + 2ixy) + (x^2 - y^2 + 2ixy) - x - iy$

$x - iy = (x^2 - y^2 - 2xy - x) + i(x^2 - y^2 + 2xy - y)$

Comparing real and imaginary part both sides.

$x = x^2 - y^2 - 2xy - x$

$\Rightarrow \quad (2x + 2xy) = x^2 - y^2 \quad$ ...(i)

And $\quad -y = x^2 - y^2 + 2xy - y$

$\Rightarrow \quad x^2 - y^2 + 2xy = 0$

$x^2 - y^2 = -2xy \quad$ ...(ii)

From (i) and (ii),

$2x + 2xy = -2xy$

$\Rightarrow \quad 2x + 4xy = 0$

$\Rightarrow \quad 2x(1 + 2y) = 0$

$\Rightarrow \quad x = 0, y \times \frac{-1}{2}$

For $x = 0$, $\quad y = 0$

If $\quad y = \frac{-1}{2}$

$\Rightarrow \quad x^2 - \frac{1}{4} = -2x \times \frac{-1}{2}$

$\Rightarrow \quad x^2 - x - \frac{1}{4} = 0$

$\Rightarrow \quad x = \frac{+1 \pm \sqrt{1 - 4 \times 1 \times \frac{-1}{4}}}{2 \times 1}$

$\Rightarrow \quad x = \frac{1 \pm \sqrt{2}}{2}$

$(0,0), \left(\frac{1-\sqrt{2}}{2}, \frac{-1}{2}\right), \left(\frac{1+\sqrt{2}}{2}, \frac{-1}{2}\right)$

$z_1 \quad z_2 \quad z_3$

Now, $|z_1|^2 + |z_2|^2 + |z_3|^2$

$= 0 + \left(\frac{1-\sqrt{2}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{1+\sqrt{2}}{2}\right)^2 + \left(\frac{-1}{2}\right)^2$

$= \frac{1+2-2\sqrt{2}}{4} + \frac{1}{4} + \frac{1+2+2\sqrt{2}}{4} + \frac{1}{4}$

$= \frac{3+2\sqrt{2}+3-2\sqrt{2}+1+1}{4} = \frac{8}{4} = 2$

12. If the sum of all the roots of the equation $e^{2x} - 11e^x - 45e^{-x} + \frac{81}{2} = 0$ is $\log_e p$, then p is equal to … **[JEE (Main) – 27th June 2022 - Shift-1]**

Sol. The correct answer is (45).

Given that

$e^{2x} - 11e^x - 45e^{-x} + \frac{81}{2} = 0$

$\Rightarrow 2e^{3x} - 22e^{2x} - 90 + 81e^x = 0$

$\Rightarrow 2(e^x)^3 - 22(e^x)^2 + 81e^x - 90 = 0$

Let $e^x = y$

$\Rightarrow 2y^3 - 22y^2 + 81y - 90 = 0$

Product of roots (y_1, y_2, y_3)

$y_1 . y_2 . y_3 = \frac{-(-90)}{2} = 45$

Let x_1, x_2, and x_3 be roots of given equation

$\Rightarrow \quad e^{x_1} . e^{x_2} . e^{x_3} = 45$

$\Rightarrow \quad e^{x_1 + x_2 + x_3} = 45$

$\Rightarrow \quad x_1 + x_2 + x_3 = \log_e 45 = \log_e p$

$\Rightarrow \quad p = 45$

13. Let α, β be the roots of the equation $x^2 - 4\lambda x + 5$ and α, γ be the roots of the equation $x^2 - (3\sqrt{2} + 2\sqrt{3})x + 7 + 3\lambda\sqrt{3} = 0$, $\lambda > 0$. If $\beta + \gamma = 3\sqrt{2}$, then $(\alpha + 2\beta + \gamma)^2$ is equal to ….

[JEE (Main) – 27th June 2022 - Shift-2]

Sol. The correct answer is (98).

Given that α, β be the roots of the equation

$x^2 - 4\lambda x + 5 = 0$

$\therefore \quad \alpha + \beta = 4\lambda \quad$ ...(i)

and α and γ be the roots of the equation

$$x^2 - \left(3\sqrt{2}+2\sqrt{3}\right)x + 7 + 3\lambda\sqrt{3} = 0$$

$\therefore \quad \alpha + \gamma = 3\sqrt{2}+2\sqrt{3}$ ...(ii)

$\beta + \gamma = 3\sqrt{2}$ (Given) ...(iii)

Adding (i), (ii) and (iii), we get

$$\alpha + \beta + \gamma = \frac{1}{2}(4\lambda + 6\sqrt{2} + 2\sqrt{3}) \text{ ...(iv)}$$

From (i), (ii), (iii) and (iv), we get

$\alpha = 2\lambda + \sqrt{3}$; $\beta = 2\lambda - \sqrt{3}$

and $\gamma = 3\sqrt{2} + \sqrt{3} - 2\lambda$

$\because \alpha\beta = 5$

$\therefore 4\lambda^2 - 3 = 5$

$\Rightarrow \lambda = \sqrt{2}$

So, $(\alpha + 2\beta + \gamma)^2$

$= (2\lambda + \sqrt{3} + 4\lambda - 2\sqrt{3} + 3\sqrt{2} + \sqrt{3} - 2\lambda)^2$

$= (3\sqrt{2} + 4\lambda)^2 = (3\sqrt{2} + 4\sqrt{2})^2$

$= (7\sqrt{2})^2 = 98$

14. The sum of the cubes of all the roots of the equation $x^4 - 3x^3 - 2x^2 + 3x + 1 = 0$ is

[JEE (Main) – 26th June 2022 - Shift-1]

Sol. The correct answer is (36).

Let $\alpha, \beta, \gamma, \delta$ be the roots of given equation

Then to evaluate

$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = ?$

Given, $x^4 - 3x^3 - 2x^2 + 3x + 1 = 0$ ...(i)

Dividing by x^2

$$x^2 - 3x - 2 + \frac{3}{x} + \frac{1}{x^2} = 0$$

$$\left(x^2 + \frac{1}{x^2}\right) - 3\left(x - \frac{1}{x}\right) - 2 = 0 \quad \text{...(ii)}$$

Let $\quad x - \frac{1}{x} = t$

$\Rightarrow \quad \left(x - \frac{1}{x}\right)^2 = t^2$

$\Rightarrow \quad x^2 + \frac{1}{x^2} - 2 = t^2$

$\Rightarrow \quad x^2 + \frac{1}{x^2} = t^2 + 2$

From (ii),

$(t^2 + 2) - 3t - 2 = 0$

$\Rightarrow t^2 + 2 - 3t - 2 = 0$

$\Rightarrow t^2 - 3t = 0$

$\therefore t = 0$ ant $t = 3$

If $\quad t = 0$

$\Rightarrow \quad x - \frac{1}{x} = 0$

$\Rightarrow \quad x^2 - 1 = 0$

$\Rightarrow \quad (x - 1)(x + 1) = 0$

$\Rightarrow x = 1, x = -1$

$\Rightarrow \gamma = 1, \delta = -1$

And if $t = 3$

$\Rightarrow \quad x - \frac{1}{x} = 3$

$\Rightarrow \quad x^2 - 1 = 3x$

$\Rightarrow \quad x^2 - 3x - 1 = 0$

Now, $\quad \alpha + \beta = 3$

$\alpha.\beta = -1$

$\therefore \alpha^3 + \beta^3 + \gamma^3 + \delta^3 = \alpha^3 + \beta^3 + 1 - 1$

$= \alpha^3 + \beta^3$

$= (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$

$= (\alpha + \beta)((\alpha + \beta)^2 - 3\alpha\beta)$

$= 3 \times (9 - 3 \times (-1)) = 3 \times 12 = 36$

Hence, $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 36$

15. Let p and q be two real numbers such that $p + q = 3$ and $p^4 + q^4 = 369$. Then $\left(\frac{1}{p} + \frac{1}{q}\right)^{-2}$ is equal to …

[JEE (Main) – 26th June 2022 - Shift-2]

Sol. The correct answer is (4).

Given $p + q = 3$ and $p^4 + q^4 = 369$

where p, q be two real numbers

$p + q = 3$

$\Rightarrow \quad (p + q)^2 = 9$

$\Rightarrow \quad p^2 + q^2 + 2pq = 9$

$\therefore \quad p^2 + q^2 = 9 - 2pq$ ...(i)

and $\quad p^4 + q^4 = 369$

$\Rightarrow (p^2)^2 + (q^2)^2 - 2p^2q^2 + 2p^2q^2 = 369$

$\Rightarrow \quad (p^2 + q^2)^2 - 2p^2q^2 = 369$

$\Rightarrow \quad (9 - 2pq)^2 - 2p^2q^2 = 369$

$\Rightarrow \quad 81 + 4p^2q^2 - 36pq - 2p^2q^2 = 369$

$\Rightarrow \quad 4p^2q^2 - 2p^2q^2 - 36pq = 369 - 81 = 288$

$\Rightarrow \quad 2p^2q^2 - 36pq = 288$

$\Rightarrow \quad p^2q^2 - 18pq = 144$

Let $pq = A$

$\Rightarrow \quad A^2 - 18A - 144 = 0$

$\Rightarrow \quad A^2 - 24A + 6A - 144 = 0$

$\Rightarrow \quad A(A - 24) + 6(A - 24) = 0$

$\Rightarrow \quad (A + 6)(A - 24) = 0$

$\Rightarrow \quad A = -6, 24$ ...(ii)

$\Rightarrow \quad A = pq$

$\Rightarrow \quad pq = -6, 24$

If $\quad pq = 24$

$p^2 + q^2 = 9 - 2pq$

$$= \quad 9 - 48 = -39$$

(–ve not possible)

If $\quad pq = -6$

$$p^2 + q^2 = 9 + 12$$

$$= 21 \quad \text{...(iii)}$$

Then the value of $\left(\frac{1}{p}+\frac{1}{q}\right)^{-2} = \left(\frac{pq}{p+q}\right)^2$

$$= \left(\frac{-6}{3}\right)^2 = (-2)^2$$

$$= 4$$

16. If $z^2 + z + 1 = 0, z \in C$, then

$\left|\sum_{n=1}^{15}\left(z^n + (-1)^n \frac{1}{z^n}\right)^2\right|$ is equal to …

[JEE (Main) – 26th June 2022 - Shift-2]

Sol. The correct answer is (2).

If $z^2 + z + 1 = 0$ where z is a complex number

Let ω and ω^2 are roots of $z = x + iy$

The root of this equation cube root $\omega^3 = 1$ of ω.

$$\omega = \frac{1}{\omega^2}$$

$$\left|\sum_{n=1}^{15}\left(z^n + (-1)^n \frac{1}{z^n}\right)^2\right|$$

$$\left|\sum_{n=1}^{15} z^{2n} + \sum_{n=1}^{15} \frac{1}{z^{2n}} + \sum_{n=1}^{15}(-1)^n\right|$$

$$\left|\sum_{n=1}^{15} \omega^{2n} + \sum_{n=1}^{15} \frac{1}{\omega^{2n}} + \sum_{n=1}^{15}(-1)^n\right|$$

$$\left|\sum_{n=1}^{15} \omega^{2n} + \sum_{n=1}^{15} \omega^{n} + \sum_{n=1}^{15}(-1)^n\right|$$

$|(\omega^2 + \omega^4 + \omega^6 + ... + \omega^{30}) + (\omega + \omega^2 + \omega^3 + ... + \omega^{15}) + (-1 + 1 - 1 + 1 - 1 + 1 .... (-1))|$

$|\omega^2(1 + \omega^2 + \omega^3 + ... + \omega^{15}) + \omega(1 + \omega + \omega^2 + ... + \omega^{14}) - 1|$

$$\left|\omega^2\left(\frac{1-\omega^{15}}{1-\omega}\right) + \omega\left(\frac{1-\omega^{14}}{1-\omega}\right) - 1\right|$$

Therefore $\omega^3 = 1$

$|\omega^2(0) + \omega(1 + \omega) - 1| = |\omega^2 + \omega - 1|$

$= |-1 - 1| = 2$

17. Let the abscissas of the two points P and Q be the roots of $2x^2 - rx + p = 0$ and the ordinates of P and Q be the roots of $x^2 - sx - q = 0$. If the equation of the circle described on PQ as diameter is $2(x^2 + y^2) - 11x - 14y - 22 = 0$, then $2r + s - 2q + p$ is equal to ______.

[JEE (Main) – 25th June 2022 - Shift-1]

Sol. The correct answer is (7).

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$

x_1, x_2 are two roots of the equations

$$2x^2 - rx + p = 0$$

Sum of the roots

$$x_1 + x_2 = \frac{r}{2}$$

$$x_1.x_2 = \frac{p}{2} \quad \text{...(i)}$$

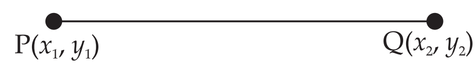
y_1, y_2 are two roots of the equation

$$x^2 - Sx - q = 0$$

$$y_1 + y_2 = S$$

$$y_1.y_2 = -q \quad \text{...(ii)}$$

$\therefore$ Coordinate of $P(x_1, y_1)$ and $Q(x_2, y_2)$



We have, equation of circle in diameter form

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\Rightarrow x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 = 0$

$$\Rightarrow x^2 + y^2 - \frac{r}{2}x - sy + \frac{p}{2} - q = 0 \quad \text{...(iii)}$$

Comparing with $2(x^2 + y^2) - 11x - 14y - 22 = 0$

$$\Rightarrow \frac{r}{2} = \frac{11}{2}$$

$\Rightarrow r = 11$

And $p - 2q = -22$

Then, the value of $2r + s - 2q + p = 22 + 7 - 22 = 7$

18. A point z moves in the complex plane such that $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$, then the minimum value of $\left|z - 9\sqrt{2} - 2i\right|^2$ is equal to ______.

[JEE (Main) – 31st Aug. 2021 - Shift-1]

Sol. The correct answer is (98).

$$\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$$

Let $z = x + iy$

$$\Rightarrow \quad \arg\left(\frac{x+iy-2}{x+iy+2}\right) = \frac{\pi}{4}$$

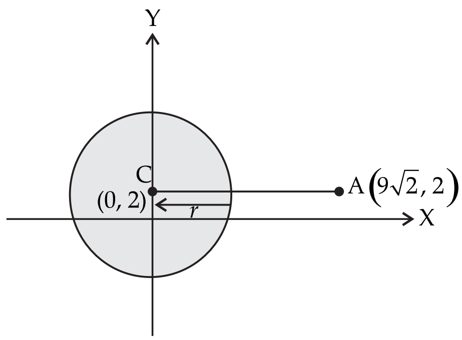
$$\Rightarrow \quad \arg\left(\left(\frac{x+iy-2}{x+iy+2}\right)\left(\frac{x+2-iy}{x+2-iy}\right)\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \arg\left(\frac{x^2+y^2-4+4iy}{(x+2)^2+y^2}\right) = \frac{\pi}{4}$$

$\Rightarrow \dfrac{x^2+y^2-4}{4y} = 1$

$\Rightarrow x^2 + (y-2)^2 = 8$ ...(i)

Equation (i) represent circle whose centre (0, 2) & radius $2\sqrt{2}$



Minimum distance, $d = |AC - r|$

Given point A $(9\sqrt{2}, 2) = |AC - r|$

$= |(9\sqrt{2} - 2\sqrt{2})| = 7\sqrt{2}$

Square of distance $= d^2 = (7\sqrt{2})^2 = 98$

19. Let z_1 and z_2 be two complex numbers such that $\arg(z_1 - z_2)$ and z_1, z_2 satisfy the equation $|z-3| = \text{Re}(z)$. Then, the imaginary part of $z_1 + z_2$ is equal to ___________.

[JEE (Main) – 27th Aug. 2021 - Shift-2]

Sol. The correct answer is (6).

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$

$\Rightarrow z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

$\because \quad \arg(z_1 - z_2) = \dfrac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{y_1 - y_2}{x_1 - x_2}\right) = \dfrac{\pi}{4}$

$\Rightarrow \quad y_1 - y_2 = x_1 - x_2$ ...(i)

Now, $\quad |z_1 - 3| = Re(z_1)$

$\Rightarrow \quad (x_1 - 3)^2 + y^2_1 = x_1^2$

And, $\quad |z_2 - 3| = Re(z_2)$

$\Rightarrow \quad (x_2 - 3)^2 + y^2_2 = x_2^2$

$\Rightarrow \quad (x_1 - 3)^2 - (x_2 - 3)^2 + (y_1^2 - y_2^2) = x_1^2 - x_2^2$

$\Rightarrow \quad (x_1 - x_2)(x_1 + x_2 - 6) + (y_1 - y_2)(y_1 + y_2) = (x_1 + x_2)(x_1 - x_2)$

$\Rightarrow \quad x_1 + x_2 - 6 + y_1 + y_2 = x_1 + x_2$

$\Rightarrow y_1 + y_2 = 6$

20. The sum of all integral values of k $(k \neq 0)$ for which the equation $\dfrac{2}{x-1} - \dfrac{1}{x-2} = \dfrac{2}{k}$ in x has no real roots, is ______.

[JEE (Main) – 26th Aug. 2021 - Shift-1]

Sol. The correct answer is (66).

$\dfrac{2}{x-1} - \dfrac{1}{x-2} = \dfrac{2}{k} \Rightarrow \dfrac{2x-4-x+1}{(x-1)(x-2)} = \dfrac{2}{k}$

$\Rightarrow \quad 2x^2 - (6+k)x + 3k + 4 = 0$

For non-real roots, $D < 0$

$\Rightarrow \quad (6+k)^2 - 8(3k+4) < 0$

$\Rightarrow \quad k^2 + 12k + 36 - 24k - 32 < 0$

$\Rightarrow \quad (k-6)^2 - 32 < 0$

Integral value of $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

Sum of $k = 66$

21. Let $z = \dfrac{1 - i\sqrt{3}}{2}, i = \sqrt{-1}$. Then the value of

$$21 + \left(z + \frac{1}{z}\right)^3 + \left(z^2 + \frac{1}{z^2}\right)^3 + \left(z^3 + \frac{1}{z^3}\right)^3 + \ldots + \left(z^{21} + \frac{1}{z^{21}}\right)^3$$

is _______________.

[JEE (Main) – 26th Aug. 2021 - Shift-1]

Sol. The correct answer is (13).

Given,

$-z = \dfrac{-1 + i\sqrt{3}}{2}$ and $i = \sqrt{-1}$

Let $-z = \omega$ or $z = \omega$

Now,

$z + \dfrac{1}{z} = -\omega - \dfrac{1}{\omega} = \dfrac{-\omega^2 - 1}{\omega} = \dfrac{\omega}{\omega} = 1$ ...(i)

And,

$z^2 + \dfrac{1}{z^2} = (-\omega)^2 + \dfrac{1}{(-\omega)^2}$

$\omega^2 + \dfrac{1}{\omega^2} = \dfrac{\omega^4 + 1}{\omega^2} = \dfrac{\omega + 1}{\omega^2} = \dfrac{-\omega^2}{\omega^2} = -1$ ...(ii)

Again,

$z^3 + \dfrac{1}{z^3} = (-\omega)^3 + \dfrac{1}{(-\omega)^3}$

$= -1 - 1 = -2$ ...(iii)

From equation (i), (ii) and (iii)

$\left(z + \dfrac{1}{z}\right)^3 + \left(z^2 + \dfrac{1}{z^2}\right)^3 + \left(z^3 + \dfrac{1}{z^3}\right)^3$

$= 1 - 1 - 8 = -8$ ...(iv)

Again,

$z^6 + \dfrac{1}{z^6} = (-\omega)^6 + \dfrac{1}{(-\omega)^6} = (1+1)^3 = 8$

$\left(z^3 + \dfrac{1}{z^3}\right)^3 + \left(z^6 + \dfrac{1}{z^6}\right)^3 = 0$ ...(v)

$$\left(z^9+\frac{1}{z^9}\right)^3+\left(z^{12}+\frac{1}{z^{12}}\right)^3=0 \quad \ldots(\text{vi})$$

$$\left(z^{15}+\frac{1}{z^{15}}\right)^3+\left(z^{18}+\frac{1}{z^{18}}\right)^3=0 \quad \ldots(\text{vii})$$

$$z^{21}+\frac{1}{z^{21}}=-8 \quad \ldots(\text{viii})$$

According to the question, by using above equations.

$\Rightarrow \quad 21-8=13$

22. The least positive integer n such that $\frac{(2i)^n}{(1+i)^{n-2}}, i=\sqrt{-1},$ is a positive integer is

[JEE (Main) – 26th Aug. 2021 - Shift-2]

Sol. The correct answer is (6).

$$\frac{(2i)^n}{(1+i)^{n-2}}=\frac{(2i)^n\,(1-i)^{n-2}}{((1+\text{i})\,(1-i))^{n-2}}$$

$$=\frac{2^n\cdot i^n\,(1-i)^n}{(1-i)^2\,2^{n-2}}$$

$$=\frac{4\cdot i^n\,(1-i)^n}{(-2i)}=\frac{4\,(1+i)^n}{(-2i)}$$

Now, $(1+i)^2=2i$
$(1+i)^4=-4$
$(1+i)^6=-8i$

$\therefore$ For $n=6$, given expression will become

$$\frac{4\,(-8i)}{(-2i)}=16$$

23. Let $\lambda \neq 0$ be in R. If α and β are the roots of the equation $x^2-x+2\lambda=0$ and α and γ are the roots of the equation $3x^2-10x+27\lambda=0$, then $\frac{\beta\gamma}{\lambda}$ is equal to __________.

[JEE (Main) – 26th Aug. 2021 - Shift-2]

Sol. The correct answer is (18).

Given: α & β are roots of the equation $x^2-x+2\lambda=0$

$\therefore \quad a+b=1 \quad \ldots(\text{i})$

$\alpha\beta=2\lambda \quad \ldots(\text{ii})$

Since, α and γ are the roots of $3x^2-10x+27\lambda=0$

$\therefore \quad \alpha+\beta=\frac{10}{3} \quad \ldots(\text{iii})$

$\alpha\gamma=9\lambda \quad \ldots(\text{iv})$

From $\frac{(\text{iv})}{(\text{iii})}$

$\Rightarrow \quad \frac{\alpha}{\beta}=\frac{9}{2} \quad \ldots(\text{v})$

From (iii) – (i)

$\Rightarrow \quad \alpha-\beta=\frac{10}{3}-1=\frac{7}{3} \quad \ldots(\text{vi})$

From (v) and (vi)

$\Rightarrow \quad b=\frac{2}{3}, r=3$

From (i) $\alpha=1-\beta=1-\frac{2}{3}=\frac{1}{3}$

From (ii) $\lambda=\frac{\alpha\beta}{2}=\frac{\frac{1}{3}\times\frac{2}{3}}{2}=\frac{1}{9}$

So, $\frac{\beta\lambda}{\lambda}=\frac{\frac{2}{3}\times 3}{\frac{1}{9}}=18$

24. If the real part of the complex number $z=\frac{3+2i\cos\theta}{1-3i\cos\theta}$, $\theta\in\left(0,\frac{\pi}{2}\right)$ is zero, then the value of $(\sin^2 3\theta+\cos^2\theta)$ is equal to _______.

[JEE (Main) – 27th July 2021 - Shift-2]

Sol. The correct answer is (1).

$$z=\left(\frac{3+2i\cos\theta}{1-3i\cos\theta}\right)=\left(\frac{3+2i\cos\theta}{1-3i\cos\theta}\right)\left(\frac{1+3i\cos\theta}{1+3i\cos\theta}\right)$$

$$\Rightarrow z=\frac{3-6\cos^2\theta+11i\cos\theta}{1+9\cos^2\theta}$$

Given, $\quad \text{Re}(z)=0 \Rightarrow \frac{3-6\cos^2\theta}{1+9\cos^2\theta}=0$

$\Rightarrow \quad \cos^2\theta=\frac{1}{2}$

$\Rightarrow \quad \theta=\frac{\pi}{4}$

$\therefore \quad \sin^2 3\theta+\cos^2\theta=\frac{1}{2}+\frac{1}{2}=1$

25. The number of real roots of the equation $e^{4x}-e^{3x}-4e^{2x}-e^x+1=0$ is equal to ______.

[JEE (Main) – 27th July 2021 - Shift-2]

Sol. The correct answer is (2).

$$e^{4x}-e^{3x}-4e^{2x}-e^x+1=0$$

$$\Rightarrow \quad e^{2x}-e^x-4-\frac{1}{e^x}+\frac{1}{e^{2x}}=0$$

$$\Rightarrow \quad \left(e^x+\frac{1}{e^x}\right)^2-2-\left(e^x+\frac{1}{e^x}\right)-4=0$$

$$\Rightarrow \quad \left(e^x+\frac{1}{e^x}\right)^2-\left(e^x+\frac{1}{e^x}\right)-6=0$$

Put $e^x + \dfrac{1}{e^x} = t$

$\Rightarrow t^2 - t - 6 = 0$

$\Rightarrow \quad t = 3, t = -2$ (not possible)

$\Rightarrow e^x + \dfrac{1}{e^x} = 3$

$\therefore \quad e^{2x} - 3e^x + 1 = 0$

Put $e^x = P \Rightarrow P^2 - 3P + 1 = 0 \Rightarrow$ number of solutions = 2

26. If α, β are roots of the equation $x^2 + 5\sqrt{2}\, x + 10 = 0$, $\alpha > \beta$ and $P_n = \alpha^n - \beta^n$ for each positive integer n, then the value of $\left(\dfrac{P_{17}P_{20} + 5\sqrt{2}\, P_{17}P_{19}}{P_{18}P_{19} + 5\sqrt{2}\, P_{18}^2}\right)$ is equal to……

[JEE (Main) – 25th July 2021 - Shift-1]

Sol. The correct answer is (1).

$$x^2 + 5\sqrt{2}x + 10 = 0, \alpha > \beta$$

Giving $P_n = \alpha^n - \beta^n$

$\Rightarrow P_{17} = \alpha^{17} - \beta^{17}, P^{18} = \alpha^{18} - \beta^{18}$

$\Rightarrow \alpha^2 + 10 = 5\sqrt{2}\alpha$ …(i)

$\Rightarrow \beta^2 + 10 = -5\sqrt{2}\beta$ … (ii)

$$\Rightarrow \frac{P_{17}(\alpha^{20} - \beta^{20} + 5\sqrt{2}(\alpha^{19} - \beta^{19}))}{P_{18}((\alpha^{15} - \beta^{19}) + 5\sqrt{2}(\alpha^{18} - \beta^{18})}$$

$$\Rightarrow \frac{P_{17}(\alpha^{18}(\alpha^2 + 5\sqrt{2}\alpha) - \beta^{18}(5\sqrt{2}\beta + \beta^2))}{P_{18}(\alpha^{17}(\alpha^2 + 5\sqrt{2}\alpha) - \beta^{17}\left(5\sqrt{2}\beta + \beta^2\right)}$$

Using equation (i) and (ii)

$$= \frac{P_{17} \cdot P_{18}(-10)}{P_{18} \cdot P_{17}(-10)} = 1$$

27. If $a + b + c = 1$, $ab + bc + ca = 2$ and $abc = 3$, then the value of $a^4 + b^4 + c^4$ is equal to ________.

[JEE (Main) – 25th July 2021 - Shift-2]

Sol. The correct answer is (13).

$(a + b + c)^2 = 1$

$\Rightarrow a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1$

$\Rightarrow a^2 + b^2 + c^2 = -3$ …(i)

$\Rightarrow ab + bc + ca = 2$ …(ii)

Squaring of equation (ii), we get

$\Rightarrow a^2b^2 + b^2c^2 + c^2a^2 + 2(ab^2c + bc^2a + ca^2b) = 4$

$\Rightarrow a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a+b+c) = 4$

$\Rightarrow a^2b^2 + b^2c^2 + c^2a^2 + 6 = 4$

$\Rightarrow a^2b^2 + b^2c^2 + c^2a^2 = -2$ …(iii)

Squaring of equation (i), we get

$\Rightarrow a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + c^2a^2) = 9$

$\Rightarrow a^4 + b^4 + c^4 - 4 = 9$

$\Rightarrow a^4 + b^4 + c^4 = 13$

28. The number of solutions of the equation $\log_{(x+1)}(2x^2 + 7x + 5) + \log_{(2x+5)}(x+1)^2 - 4 = 0$, $x > 0$, is ……..

[JEE (Main) – 20th July 2021 - Shift-2]

Sol. The correct answer is (1).

$$\log_{(x+1)}((2x+5)(x+1)) + \log_{(2x+5)}(x+1)^2 = 4$$

$$\therefore \quad 1 + \log_{(x+1)}(2x+5) + 2\log_{(2x+5)}(x+1) = 4$$

Put $\quad \log_{(x+1)}(2x+5) = t$

So, $1 + t + \dfrac{2}{t} = 4$

$\Rightarrow t^2 + t + 2 = 4t$

$\Rightarrow t^2 - 3t + 2 = 0$

$\Rightarrow t = 1, t = 2$

For $t = 1 \quad \Rightarrow \quad 2x + 5 = x + 1$

$\Rightarrow x = -4$ (rejected), as $x + 1 > 0$ for $\log(x+1)$ to be defined

For $t = 2 \quad \Rightarrow \quad 2x + 5 = (x+1)^2$

$\Rightarrow x = 2, x = -2$ (rejected)

29. Let z_1, z_2 be the roots of the equations $z^2 + az + 12 = 0$ and z_1, z_2 form an equilateral triangle with origin. Then, the value of $|a|$ is ______.

[JEE (Main) – 18th March 2021 - Shift-1]

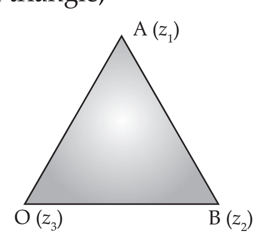
Sol. The correct answer is (6).

We have to find the value of $|a|$ where z_1, z_2 are the roots of the equations $z^2 + az + 12 = 0$ and z_1, z_2 are forming an equilateral triangle with origin.

Applying roots and coefficients relation for given quadratic equation,

$$z^2 + az + 12 = 0$$
$$z_1 + z_2 = -a$$
$$z_1 z_2 = 12$$

Drawing a triangle,



Since third vertex is given origin therefore $z_3 = 0$

We know the condition of equilateral Δ for complex plane,

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$\Rightarrow z_1^2 + z_2^2 + 0 = z_1 z_2 + 0 + 0$

$\Rightarrow (z_1 + z_2)^2 - 2z_1 z_2 = z_1 z_2$

$\Rightarrow (-a)^2 - 2 \times 12 = 12$

$\Rightarrow a^2 = 12 + 24$

$\Rightarrow a^2 = 36$

$\Rightarrow a = \pm\sqrt{36}$

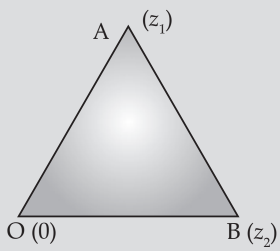
$\Rightarrow |a| = |\pm 6|$

$\Rightarrow |a| = 6$

Hence , value of $|a|$ is 6.

Hint:
(i) Use roots and coefficients relation for quadratic equation,
(ii) Use the condition of equilateral triangle in complex plane for given vertices.

Shortcut Method:



In equilateral triangle ,

$z_1^2 + z_2^2 = z_1 z_2$

$\Rightarrow (z_1 + z_2)^2 = 3z_1 z_2$

$\Rightarrow |a| = 6$

30. If $f(x)$ and $g(x)$ are two polynomials such that the polynomial $p(x) = f(x^3) + x\,g(x^3)$ is divisible by $x^2 + x + 1$, then $p(1)$ is equal to _______.

[JEE (Main) – 18th March 2021 - Shift-2]

Sol. The correct answer is (0).

Given

$f(x)$ and $g(x)$ are two polynomials such that the polynomial $p(x) = f(x^3) + xg(x^3)$ is divisible by $x^2 + x + 1$.

$x^2 + x + 1 = 0$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow \quad x = \dfrac{-1 + \sqrt{3}i}{2} = \omega$

or $\quad x = \dfrac{-1 - \sqrt{3}i}{2} = \omega^2$

$p(\omega) = f(\omega^3) + \omega g(\omega^3)$

$p(\omega) = f(1) + \omega g(1) = 0$ …(i)

$\because\ x^2 + x + 1$ dividing $f(x)$ $(\because \omega^3 = 1)$ …(ii)

$(\because (\omega^3)^2 = 1)$

$\because\ f(1)$ and $g(1)$ are real $f(1) = g(1) = 0$

$\therefore\ f(1) = 0$

Hint:
(i) Roots of $x^2 + x + 1 = 0$ are ω and ω^2.
(ii) $p(x)$ is divisible by $x^2 + x + 1$. So, $p(w) = p(w^2) = 0$.
(iii) Compare real and imaginary parts to find $f(1)$ and $g(1)$.
(iv) Finally, compute $p(1)$.

Shortcut Method:

$x^2 + x + 1 = 0$

$x = \omega, \omega^2$

$p(1) = f(1) + g(1)$

$p(\omega) = f(1) + \omega g(1)$

$p(\omega^2) = f(1) + \omega^2 g(1)$

$p(1) + p(\omega) + p(\omega^2) = 3f(1) + 0$

$f(1) = 0$

$\therefore g(1) = 0$

$p(1) = 0 + 0 = 0$

31. Let z and ω be two complex numbers such that $\omega = z\bar{z} - 2z + 2, \left|\dfrac{z+i}{z-3i}\right| = 1$ and Re (ω) has minimum value. Then, the minimum value of $n \in \mathbb{N}$ for which ω^n is real, is equal to ____.

[JEE (Main) – 16th March 2021 - Shift-1]

Sol. The correct answer is (4).

Given that

$\omega = z\bar{z} - 2z + 2$ and $\left|\dfrac{z+i}{z-3i}\right| = 1$ ...(i)

Using $\left|\dfrac{z_1}{z_2}\right| = 1 \Rightarrow |z_1| = |z_2|$,

We can write $\left|\dfrac{z+i}{z-3i}\right| = 1$ as,

$|z+i| = |z-3i|$ …(ii)

Put $z = x + iy$ in equation (ii)

$|x + iy + i| = |x + iy - 3i|$

$\Rightarrow |x + (y+1)i| = |x + (y-3)i|$ …(iii)

Since, $|z| = \sqrt{x^2 + y^2}$,

So from equation (iii)

$$\sqrt{x^2 + (y+1)^2} = \sqrt{x^2 + (y-3)^2}$$

$\Rightarrow x^2 + y^2 + 2y + 1 = x^2 + y^2 - 6y + 9$

$\Rightarrow 8y = 8 \Rightarrow y = 1$ ...(iv)

Now using equation (i)

$$\omega = (x+iy)(\overline{x+iy}) - 2(x+iy) + 2$$

using $z.\bar{z} = |z|^2$, we can write

$$\omega = x^2 + y^2 - 2(x+iy) + 2$$

$$\Rightarrow \omega = (x^2 + y^2 - 2x + 2) + 2yi$$

$$\omega = (x^2 - 2x + 3) - 2i;\ y = 1 \quad \text{...(v)}$$

Now,

$$\text{Re}(\omega) = x^2 - 2x + 3$$

$$\Rightarrow \text{Re}(\omega) = 2 + (x-1)^2$$

Re(ω) will be minimum when $x = 1$.
Hence $z = 1 + i;\ x = y = 1$
Now using equation (v)

$$\omega = 1 - 2 + 3 - 2i \Rightarrow \omega = 2 - 2i$$

$$\Rightarrow \omega = 2(1-i) = \omega \Rightarrow 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

$$\omega = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)i\right)$$

$$\Rightarrow \omega = 2\sqrt{2}\left(\cos\frac{\pi}{4} + \sin\left(-\frac{\pi}{4}\right)i\right)$$

$$\omega = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)i\right)$$

$$\Rightarrow \omega = 2\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)} \Rightarrow \omega^n = \left(2\sqrt{2}\right)^n e^{i\left(-\frac{n\pi}{4}\right)}$$

ω^n is real and minimum when $n = 4$.

Shortcut Method:

Let $z = x + iy;\ \bar{z} = x - iy;$

$$z.\bar{z} = x^2 + y^2;\ |z| = \sqrt{x^2 + y^2}$$

$$\omega = z.\bar{z} - 2z + 2$$

$$\Rightarrow \omega = x^2 + y^2 + 2 - 2x - 2yi \quad \text{...(i)}$$

$$\left|\frac{z+i}{z-3i}\right| = 1 \Rightarrow |z+i| = |z-3i| \Rightarrow y = 1 \quad \text{...(ii)}$$

So, $\omega = x^2 - 2x + 3 - 2i$

$$\Rightarrow \text{Re}(\omega) = x^2 - 2x + 3 = 2 + (x-1)^2$$

Re(ω) min at $x = 1$.

$z = 1 + i$

$$\omega = 2\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)} \Rightarrow \omega^n = \left(2\sqrt{2}\right)^n e^{\left(\frac{-n\pi}{4}\right)i}$$

ω^n is real and minimum when $n = 4$.

32. The number of solutions of the equation $\log_4(x - 1) = \log_2(x - 3)$ is ______.

[JEE (Main) – 26th Feb. 2021 - Shift-1]

Sol. The correct answer is (1).

Given: $\log_4(x - 1) = \log_2(x - 3)$

$$\log_{a^m} b = \frac{1}{m}\log_a b$$

$$\log_{2^2}(x-1) = \frac{1}{2}\log_2(x-1)$$

$$\frac{1}{2}\log_2(x-1) = \log_2(x-3)$$

$\Rightarrow \quad \log_2(x - 1) = 2\log_2(x - 3)$

$m \log_a b = \log_a b^m$

$\Rightarrow \log_2(x - 1) = \log_2(x - 3)^2$

$\log_a x = \log_a y$

$\Rightarrow \quad x = y$

$\therefore \quad x - 1 = (x - 3)^2$

$\Rightarrow \quad x - 1 = x^2 + 9 - 6x$

$\Rightarrow \quad x^2 - 7x + 10 = 0$

$\Rightarrow \quad x^2 - 5x - 2x + 10 = 0$

$\Rightarrow \quad x(x - 5) - 2(x - 5) = 0$

$\Rightarrow \quad (x - 2)(x - 5) = 0$

$\Rightarrow \quad x = 2, 5$

For $\log_4(x - 1)$ to be defined,
$x - 1 > 0 \Rightarrow x > 1$
For $\log_4(x - 3)$ to be defined,
$x - 3 > 0 \Rightarrow x > 3$
So, only answer is $x = 5$
Number of solution = 1.

Hint:

(i) $\log_{a^n} b = \frac{1}{n}\log_a b$

(ii) $m \log_a b = \log_a b^m$

(iii) $\log_a x = \log_a y \Rightarrow x = y$

Shortcut Method:

$\log_4(x - 1) = \log_2(x - 3)$

$\log_4(x - 1) = 2\log_4(x - 3)$

$\Rightarrow \quad x - 1 = (x - 3)^2$

$\Rightarrow \quad x^2 + 9 - 6x - x + 1 = 0$

$\Rightarrow \quad x^2 - 7x + 10 = 0$

$x = 2, 5$

For $\quad x = 2$

$\log_2(x-3)$ is not defined.

$\therefore \quad x = 5$

Number of solution is 1.

33. The sum of 162^{nd} power of the roots of the equation $x^3 - 2x^2 + 2x - 1 = 0$ is ______.

[JEE (Main) – 26th Feb. 2021 - Shift-1]

Sol. The correct answer is (3).

Given: $\quad x^3 - 2x^2 + 2x - 1 = 0$

$\Rightarrow \quad (x^3 - 1) - 2x(x-1) = 0$

$\because \quad a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$

$x^3 - 1 = (x-1)(x^2 + 1 + x)$

$(x-1)(x^2 + 1 + x) - 2x(x-1) = 0$

$\Rightarrow \quad (x-1)(x^2 + 1 + x - 2x) = 0$

$\Rightarrow \quad (x-1)(x^2 - x + 1) = 0$

$x = 1, x^2 - x + 1 = 0$

$$\Rightarrow x = \frac{1 \pm \sqrt{1^2 - 4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$x = \frac{1 \pm \sqrt{3}i}{2} \left(i = \sqrt{-1}\right)$$

$$x = \frac{1 + \sqrt{3}i}{2} \text{ or } \frac{1 - \sqrt{3}i}{2}$$

$$\omega = \frac{-1 + \sqrt{3}i}{2}, \omega^2 = \frac{-1 - \sqrt{3}i}{2}$$

where ω and ω^2 are cube roots of unity.

$\therefore \; x = 1, -\omega, -\omega^2$

Sum of 162^{nd} power of the roots

$= 1^{162} + (-\omega)^{162} + (-\omega^2)^{162}$

$= 1 + \omega^{162} + \omega^{324}$

$\therefore \; 1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

Sum $= 1 + (\omega^3)^{54} + (\omega^3)^{108}$

$= 1 + 1 + 1 = 3.$

Hint:

(i) Solution of $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(ii) $\omega = \dfrac{-1 + \sqrt{3}i}{2}, \omega^2 = \dfrac{-1 - \sqrt{3}i}{2}, i = \sqrt{-1}$

(iii) $\omega^3 = 1$

Shortcut Method:

$x^3 - 2x^2 + 2x - 1 = 0$

$\Rightarrow \quad (x-1)(x^2 - x + 1) = 0$

$$\Rightarrow \quad x = 1, \frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}$$

$x = 1, -\omega^2, -\omega$

Sum of 162^{nd} power of roots

$= 1^{162} + (-\omega^2)^{162} + (-\omega)^{162}$

$= 1 + (\omega)^{324} + (\omega)^{162}$

$= 1 + 1 + 1 = 3.$

34. Let α and β be two real numbers such that $\alpha + \beta = 1$ and $a\beta = -1$. Let $P_n = (\alpha)^n + (\beta)^n$, $P_{n-1} = 11$ and $P_{n+1} = 29$ for some integer $n \geq 1$. Then, the value of P_n^2 is ______.

[JEE (Main) – 26th Feb. 2021 - Shift-2]

Sol. The correct answer is (324).

Given that α and β are two real numbers.

Also, $\quad \alpha + \beta = 1 \quad$ ...(i)

$\alpha\beta = -1 \quad$ ...(ii)

Let α and β are the roots of a quadratic equation, then

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Using equation (i) and (ii), we can write

$x^2 - (1)x + (-1) = 0$

$\Rightarrow \quad x^2 - x - 1 = 0 \quad$ ...(iii)

As α and β are the roots of the equation $x^2 - x - 1 = 0$ then α and β will satisfy it.

$\alpha^2 - \alpha - 1 = 0$ and $\beta^2 - \beta - 1 = 0$

$\Rightarrow \quad \alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$

$\Rightarrow \quad (\alpha^2)(\alpha^{n-1}) = (\alpha + 1)(\alpha^{n-1})$

and $(\beta^2)(\beta^{n-1}) = (\beta + 1)(\beta^{n-1})$

$\Rightarrow \quad \alpha^{n+1} = \alpha^n + \alpha^{n-1}$ and

$\beta^{n+1} = \beta^n + \beta^{n-1}$

$\Rightarrow \alpha^{n+1} + \beta^{n+1} = \left(\alpha^n + \beta^n\right) + \left(\alpha^{n-1} + \beta^{n-1}\right) \quad$ ...(iv)

Also given that $\quad P_n = \alpha^n + \beta^n \quad$ ...(v)

$P_{n-1} = 11 \quad$ ...(vi)

$P_{n+1} = 29 \quad$ ...(vii)

Now using equation (iv) and (v) we can write

$P_{n+1} = P_n + P_{n-1}$

Now using equation (vi) and (vii)

$29 = P_n + 11$

$\Rightarrow \quad P_n = 29 - 11$

$\Rightarrow \quad P_n = 18$

$\Rightarrow \quad (P_n)^2 = (18)^2$

$\Rightarrow \quad P^2{}_n = 324$

Hint:

(i) Quadratic equation whose roots are 'α' and 'β' will be

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

(ii) Root of any equation, always satisfy its equation.

(iii) Using manipulation find value of $\alpha^{n+1} + \beta^{n+1}$

Shortcut Method:

Given, $\alpha + \beta = 1$ and $\alpha\beta = -1$ then

$\alpha^2 - \alpha - 1 = 0$ and $\beta^2 - \beta - 1 = 0$

$\Rightarrow \quad \alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$

$\Rightarrow \quad \alpha^{n+1} + \beta^{n+1} = (\alpha^n + \beta^n) + (\alpha^{n-1} + \beta^{n-1})$

$\Rightarrow \quad P_{n+1} = P_n + P_{n-1}$

$\Rightarrow \quad P_n = 18 \Rightarrow P^2_n = 324$

35. If the least and the largest real values of α, for which the equation $z + \alpha\,|z-1| + 2i = 0$ $\left(z \in \mathrm{C} \text{ and } i = \sqrt{-1}\right)$ has a solution, are p and q respectively; then $4(p^2 + q^2)$ is equal to ___.

[JEE (Main) – 24th Feb. 2021 - Shift-1]

Sol. The correct answer is (10).

Given:

$z + \alpha\,|z-1| + 2i = 0 \left(z \in \mathrm{C} \text{ and } i = \sqrt{-1}\right)$ has a solution.

Put $z = x + iy$

$$(x+iy) + \alpha\,|x+iy-1| + 2i = 0$$

$\because$ Modulus of $z = x + iy$ is $|z| = \sqrt{x^2+y^2}$

$\therefore \quad (x+iy) + \alpha\sqrt{(x-1)^2+y^2} + 2i = 0$

Compare real and imaginary parts,

$\therefore \quad y + 2 = 0$ and $x + \alpha\sqrt{(x-1)^2+y^2} = 0$

$\Rightarrow \quad y = -2$ and $x + \alpha\sqrt{(x-1)^2+4} = 0$

$\Rightarrow \quad x^2 = \alpha^2(x-1)^2 + 4\alpha^2$

$\Rightarrow \quad \alpha^2 = \dfrac{x^2}{x^2-2x+5}$

$\Rightarrow \quad x^2(\alpha^2-1) - 2x\alpha^2 + 5\alpha^2 = 0$

is a quadratic in x whose roots are real.

$\therefore \quad D \geq 0$

$\Rightarrow \quad (2\alpha^2)^2 - 4(\alpha^2-1)(5\alpha^2) \geq 0$

$\Rightarrow \quad 4\alpha^4 - 4(\alpha^2-1)(5\alpha^2) \geq 0$

$\Rightarrow \quad \alpha^2\left(-4\alpha^2+5\right) \geq 0 \Rightarrow \alpha^2\left(\alpha^2 - \dfrac{5}{4}\right) \leq 0$

$\Rightarrow \quad \alpha^2 \in \left[0, \dfrac{5}{4}\right] \Rightarrow \alpha \in \left[-\dfrac{\sqrt{5}}{2}, \dfrac{\sqrt{5}}{2}\right]$

So, $p = -\dfrac{\sqrt{5}}{2}, q = \dfrac{\sqrt{5}}{2}$

$\Rightarrow \quad 4(p^2 + q^2) = 4\left(\dfrac{5}{4} + \dfrac{5}{4}\right) = 10$

Hint:

(i) Put $z = x + iy$ in given equation

(ii) Compare real and imaginary parts both sides.

(iii) For the roots to be real. $D \geq 0$

(iv) Find minimum and maximum value of α to find required value.

Shortcut Method:

$$\alpha = \frac{-z-2i}{|z-1|}$$

$$z = x + iy$$

$$\alpha = \frac{-x-(y+2)i}{|z-1|} \quad \alpha \text{ is real}$$

$$\alpha = \frac{-x}{\sqrt{(x-1)^2+y^2}}$$

$$\alpha^2 = \frac{x^2}{x^2-2x+5} \quad \therefore\ y = -2$$

$\Rightarrow \quad x^2(\alpha^2-1) - 2x\alpha^2 + 5\alpha^2 = 0$

$x \in \mathrm{R}, D \geq 0$

$\Rightarrow \quad \alpha^2\left(\alpha^2 - \dfrac{5}{4}\right) \leq 0$

$$\alpha \in \left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$$

$$4(p^2 + q^2) = 4\left(\frac{5}{4} + \frac{5}{4}\right) = 10$$

36. The number of the real roots of the equation $(x+1)^2 + |x-5| = \dfrac{27}{4}$ is ____

[JEE (Main) – 24th Feb. 2021 - Shift-2]

Sol. The correct answer is (2).

Given: $(x+1)^2 + |x-5| = \dfrac{27}{4}$

Consider two cases:

Case I: $x \geq 5$

$\therefore |x-5| = x - 5$

Equation becomes

$$(x+1)^2 + x - 5 = \frac{27}{4}$$

$\Rightarrow \quad x^2 + 2x + 1 + x - 5 = \dfrac{27}{4}$

$\Rightarrow \quad x^2 + 3x - 4 - \dfrac{27}{4} = 0$

$\Rightarrow \quad 4x^2 + 12x - 16 - 27 = 0$

$\Rightarrow \quad 4x^2 + 12x - 43 = 0$

$\Rightarrow \quad x = \dfrac{-12 \pm \sqrt{144+688}}{8}$

$x = \dfrac{-12 \pm \sqrt{832}}{8}$

$x = \dfrac{-12 \pm 28.8}{8} = \dfrac{-3 \pm 7.2}{2}$

None of which is greater than or equal to 5. So, no solution in this case.

Case II: $x \le 5$

Equation becomes,

$(x + 1)^2 - (x - 5) = \dfrac{27}{4}$

$\Rightarrow x^2 + x + 6 - \dfrac{27}{4} = 0 \Rightarrow 4x^2 + 4x - 3 = 0$

$\Rightarrow x = \dfrac{-4 \pm \sqrt{16+48}}{8} \Rightarrow x = \dfrac{-4 \pm 8}{8}$

$\Rightarrow x = \dfrac{-12}{8}, \dfrac{4}{8} \Rightarrow x = -\dfrac{3}{2}, \dfrac{1}{2}$

Both are less than 5.

So, total number of real roots = 2.

Hint:

(i) If $x \ge 5$ then $|x-5| = x - 5$.

(ii) If $x < 5$ then $|x-5| = 5 - x$.

Shortcut Method:

For $x \ge 5$

$(x + 1)^2 + x - 5 = \dfrac{27}{4}$

$\Rightarrow \quad x^2 + 1 + 3x - 5 = \dfrac{27}{4}$

$\Rightarrow \quad x^2 + 3x - 4 = \dfrac{27}{4}$

$\Rightarrow \quad x^2 + 3x - \dfrac{43}{4} = 0$

If roots are more than 5, its sum cannot be –3

Sum of roots = –3

So, no solution. For $x < 5$

$\Rightarrow x \quad (x+1)^2 - x + 5 - \dfrac{27}{4} = 0$

$\Rightarrow x = \dfrac{-3}{2}, \dfrac{1}{2}$

So, 2 roots.

37. Let $i = \sqrt{-1}$, If $\dfrac{\left(-1+i\sqrt{3}\right)^{21}}{(1-i)^{24}} + \dfrac{\left(1+i\sqrt{3}\right)^{21}}{(1+i)^{24}} = k$, and $n = [|k|]$ be the greatest integral part of $|k|$. Then $\sum_{j=0}^{n+5}(j+5)^2 - \sum_{j=0}^{n+5}(j+5)$ is equal to _______.

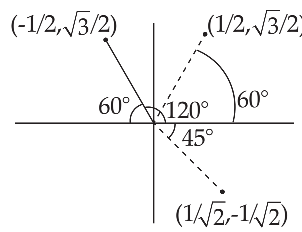
[JEE (Main) – 24th Feb. 2021 - Shift-2]

Sol. The correct answer is (310).

Given: $i = \sqrt{-1}$

$$\frac{\left(-1+i\sqrt{3}\right)^{21}}{(1-i)^{24}} + \frac{\left(1+i\sqrt{3}\right)^{21}}{(1+i)^{24}} = k$$

$$\Rightarrow \frac{\left(2\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)\right)^{21}}{\left(\sqrt{2}\left(\frac{1}{2}-\frac{i}{\sqrt{2}}\right)\right)^{24}} + \frac{\left(2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)^{21}}{\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right)^{24}} = k$$



$$\Rightarrow -\frac{1}{2}+i\frac{\sqrt{3}}{2} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = e^{i\frac{2\pi}{3}}$$

$$\Rightarrow \frac{1}{2}+i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = e^{i\frac{\pi}{3}}$$

$$\Rightarrow \frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{i\frac{\pi}{4}}$$

$$\Rightarrow \frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = e^{-i\frac{\pi}{4}}$$

$$\therefore \quad k = \frac{2^{21}.\left(e^{i\frac{2\pi}{3}}\right)^{21}}{2^{12}\left(e^{-i\frac{\pi}{4}}\right)^{24}} + \frac{2^{21}.\left(e^{i\frac{\pi}{3}}\right)^{21}}{2^{12}\left(e^{i\frac{\pi}{4}}\right)^{24}}$$

$= 2^9.e^{i20\pi} + 2^9.e^{i\pi}$

$= 2^9(e^{i20\pi} + e^{i\pi})$

$= 2^9(\cos 20\pi + i \sin 20\pi + \cos \pi + i \sin \pi)$

$= 2^9(1 + i(0) + (-1) + i(0)) = 2^9 \times 0 = 0$

$n = [|k|]$

$\Rightarrow \quad n = 0$

$$S = \sum_{j=0}^{n+5}(j+5)^2 - \sum_{j=0}^{n+5}(j+5)$$

$$= \sum_{j=0}^{5}(j+5)^2 - \sum_{j=0}^{5}(j+5)$$

$$= \sum_{j=0}^{5}[(j+5)^2 - (j+5)]$$

$$= \sum_{j=0}^{5}[j^2 + 25 + 10j - j - 5]$$

$$= \sum_{j=0}^{5}(j^2 + 9j + 20)$$

$$= \sum_{j=0}^{5} j^2 + 9\sum_{j=0}^{5} j + 20\sum_{j=0}^{5} 1$$

$$= \sum_{j=1}^{5} j^2 + 9\sum_{j=1}^{5} j + 20 \times 6$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$S = \frac{5\times6\times11}{6} + 9\times\frac{5\times6}{2} + 120$$

$= 55 + 135 + 120 = 310$

Hint:

(i) $\cos\theta + i\sin\theta = ei^{\theta}$

(ii) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(iii) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Shortcut Method:

$$\frac{2^{21}\left(e^{i\frac{2\pi}{3}}\right)^{21}}{2^{12}\left(e^{-i\frac{\pi}{4}}\right)^{24}} + \frac{2^{21}.\left(e^{i\frac{\pi}{3}}\right)^{21}}{2^{12}\left(e^{i\frac{\pi}{4}}\right)^{24}}$$

$= 2^9(e^{i20\pi} + e^{i\pi}) = 0 = k$

$n = [|k|] = 0$

$$S = \sum_{j=0}^{n+5}(j+5)^2 - \sum_{j=0}^{n+5}(j+5)$$

$$= \sum_{j=0}^{5}(j^2 + 9j + 20)$$

$$= \frac{5\times6\times11}{6} + \frac{9\times5\times6}{2} + 120$$

$= 55 + 135 + 120$

$S = 310.$

❐❐❐