Differential Equations

Chapter 9

Syllabus

Ordinary differential equations, their order and degree, the formation of differential equations, solution of differential equation by the method of separation of variables, solution of a homogeneous and linear differention equation of the type $\frac{dy}{dx}+p(x)y=q(x)$

Topic
Differential Equations

JEE (Main) Previous Year Questions

Multiple Choice Questions

1. Let $\alpha x = \exp(x^{\beta}y^{\gamma})$ be the solution of the differential equation $2x^2y\,dy - (1 - xy^2)\,dx = 0$, $x > 0$, $y(2) = \sqrt{\log_e 2}$. Then $\alpha + \beta - \gamma$ equals :

(1) 1 **(2)** –1 **(3)** 3 **(4)** 0

[JEE (Main) – 1st February 2023 - Shift-2]

2. If $y = y(x)$ is the solution curve of the differential equation $\frac{dy}{dx}+y\tan x = x\sec x, 0 \le x \le \frac{\pi}{3}$, $y(0) = 1$, then $y\left(\frac{\pi}{6}\right)$ is equal to

(1) $\frac{\pi}{12}-\frac{\sqrt{3}}{2}\log_e\left(\frac{2\sqrt{3}}{e}\right)$ **(2)** $\frac{\pi}{12}-\frac{\sqrt{3}}{2}\log_e\left(\frac{2}{e\sqrt{3}}\right)$

(3) $\frac{\pi}{12}+\frac{\sqrt{3}}{2}\log_e\left(\frac{2}{e\sqrt{3}}\right)$ **(4)** $\frac{\pi}{12}+\frac{\sqrt{3}}{2}\log_e\left(\frac{2\sqrt{3}}{e}\right)$

[JEE (Main) – 1st February 2023 - Shift-1]

3. The solution of the differential equation $\frac{dy}{dx}=-\left(\frac{x^2+3y^2}{3x^2+y^2}\right)$, $y(1) = 0$ is

(1) $\log_e|x+y|-\frac{xy}{(x+y)^2}=0$

(2) $\log_e|x+y|+\frac{2xy}{(x+y)^2}=0$

(3) $\log_e|x+y|-\frac{2xy}{(x+y)^2}=0$

(4) $\log_e|x+y|+\frac{xy}{(x+y)^2}=0$

[JEE (Main) – 30th January 2023 - Shift-2]

4. Let $y = y(x)$ be the solution of the differential equation $x\log_e x\frac{dy}{dx}+y = x^2\log_e x$, $(x > 1)$. If $y(2) = 2$, then $y(e)$ is equal to

(1) $\frac{1+e^2}{2}$ **(2)** $\frac{4+e^2}{4}$ **(3)** $\frac{2+e^2}{2}$ **(4)** $\frac{1+e^2}{4}$

[JEE (Main) – 29th January 2023 - Shift-2]

5. Let $y = y(x)$ be the solution curve of the differential equation

$\frac{dy}{dx}=\frac{y}{x}(1+xy^2(1+\log_e x))$, $x > 0$, $y(1) = 3$. Then $\frac{y^2(x)}{9}$ is equal to :

(1) $\frac{x^2}{2x^3(2+\log_e x^3)-3}$ **(2)** $\frac{x^2}{3x^3(1+\log_e x^2)-2}$

(3) $\frac{x^2}{7-3x^3(2+\log_e x^2)}$ **(4)** $\frac{x^2}{5-2x^3(2+\log_e x^3)}$

[JEE (Main) – 25th January 2023 - Shift-1]

6. Let $y = y(x)$ be the solution of the differential equation $x^3dy + (xy - 1)\,dx = 0$, $x > 0$, $y\left(\frac{1}{2}\right) = 3 - e$. Then $y(1)$ is equal to

(1) 1 **(2)** e **(3)** 3 **(4)** $2 - e$

[JEE (Main) – 24th January 2023 - Shift-1]

7. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} + \frac{5}{x(x^5+1)}y = \frac{(x^5+1)^2}{x^7}, x > 0$. If $y(1) = 2$, then $y(2)$ is equal to:

(1) $\frac{693}{128}$ **(2)** $\frac{637}{128}$ **(3)** $\frac{697}{128}$ **(4)** $\frac{679}{128}$

[JEE (Main) – 11th April 2023 - Shift-2]

8. Let the solution curve $y = y(x)$ of the differential equation $\left(1 + e^{2x}\right)\left(\frac{dy}{dx} + y\right) = 1$ pass through the point $\left(0, \frac{\pi}{2}\right)$. Then, $\lim\limits_{\delta x \to \infty} e^x y(x)$ is equal to

(1) $\frac{\pi}{4}$ **(2)** $\frac{3\pi}{4}$ **(3)** $\frac{\pi}{2}$ **(4)** $\frac{3\pi}{2}$

[JEE (Main) – 29th July 2022 - Shift-1]

9. If the solution curve of the differential equation $\frac{dy}{dx} = \frac{x+y-2}{x-y}$ passes through the point (2, 1) and $(k+1, 2)$, $k>0$, then

(1) $2\tan^{-1}\left(\frac{1}{k}\right) = \log_e(k^2 + 1)$

(2) $\tan^{-1}\left(\frac{1}{k}\right) = \log_e(k^2 + 1)$

(3) $2\tan^{-1}\left(\frac{1}{k+1}\right) = \log_e(k^2 + 2k + 2)$

(4) $2\tan^{-1}\left(\frac{1}{k}\right) = \log_e\left(\frac{k^2+1}{k^2}\right)$

[JEE (Main) – 29th July 2022 - Shift-2]

10. Let $y = y(x)$ be the solution curve of the differential equation

$$\frac{dy}{dx} + \left(\frac{2x^2+11x+13}{x^3+6x^2+11x+6}\right) y = \frac{(x+3)}{x+1}, x > -1,$$

which passes through the point (0,1). Then, $y(1)$ is equal to:

(1) $\frac{1}{2}$ **(2)** $\frac{3}{2}$ **(3)** $\frac{5}{2}$ **(4)** $\frac{7}{2}$

[JEE (Main) – 29th July 2022 - Shift-2]

11. Let the solution curve of the differential equation $x dy = \left(\sqrt{x^2+y^2} + y\right) dx$, $x > 0$, intersect the line $x = 1$ at $y = 0$ and the line $x = 2$ at $y = \alpha$. then the value of α is :

(1) $\frac{1}{2}$ **(2)** $\frac{3}{2}$ **(3)** $\frac{-3}{2}$ **(4)** $\frac{5}{2}$

[JEE (Main) – 28th July 2022 - Shift-1]

12. If $y = y(x)$, $x \in \left(0, \frac{\pi}{2}\right)$ be the solution curve of the differential equation $\left(\sin^2 2x\right)\frac{dy}{dx} + \left(8\sin^2 2x + 2\sin 4x\right) y = 2e^{-4x}\left(2\sin 2x + \cos 2x\right)$, with $y(\pi/4) = e^{-\pi}$, then $y(\pi/6)$ is equal to

(1) $\frac{2}{\sqrt{3}}e^{-2\pi/3}$ (2) $\frac{2}{\sqrt{3}}e^{2\pi/3}$

(3) $\frac{1}{\sqrt{3}}e^{-2\pi/3}$ (4) $\frac{1}{\sqrt{3}}e^{2\pi/3}$

[JEE (Main) – 28th July 2022 - Shift-1]

13. Let $y = y(x)$ be the solution curve of the differential equation $\frac{dy}{dx} + \frac{1}{x^2-1}y = \left(\frac{x-1}{x+1}\right)^{1/2}$. $x > 1$ passing through the point $\left(2, \sqrt{\frac{1}{3}}\right)$. Then, $\sqrt{7}y(8)$ is equal to:

(1) $11 + 6\log_e 3$ (2) 19

(3) $12 - 2\log_e 3$ (4) $19 - 6\log_e 3$

[JEE (Main) – 28th July 2022 - Shift-2]

14. The differential equation of the family of circles passing through the points (0, 2) and (0, −2) is:

(1) $2xy\frac{dy}{dx} + \left(x^2 - y^2 + 4\right) = 0$

(2) $2xy\frac{dy}{dx} + (x^2 + y^2 - 4) = 0$

(3) $2xy\frac{dy}{dx} + (y^2 - x^2 + 4) = 0$

(4) $2xy\frac{dy}{dx} - (x^2 - y^2 + 4) = 0$

[JEE (Main) – 28th July 2022 - Shift-2]

15. Let $y = y_1(x)$ and $y = y_2(x)$ be two distinct solutions of the differential equation $\frac{dy}{dx} = x + y$, with $y_1(0) = 0$ and $y_2(0) = 1$ respectively. Then, the number of points of intersection of $y = y_1(x)$ and $y = y_2(x)$ is.

(1) 0 **(2)** 1 **(3)** 2 **(4)** 3

[JEE (Main) – 27th July 2022 - Shift-1]

16. If $\frac{dy}{dx}+2y\tan x = \sin x$, $0 < x < \frac{\pi}{2}$ and $y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of $y(x)$ is :

(1) $\frac{1}{8}$ **(2)** $\frac{3}{4}$ **(3)** $\frac{1}{4}$ **(4)** $\frac{3}{8}$

[JEE (Main) – 26th July 2022 - Shift-1]

17. Let the solution curve $y = f(x)$ of the differential equation

$$\frac{dy}{dx}+\frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$$ $x \in (-1, 1)$ pass through the origin. Then $\int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)\ dx$ is equal to

(1) $\frac{\pi}{3}-\frac{1}{4}$ **(2)** $\frac{\pi}{3}-\frac{\sqrt{3}}{4}$

(3) $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$ **(4)** $\frac{\pi}{6}-\frac{\sqrt{3}}{2}$

[JEE (Main) – 26th July 2022 - Shift-2]

18. The slope of the tangent to a curve C : $y = y(x)$ at any point (x, y) on it is $\frac{2e^{2x}-6e^{-x}+9}{2+9e^{-2x}}$. If C passes through the points $\left(0, \frac{1}{2}+\frac{\pi}{2\sqrt{2}}\right)$ $\left(\alpha, \frac{1}{2}e^{2\alpha}\right)$, then e^{α} is equal to :

(1) $\frac{3+\sqrt{2}}{3-\sqrt{2}}$ **(2)** $\frac{3}{\sqrt{2}}\left(\frac{3+\sqrt{2}}{3-\sqrt{2}}\right)$

(3) $\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$ **(4)** $\frac{\sqrt{2}+1}{\sqrt{2}-1}$

[JEE (Main) – 25th July 2022 - Shift-1]

19. The general solution of the differential equation $(x - y^2)dx + y(5x + y^2)dy = 0$ is :

(1) $(y^2 + x)^4 = C|(y^2 + 2x)^3|$

(2) $(y^2 + 2x)^4 = C|(y^2 + x)^3|$

(3) $|(y^2 + x)^3| = C(2y^2 + x)^4$

(4) $|(y^2 + 2x)^3| = C(2y^2 + x)^4$

[JEE (Main) – 25th July 2022 - Shift-1]

20. Let the solution curve $y = y(x)$ of the differential equation

$$\left[\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right]x\frac{dy}{dx} = x+\left[\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right]y$$

pass through the point $(1, 0)$ and $(2\alpha, \alpha)$, $\alpha > 0$. Then α is equal to

(1) $\frac{1}{2}\exp\left(\frac{\pi}{6}+\sqrt{e}-1\right)$ **(2)** $\frac{1}{2}\exp\left(\frac{\pi}{3}+e-1\right)$

(3) $\exp\left(\frac{\pi}{6}+\sqrt{e}+1\right)$ **(4)** $2\exp\left(\frac{\pi}{3}+\sqrt{e}-1\right)$

[JEE (Main) – 28th June 2022 - Shift-1]

21. Let $y = y(x)$ be the solution of the differential equation $x(1 - x^2)\frac{dy}{dx} + (3x^2y - y - 4x^3) = 0$, $x > 1$, with $y(2) = -2$. Then $y(3)$ is equal to

(1) –18 **(2)** –12 **(3)** –6 **(4)** –3

[JEE (Main) – 28th June 2022 - Shift-1]

22. Let $\frac{dy}{dx}=\frac{ax-by+a}{bx+cy+a}$ where a, b, c are constants, represent a circle passing through point (2, 5), then the shortest distance of the point (11, 6) from this circle is :

(1) 10 **(2)** 8 **(3)** 7 **(4)** 5

[JEE (Main) – 27th June 2022 - Shift-1]

23. If $\frac{dy}{dx}+\frac{2^{x-y}(2^y-1)}{2^x-1} = 0$, $x, y > 0$, $y' = 1$, then $y(2)$ is equal to:

(1) $2 + \log_2 3$ **(2)** $2 + \log_3 2$

(3) $2 - \log_3 2$ **(4)** $2 - \log_2 3$

[JEE (Main) – 27th June 2022 - Shift-1]

24. If the solution curve of the differential equation $((\tan^{-1}y) - x)dy = (1 + y^2)dx$ passes through the point (1, 0), then the abscissa of the point on the curve whose ordinate is tan(1) is

(1) $2e$ **(2)** $2/e$ **(3)** 2 **(4)** $1/e$

[JEE (Main) – 27th June 2022 - Shift-2]

25. If $y = y(x)$ is the solution of the differential equation $x\frac{dy}{dx} + 2y = xe^x$, $y(1) = 0$ then the local maximum value of the function $z(x) = x^2y(x) - e^x$, $x \in R$ is:

(1) $1 - e$ **(2)** 0 **(3)** 1/2 (4) $\frac{4}{e} - e$

[JEE (Main) – 26th June 2022 - Shift-2]

26. If the solution of the differential equation $\frac{dy}{dx} + e^x(x^2 - 2)y = (x^2 - 2x)(x^2 - 2)e^{2x}$ satisfies $y(0) = 0$ then the value of $y(2)$ is ….

(1) –1 **(2)** 1 **(3)** 0 **(4)** e

[JEE (Main) – 26th June 2022 - Shift-2]

27. Let $y = y(x)$ be the solutions of the differential equation $(x + 1)y' - y = e^{3x}(x + 1)^2$, with $y(0) = \frac{1}{3}$. Then, the point $x = \frac{-4}{3}$ for the curve $y = y(x)$ is:

(1) not a critical point

(2) a point of local minima

(3) a point of local maxima

(4) a point of inflection

[JEE (Main) – 25th June 2022 - Shift-1]

28. If the solution curve $y = y(x)$ of the differential equation $y^2dx + (x^2 - xy + y^2)dy = 0$ which passes through the point (1, 1) and intersects the line $y = \sqrt{3}x$ at the point $(\alpha, \sqrt{3}\alpha)$, then value of $\log_e(\sqrt{3}\alpha)$ is equal to:

(1) $\frac{\pi}{3}$ **(2)** $\frac{\pi}{2}$ **(3)** $\frac{\pi}{12}$ **(4)** $\frac{\pi}{6}$

[JEE (Main) – 25th June 2022 - Shift-1]

29. If $y = y(x)$ is the solution of the differential equation $2x^2\frac{dy}{dx} - 2xy + 3y^2 = 0$ such that $y(e) = \frac{e}{3}$, then $y(1)$ is equal to

(1) $\frac{1}{3}$ **(2)** $\frac{2}{3}$ **(3)** $\frac{3}{2}$ **(4)** 3

[JEE (Main) – 25th June 2022 - Shift-2]

30. The surface area of a balloon of spherical shape being inflated, increases at a constant rate. If initially, the radius of balloon is 3 units and after 5 seconds, it becomes 7 units, then its radius after 9 seconds is :

(1) 9 **(2)** 10 **(3)** 11 **(4)** 12

[JEE (Main) – 24th June 2022 - Shift-1]

31. If $y = y(x)$ is the solution curve of the differential equation $x^2\,dy + \left(y - \frac{1}{x}\right)dx = 0$; $x > 0$ and, $y(1) = 1$, then $y\left(\frac{1}{2}\right)$ is equal to :

(1) $3+\frac{1}{\sqrt{e}}$ **(2)** $\frac{3}{2}-\frac{1}{\sqrt{e}}$ **(3)** $3 + e$ **(4)** $3 - e$

[JEE (Main) – 1st Sep 2021 - Shift-2]

32. If $\frac{dy}{dx} = \frac{2^{x+y} - 2x}{2^y}$, $y(0) = 1$, then $y(1)$ is equal to :

(1) $\log_2(2 + e)$ **(2)** $\log_2(1 + e)$

(3) $\log_2(2e)$ **(4)** $\log_2(1 + e^2)$

[JEE (Main) – 31st Aug 2021 - Shift-1]

33. If $\frac{dy}{dx} = \frac{2^x y + 2^y.2^x}{2^x + 2^{x+y}\log_e 2}$, $y(0) = 0$, then for $y = 1$, the value of x lies in the interval:

(1) $\left(\frac{1}{2},1\right]$ **(2)** $\left(0,\frac{1}{2}\right]$ **(3)** (1, 2) **(4)** (2, 3)

[JEE (Main) – 31st Aug 2021 - Shift-2]

34. If $y\frac{dy}{dx} = x\left[\frac{y^2}{x^2} + \frac{\phi\left(\frac{y^2}{x^2}\right)}{\phi'\left(\frac{y^2}{x^2}\right)}\right]$, $x > 0$, $\phi > 0$ and y **(1)** $= -1$, then $\phi\left(\frac{y^2}{4}\right)$ is equal to:

(1) $\phi(1)$ **(2)** $4\,\phi(2)$ **(3)** $2\,\phi(1)$ **(4)** $4\,\phi(1)$

[JEE (Main) – 31st Aug 2021 - Shift-2]

35. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = 2\,(y + 2\sin x - 5)\,x - 2\cos x$ such that $y(0) = 7$. Then, $y(\pi)$ is equal to:

(1) $3e^{\pi^2} + 5$ **(2)** $7e^{\pi^2} + 5$ **(3)** $2e^{\pi^2} + 5$ **(4)** $e^{\pi^2} + 5$

[JEE (Main) – 27th Aug 2021 - Shift-1]

36. Let $y = y(x)$ be a solution curve of the differential equation $(y + 1)\tan^2x\,dx + \tan x\,dy + ydx = 0$, $x \in \left(0,\frac{\pi}{2}\right)$. If $\lim\limits_{x\to 0^+} x\,y(x) = 1$, then the value of $y\left(\frac{\pi}{4}\right)$ is:

(1) $\frac{\pi}{4}-1$ **(2)** $\frac{\pi}{4}+1$ **(3)** $\frac{\pi}{4}$ **(4)** $-\frac{\pi}{4}$

[JEE (Main) – 26th Aug 2021 - Shift-1]

37. Let $y(x)$ be the solution of the differential equation $2x^2dy + (e^y - 2x)\,dx = 0$, $x > 0$. If $y(e) = 1$. Then $y(1)$ is equal to:

(1) 0 **(2)** 2 **(3)** $\log_e(2e)$ **(4)** $\log_e 2$

[JEE (Main) – 26th Aug 2021 - Shift-2]

38. Let $y = y(x)$ be solution of the differential equation $\log_e\left(\frac{dy}{dx}\right) = 3x + 4y$, with $y(0) = 0$. If $y\left(-\frac{2}{3}\log_e 2\right) = \alpha\,\log_e 2$, then the value of α is equal to:

(1) $-\frac{1}{2}$ **(2)** $-\frac{1}{4}$ **(3)** 2 **(4)** $\frac{1}{4}$

[JEE (Main) – 27th July 2021 - Shift-1]

39. Let $y = y(x)$ be the solution of the differential equation $(x - x^3)dy = (y + yx^2 - 3x^4)dx$, $x > 2$. If $y(3) = 3$, then $y(4)$ is equal to :

(1) 8 **(2)** 12 **(3)** 16 **(4)** 4

[JEE (Main) – 27th July 2021 - Shift-2]

40. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = 1 + xe^{y-x}, -\sqrt{2} < x < \sqrt{2}$, $y(0) = 0$, then the minimum value of $y(x)$, $x \in \left(-\sqrt{2}, \sqrt{2}\right)$ is equal to:

(1) $\left(1-\sqrt{3}\right) - \log_e\left(\sqrt{3}-1\right)$

(2) $\left(2+\sqrt{3}\right) + \log_e 2$

(3) $\left(2-\sqrt{3}\right) - \log_e 2$

(4) $\left(1+\sqrt{3}\right) - \log_e\left(\sqrt{3}-1\right)$

[JEE (Main) – 25th July 2021 - Shift-1]

41. Let $y = f(x)$ be the solution of the differential equation $xdy = (y + x^3\cos x)dx$ with $y(\pi) = 0$, then $f\left(\frac{\pi}{2}\right)$ is equal to :

(1) $\frac{\pi^2}{2} - \frac{\pi}{4}$ **(2)** $\frac{\pi^2}{4} + \frac{\pi}{2}$ **(3)** $\frac{\pi^2}{4} - \frac{\pi}{2}$ **(4)** $\frac{\pi^2}{2} + \frac{\pi}{4}$

[JEE (Main) – 25th July 2021 - Shift-2]

42. Let $y = y(x)$ be the solution of the differential equation

$\operatorname{cosec}^2 x\, dy + 2dx = (1 + y\cos 2x)\operatorname{cosec}^2 x\, dx$, with $y\left(\frac{\pi}{4}\right)$. Then, the value of $(y(0) + 1)^2$ is equal to :

(1) $e^{\frac{1}{2}}$ **(2)** $e^{\frac{-1}{2}}$ **(3)** e^{-1} **(4)** e

[JEE (Main) – 22nd July 2021 - Shift-2]

43. Let $y = y(x)$ be the solution of the differential equation

$x\tan\left(\frac{y}{x}\right)dy = \left(y\tan\left(\frac{y}{x}\right) - x\right)dx$, $-1 \le x \le 1$, $y\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Then the area of the region bounded by the curves $x = 0$, $x = \frac{1}{\sqrt{2}}$ and $y = y(x)$ in the upper half plane is :

(1) $\frac{1}{12}(\pi - 3)$ **(2)** $\frac{1}{6}(\pi - 1)$

(3) $\frac{1}{8}(\pi - 1)$ **(4)** $\frac{1}{4}(\pi - 2)$

[JEE (Main) – 20th July 2021 - Shift-1]

44. Let $y = y(x)$ be the solution of the differential equation $e^x\sqrt{1-y^2}\,dx + \left(\frac{y}{x}\right)dy = 0$, $y(1) = -1$. Then, the value of $(y(3))^2$ is equal to:

(1) $1 + 4e^3$ **(2)** $1 + 4e^6$ **(3)** $1 - 4e^6$ **(4)** $1 - 4e^3$

[JEE (Main) – 20th July 2021 - Shift-1]

Q 45. Let $y = y(x)$ satisfies the equation $\frac{dy}{dx} - |A| = 0$ for all $x > 0$, where $A = \begin{bmatrix} y & \sin x & 1 \\ 0 & -1 & 1 \\ 2 & 0 & \frac{1}{x} \end{bmatrix}$. If $y(\pi) = \pi + 2$, then the value of $y\left\{\frac{\pi}{2}\right\}$ is:

(1) $\frac{\pi}{2} - \frac{4}{\pi}$ **(2)** $\frac{3\pi}{2} - \frac{1}{\pi}$ **(3)** $\frac{\pi}{2} - \frac{1}{\pi}$ **(4)** $\frac{\pi}{2} + \frac{4}{\pi}$

[JEE (Main) – 20th July 2021 - Shift-2]

46. Let $y = y(x)$ be the solution of the differential equation $\frac{dy}{dx} = (y+1)\left((y+1)e^{\frac{x^2}{2}} - x\right)$, $0 < x < 2.1$, with $y(2) = 0$. Then the value of $\frac{dy}{dx}$ at $x = 1$ is equal to:

(1) $\frac{e^{\frac{5}{2}}}{\left(1+e^2\right)^2}$ **(2)** $\frac{5e^{\frac{1}{2}}}{\left(e^2+1\right)^2}$

(3) $-\frac{2e^2}{\left(1+e^2\right)^2}$ **(4)** $\frac{-e^{\frac{3}{2}}}{\left(e^2+1\right)^2}$

[JEE (Main) – 18th March 2021 - Shift-2]

47. Which of the following is true for $y(x)$ that satisfies the differential equation

$\frac{dy}{dx} = xy - 1 + x - y; y(0) = 0$:

(1) $y(1) = 1$ **(2)** $y(1) = e^{\frac{1}{2}} - 1$

(3) $y(1) = e^{\frac{1}{2}} - e^{-\frac{1}{2}}$ **(4)** $y(1) = e^{-\frac{1}{2}} - 1$

[JEE (Main) – 17th March 2021 - Shift-1]

48. Let $y = y(x)$ be the solution of the differential equation $\cos x\,(3\sin x + \cos x + 3)\,dy = (1 + y\sin x\,(3\sin x + \cos x + 3))\,dx$, $0 \le x \le \frac{\pi}{2}$, $y(0) = 0$. Then, $y\left(\frac{\pi}{3}\right)$ is equal to :

(1) $2\log_e\left(\frac{2\sqrt{3}+10}{11}\right)$ **(2)** $2\log_e\left(\frac{\sqrt{3}+7}{2}\right)$

(3) $2\log_e\left(\frac{3\sqrt{3}-8}{4}\right)$ **(4)** $2\log_e\left(\frac{2\sqrt{3}+9}{6}\right)$

[JEE (Main) – 17th March 2021 - Shift-2]

49. If the curve $y = y(x)$ is the solution of the differential equation

$$2\left(x^2 + x^{5/4}\right)dy - y\left(x + x^{1/4}\right)dx = 2x^{3/4}dx, x > 0$$

which passes through the point $\left(1, 1 - \frac{4}{3}\log_e 2\right)$ then the value of $y(16)$ is equal to :

(1) $\left(\frac{31}{3} - \frac{8}{3}\log_e 3\right)$ **(2)** $4\left(\frac{31}{3} + \frac{8}{3}\log_e 3\right)$

(3) $\left(\frac{31}{3} + \frac{8}{3}\log_e 3\right)$ **(4)** $4\left(\frac{31}{3} - \frac{8}{3}\log_e 3\right)$

[JEE (Main) – 17th March 2021 - Shift-2]

50. If $y = y(x)$ is the solution of the differential equation, $\frac{dy}{dx} + 2y\tan x = \sin x, y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of the function $y(x)$ over R is equal to :

(1) 8 **(2)** $\frac{1}{2}$ **(3)** $-\frac{15}{4}$ **(4)** $\frac{1}{8}$

[JEE (Main) – 16th March 2021 - Shift-1]

51. If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} + (\tan x)y = \sin x,\ 0 \le x \le \frac{\pi}{3}$, with $y(0) = 0$ then $y\left(\frac{\pi}{4}\right)$ equal to :

(1) $\log_e 2$ **(2)** $\frac{1}{2}\log_e 2$

(3) $\left(\frac{1}{2\sqrt{2}}\right)\log_e 2$ **(4)** $\frac{1}{4}\log_e 2$

[JEE (Main) – 16th March 2021 - Shift-2]

52. Let C_1 be the curve obtained by solution of differential equation $2xy\frac{dy}{dx} = y^2 - x^2, x > 0$ Let the curve C_2 be the solution of $\frac{2xy}{x^2 - y^2} = \frac{dy}{dx}$, If both the curves pass through (1, 1) then the area enclosed by the curves C_1 and C_2 is equal to :

(1) $\frac{\pi}{2} - 1$ **(2)** $\frac{\pi}{4} + 1$ **(3)** $\pi - 1$ **(4)** $\pi + 1$

[JEE (Main) – 16th March 2021 - Shift-2]

53. The rate of growth of bacteria in a culture is proportional to the number of bacteria present and the bacteria count is 1000 at initial time $t = 0$. The number of bacteria is increased by 20% in 2 hours. If the population of bacteria is 2000 after $\frac{k}{\log_e\left(\frac{6}{5}\right)}$ hours, then $(k/\log 2)^2$ is equal to :

(1) 4 **(2)** 2 **(3)** 16 **(4)** 8

[JEE (Main) – 26th Feb 2021 - Shift-1]

54. The population $P = P(t)$ at time 't' of a certain species follows the differential equation $\frac{dP}{dt} = 0.5P - 450$. If $P(0) = 850$, then the time at which population becomes zero is :

(1) $\frac{1}{2}\log_e 18$ **(2)** $2\log_e 18$

(3) $\log_e 9$ **(4)** $\log_e 18$

[JEE (Main) – 24th Feb 2021 - Shift-1]

55. Let f be a twice differentiable function defined on R such that $f(0) = 1, f'(0) = 2$ and $f'(x) \neq 0$ for all $x \in R$. If $\begin{vmatrix} f(x) & f\ (x) \\ f\ (x) & f\ \ (x) \end{vmatrix} = 0$, for all $x \in R$ then the value of $f(1)$ lies in the interval:

(1) (9, 12) **(2)** (6, 9) **(3)** (3, 6) **(4)** (0, 3)

[JEE (Main) – 24th Feb 2021 - Shift-2]

56. If a curve $y = f(x)$ passes through the point (1,2) and satisfies $x\frac{dy}{dx} + y = bx^4$, then for what value of $b, \int_1^2 f(x)dx = \frac{62}{5}$?

(1) 5 **(2)** $\frac{62}{5}$ **(3)** $\frac{31}{5}$ **(4)** 10

[JEE (Main) – 24th Feb 2021 - Shift-2]

ANSWER – KEY

1. (1)	2. (2)	3. (2)	4. (2)
5. (4)	6. (1)	7. (1)	8. (2)
9. (1)	10. (2)	11. (2)	12. (1)
13. (4)	14. (1)	15. (1)	16. (1)
17. (2)	18. (2)	19. (1)	20. (1)
21. (1)	22. (2)	23. (4)	24. (2)
25. (4)	26. (3)	27. (2)	28. (3)
29. (2)	30. (1)	31. (4)	32. (2)
33. (3)	34. (4)	35. (3)	36. (3)
37. (4)	38. (2)	39. (2)	40. (1)
41. (2)	42. (3)	43. (3)	44. (3)
45. (4)	46. (4)	47. (4)	48. (1)
49. (4)	50. (4)	51. (3)	52. (1)
53. (1)	54. (2)	55. (2)	56. (4)

ANSWERS WITH EXPLANATIONS

1. **Option (1) is correct.**

The given d.e. is :

$$2x^2y\frac{dy}{dx}-(1-xy^2)=0$$

$$\Rightarrow 2y\frac{dy}{dx}-\frac{1}{x^2}+\frac{y^2}{x}=0$$

Let $y^2 = u \Rightarrow 2y\frac{dy}{dx}=\frac{du}{dx}$

So, $\frac{du}{dx}+\frac{u}{x}=\frac{1}{x^2}$ (linear d.e)

I.F. $= e^{\int\frac{1}{x}dx}=e^{\ln x}=x$

$\therefore$ The solution is

$u \times x = \int\frac{1}{x^2}\times x\,dx$

$\Rightarrow ux = \log_e x + C \Rightarrow y^2x = \log_e x + C$

Now, using $y(2) = \sqrt{\log_e 2}$, we get

$2\log_e 2 = \log_e 2 + C \Rightarrow C = \log_e 2$

$\therefore$ The solution is :

$y^2x = \ln x + \ln 2 = \ln 2x \Rightarrow 2x = \exp(xy^2)$

On comparing with $\alpha x = \exp(x^\beta y^\gamma)$, we get

$\alpha = 2, \beta = 1, \gamma = 2$

$\therefore \alpha + \beta - \gamma = 2 + 1 - 2 = 1$

HINT:

Substitute $y^2 = u$ and make a linear differential equation.

2. **Option (2) is correct.**

The given differential equation is :

$$\frac{dy}{dx}+y\tan x=x\sec x, 0\le x\le\frac{\pi}{3}$$

As the above differential equation is linear i.e.,

$$\frac{dy}{dx}+P(x)y=Q(x)$$

So, integrating factor (I.F.) $= e^{\int P(x)dx}$

$$=e^{\int\tan x\,dx}=e^{\ln\sec x}=\sec x$$

Now, the solution is :

$y \times \text{I.F.} = \int Q(x)\times \text{I.F.}\,dx$

$$\Rightarrow y\times\sec x=\int x\sec^2 x\,dx$$

$$=x\int\sec^2 dx-\int\left(\frac{d}{dx}(x)\int\sec^2 x\,dx\right)dx$$

$$=x\tan x-\int\tan x\,dx$$

$\Rightarrow y \sec x = x \tan x - \ln \sec x + C$ ...(i)

Using initial condition, $y(0) = 1$, we get

$(1)\sec 0 = 0 - \ln\sec 0 + C$

$\Rightarrow 1 = -\ln 1 + C \quad \Rightarrow C = 1$

Now, put $C = 1$ in equation (i) we get

$y \sec x = x \tan x - \ln \sec x + 1$ ...(i)

Put $x=\frac{\pi}{6}$ in equation (ii) we get

$$y\times\sec\frac{\pi}{6}=\frac{\pi}{6}\times\tan\frac{\pi}{6}-\ln\sec\frac{\pi}{6}+1$$

$$\Rightarrow y\times\frac{2}{\sqrt{3}}=\frac{\pi}{6}\times\frac{1}{\sqrt{3}}-\ln\left(\frac{2}{\sqrt{3}}\right)+1$$

$$\Rightarrow y=\frac{\pi}{12}-\frac{\sqrt{3}}{2}\ln\left(\frac{2}{\sqrt{3}}\right)+\frac{\sqrt{3}}{2}\ln e$$

$$=\frac{\pi}{12}-\frac{\sqrt{3}}{2}\left(\ln\left(\frac{2}{e\sqrt{3}}\right)\right) \qquad \left(\ln a-\ln b=\ln\frac{a}{b}\right)$$

3. **Option (2) is correct.**

Given differential equation is

$$\frac{dy}{dx}=-\left(\frac{x^2+3y^2}{3x^2+y^2}\right)$$

Let $y = vx$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\therefore v+x\frac{dv}{dx}=-\left(\frac{x^2+3v^2x^2}{3x^2+v^2x^2}\right)$$

$$\Rightarrow v+x\frac{dv}{dx}=-\left(\frac{1+3v^2}{3+v^2}\right)$$

$$\Rightarrow\frac{xdv}{dx}=\frac{-1-3v^2-3v-v^3}{3+v^2}$$

$$\Rightarrow\int\frac{3+v^2}{v^3+3v^2+3v+1}dv=-\int\frac{1}{x}dx$$

$$\Rightarrow\int\frac{v^2+3}{(v+1)^3}dv=-\ln x+C$$

Let $v + 1 = t \Rightarrow dv = dt$

$$\int\frac{(t-1)^2+3}{t^3}dt=-\ln x+C$$

$$\Rightarrow \int \frac{t^2-2t+4}{t^3}dt = -\ln x + C$$

$$\Rightarrow \int \left(\frac{1}{t}-\frac{2}{t^2}+\frac{4}{t^3}\right)dt = -\ln x + C$$

$$\Rightarrow \ln t + \frac{2}{t} - \frac{2}{t^2} = -\ln x + C$$

$$\Rightarrow \ln(1+v) + \frac{2}{1+v} - \frac{2}{(1+v)^2} = -\ln x + C$$

$$\Rightarrow \ln\frac{(y+x)}{x} + \frac{2x}{x+y} - \frac{2x^2}{(x+y)^2} = -\ln x + C$$

$$\Rightarrow \ln|x+y| - \ln x + \frac{2x}{x+y} - \frac{2x^2}{(x+y)^2} = -\ln x + C$$

$$\Rightarrow \ln(x+y) + \frac{2x(x+y)-2x^2}{(x+y)^2} = C$$

$$\Rightarrow \ln(x+y) + \frac{2xy}{(x+y)^2} = C$$

Using $y(1) = 0$, we get

$\ln(1 + 0) + \frac{0}{(1)^2} = c \Rightarrow c = 0$

So, the solution is:

$$\ln(x+y) + \frac{2xy}{(x+y)^2} = 0$$

4. **Option (2) is correct.**

Given differential equation is

$x \log_e x \frac{dy}{dx} + y = x^2 \log_e x;\ x > 1.$

$\Rightarrow \frac{dy}{dx} + \frac{1}{x\log_e x} y = x$ which is a linear differential equation

Now, I.F. $= e^{\int \frac{1}{x\log_e x}dx}$

$\Rightarrow$ I.F. $= e^{\log_e(\log_e x)} \Rightarrow$ I.F. $= \log_e x$

So, solution of given differential equation is

$$y(\log_e x) = \int x(\log_e x)dx + C$$

$$\Rightarrow y(\log_e x) = \log_e x\left(\frac{x^2}{2}\right) - \int\left(\frac{1}{x}.\frac{x^2}{2}\right)dx + C$$

$$\Rightarrow y(\log_e x) = \frac{x^2}{2}\log_e x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C$$

$$\Rightarrow y(\log_e x) = \frac{x^2}{2}\log_e x - \frac{1}{4}x^2 + C$$

$\because$ At $x = 2, y = 2$

$\Rightarrow 2\ (\log_e 2) = 2 \log_e 2 - 1 + C \Rightarrow C = 1$

$$\therefore y\ (\log_e x) = \frac{x^2}{2}\log_e x - \frac{1}{4}x^2 + 1$$

$$\Rightarrow y = \frac{1}{\log_e x}\left[\frac{x^2}{2}\log_e x - \frac{1}{4}x^2 + 1\right]$$

$$\Rightarrow y(e) = \frac{e^2}{2} - \frac{1}{4}e^2 + 1 \ \Rightarrow y(e) = \frac{4+e^2}{4}$$

5. **Option (4) is correct.**

Given, $\frac{dy}{dx} = \frac{y}{x}(1 + xy^2(1 + \ln x)), y(1) = 3$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + y^3(1+\ln x)$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} - \frac{1}{y^2}\left(\frac{1}{x}\right) = 1 + \ln x$$

Let $-\frac{1}{y^2} = u \Rightarrow \frac{1}{y^3}\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx}$

$$\Rightarrow \frac{1}{2}\frac{du}{dx} + \frac{u}{x} = 1 + \ln x$$

$\Rightarrow \frac{du}{dx} + \frac{2u}{x} = 2(1 + \ln x)$, which is linear differential equation,

Now, I.F. $= e^{\int \frac{2}{x}dx}$

$\Rightarrow$ I.F. $= e^{2\ln x} = x^2$

So, solution of given differential equation is

u (I.F.) $= \int 2(1+\ln x)(\text{I.F.})dx + C$

$$\Rightarrow ux^2 = \int 2x^2(1+\ln x)dx + C$$

$$\Rightarrow ux^2 = 2(1+\ln x)\frac{x^3}{3} - 2\int\left(\frac{1}{x}\right).\frac{x^3}{3}dx + C$$

$$\Rightarrow ux^2 = \frac{2x^3}{3}(1+\ln x) - \frac{2x^3}{9} + C$$

$$\Rightarrow -\frac{1}{y^2}x^2 = \frac{2x^3}{3}(1+\ln x) - \frac{2x^3}{9} + C$$

Put $x = 1$ in above equation, we get

$$-\frac{1}{9}(1) = \frac{2}{3}(1+0) - \frac{2}{9} + C$$

$$\Rightarrow C = -\frac{5}{9}$$

$$\therefore -\frac{x^2}{y^2} = \frac{2x^3}{3}(1+\ln x) - \frac{2x^3}{9} - \frac{5}{9}$$

$$\Rightarrow \frac{x^2}{y^2} = -\frac{1}{9}[6x^3(1+\ln x) - 2x^3 - 5]$$

$$\Rightarrow \frac{x^2}{y^2} = \frac{1}{9}[5 - 4x^3 - 6x^3\ln x]$$

$$\Rightarrow \frac{x^2}{y^2} = \frac{1}{9}[5 - 2x^3(2 + \ln x^3)]$$

$$\Rightarrow \frac{y^2}{9} = \frac{x^2}{5 - 2x^3(2+\ln x^3)}$$

6. **Option (1) is correct.**

$x^3dy + (xy - 1)\ dx = 0$

Also, $y\left(\frac{1}{2}\right) = 3 - e \ \&\ x > 0$

Now, $x^3\frac{dy}{dx} + xy - 1 = 0$

$$\Rightarrow x^3\frac{dy}{dx} = 1 - xy \Rightarrow x^3\frac{dy}{dx} + xy = 1$$

$$\Rightarrow \frac{dy}{dx}+\left(\frac{1}{x^2}\right)y=\frac{1}{x^3}$$

This is a linear differential equation

I.F. = $e^{\int \frac{1}{x^2}dx}=e^{\frac{-1}{x}}$

So, differential equation becomes,

$$ye^{\frac{-1}{x}}=\int e^{\frac{-1}{x}}.\frac{1}{x^3}dx$$

Put $\frac{-1}{x}=t \Rightarrow \frac{1}{x^2}dx=dt$

R.H.S. = $\int -\underbrace{t}_{\text{I}}\underbrace{e^t}_{\text{II}}dt$

Integrating by parts

R.H.S. = – $[te^t-\int e^t dt]$ = – te^t + e^t + c

So, solution of differential equation is

$$ye^{\frac{-1}{x}}=-e^{\frac{-1}{x}}\left(\frac{-1}{x}-1\right)+c$$

$$\Rightarrow y=\left(\frac{1}{x}+1\right)+ce^{\frac{1}{x}}$$

At $x=\frac{1}{2}, y=3-e$

$$\Rightarrow 3-e=\frac{1}{\left(\frac{1}{2}\right)}+1+ce^{\frac{1}{\left(\frac{1}{2}\right)}}$$

$\Rightarrow 3-e=(2+1)+ce^2$

$\Rightarrow ce^2=-e$

$$\Rightarrow c=\frac{-1}{e}$$

So, $y=\left(\frac{1}{x}+1\right)+\left(-\frac{1}{e}\right)e^{\frac{1}{x}}$

Now, at $x=1$, $y=\left(\frac{1}{1}+1\right)+\left(\frac{-1}{e}\right)e^{\frac{1}{1}}$

$\Rightarrow y=2-1=1$

7. **Option (1) is correct.**

Given that

$$\frac{dy}{dx}+\frac{5}{x(x^5+1)}y=\frac{(x^5+1)^2}{x^7}, x>0$$

which is a linear differential eqn.

I.F. = $e^{\int\frac{5}{x(x^5+1)}dx}=e^{\int\frac{5}{x^6\left(1+\frac{1}{x^5}\right)}dx}$

Let $t=1+\frac{1}{x^5}$

$dt=-\frac{5}{x^6}dx$ or $-dt=\frac{5dx}{x^6}$

I.F. = $e^{-\int\frac{1}{t}dt}=e^{-\log t}=\frac{1}{t}=\frac{x^5}{1+x^5}$

Solution of differential eqn. is given by

$$y\times\frac{x^5}{1+x^5}=\int\frac{(x^5+1)^2}{x^7}\times\frac{x^5}{(1+x^5)}dx+c$$

$$=\int\frac{x^5+1}{x^2}dx+c=\int(x^3+x^{-2})dx+c$$

$$\Rightarrow\frac{yx^5}{1+x^5}=\frac{x^4}{4}-\frac{1}{x}+c$$

Putting $x=1$, $y=2$, we get

$$\Rightarrow\frac{2}{1+1}=\frac{1}{4}-1+\mathrm{C}\Rightarrow\mathrm{C}=\frac{7}{4}\Rightarrow\frac{yx^5}{1+x^5}=\frac{x^4}{4}-\frac{1}{x}+\frac{7}{4}$$

Putting $x=2$

$$\Rightarrow\frac{y\times 32}{1+32}=\frac{16}{4}-\frac{1}{2}+\frac{7}{4}$$

$$\rightarrow y=\frac{33}{32}\left[\frac{16+7-2}{4}\right]=\frac{33\times 21}{32\times 4}\Rightarrow y=\frac{693}{128}$$

8. **Option (2) is correct.**

Given differential equation

$$\left(1+e^{2x}\right)\left(\frac{dy}{dx}+y\right)=1$$

$$\frac{dy}{dx}+y=\frac{1}{1+e^{2x}}$$

which is linear differential equation of first order

So, P=1 and $\mathrm{Q}=\frac{1}{1+e^{2x}}$

Now, $\mathrm{I.F}=e^{\int pdx}$

$\Rightarrow \mathrm{I.F}=e^{\int 1dx}=e^x$

Solution for given differential equation is

$$y.(\mathrm{I.F})=\int(\mathrm{I.F}).\mathrm{Q}dx+c\Rightarrow y.e^x=\int\frac{e^x}{1+e^{2x}}dx+c$$

Let $e^x=t$

$\therefore e^x dx=dt$

$$\Rightarrow ye^x=\int\frac{1}{1+t^2}dt+c\Rightarrow ye^x=\tan^{-1}t+c$$

$$\Rightarrow ye^x=\tan^{-1}\left(e^x\right)+c$$

$\because$ It passes through the point (0,π/2)

$$\Rightarrow\frac{\pi}{2}=\tan^{-1}(1)+c\Rightarrow\frac{\pi}{2}=\frac{\pi}{4}+c$$

$$\therefore c=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

$$\therefore ye^x=\tan^{-1}\left(e^x\right)+\frac{\pi}{4}$$

Now, $\lim\limits_{x\to\infty}(ye^x)=\lim\limits_{x\to\infty}\left(\tan^{-1}\left(e^x\right)+\frac{\pi}{4}\right)$

$$=\tan^{-1}\left(e^{\infty}\right)+\frac{\pi}{4}=\tan^{-1}\infty+\frac{\pi}{4}=\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}$$

9. **Option (1) is correct.**

$$\frac{dy}{dx}=\frac{x+y-2}{x-y}=\frac{(x-1)+(y-1)}{(x-1)-(y-1)}$$

Let $x-1$ = X and $y-1$ = Y

$\Rightarrow dx$ = dX and dy=dY

$$\therefore \frac{dy}{dx}=\frac{dY}{dX}$$

$$\text{Let } Y=VX\Rightarrow \frac{dY}{dX}=V+X\frac{dV}{dX}$$

$$\Rightarrow\ V+X\frac{dV}{dX}=\frac{1+V}{1-V}\ \Rightarrow X\frac{dV}{dX}=\frac{1+V^2}{1-V}$$

$$\Rightarrow \int\frac{dV}{1+V^2}-\int\frac{V}{1+V^2}\,dV=\int\frac{dX}{X}$$

$$\Rightarrow \tan^{-1}V-\frac{1}{2}\log_e\left(1+V^2\right)=\log_e X+C$$

$$\Rightarrow \tan^{-1}\left(\frac{Y}{X}\right)-\frac{1}{2}\log_e\left(1+\frac{Y^2}{X^2}\right)=\log_e X+C$$

$$\Rightarrow \tan^{-1}\left(\frac{y-1}{x-1}\right)-\frac{1}{2}\log_e\left((x-1)^2+(y-1)^2+\log_e(x-1)\right)$$

$$=\log_e(x\text{-}1)+C$$

$$\Rightarrow \tan^{-1}\left(\frac{y-1}{x-1}\right)-\frac{1}{2}\log\left((x-1)^2+(y-1)^2\right)=C$$

$\therefore$ Passes through the point (2,1)

$$\Rightarrow 0-\frac{1}{2}\log_e(1)=C$$

$$\Rightarrow\qquad C=0$$

$$\therefore \tan^{-1}\left(\frac{y-1}{x-1}\right)-\frac{1}{2}\log_e\left((x-1)^2+(y-1)^2\right)=0$$

Passes through (k+1,2)

$$\therefore\ \tan^{-1}\left(\frac{1}{k}\right)-\frac{1}{2}\log_e\left(k^2+1\right)=0$$

$$\Rightarrow 2\tan^{-1}\left(\frac{1}{k}\right)=\log_e\left(k^2+1\right)$$

10. **Option (2) is correct.**

$$\text{Given } \frac{dy}{dx}+\left(\frac{2x^2+11x+13}{x^3+6x^2+11x+6}\right)y=\frac{x+3}{x+1}\quad x>1$$

Which is linear differential equation of first order

$$\text{Now, I.F}=e^{\int p\,dx}=e^{\left(\frac{2x^2+11x+13}{x^3+6x^2+11x+6}\right)dx}$$

$$=e^{\int\frac{2x^2+11x+13}{(x+1)(x+2)(x+3)}dx.}\Rightarrow \text{IF}=e^{\int\left(\frac{2}{x+1}+\frac{1}{x+2}-\frac{1}{x+3}\right)dx}$$

$$=e^{2\log_e(x+1)+\log_e e\,(x+2)-\log_e(x+3)}$$

$$=\frac{(x+1)^2\times(x+2)}{(x+3)}$$

Solution of given different equation is

$$y\,(\text{I.F})=\int(\text{I.F})\,Q\,dx+c$$

$$\Rightarrow y\frac{(x+1)^2(x+2)}{(x+3)}=\int\frac{(x+1)^2(x+2)}{(x+3)}\times\frac{(x+3)}{(x+1)}dx+c$$

$$\Rightarrow \frac{y(x+1)^2(x+2)}{(x+3)}=\int\left(x^2+3x+2\right)dx+c$$

$$\Rightarrow \frac{y(x+1)^2(x+2)}{(x+3)}=\frac{x^3}{3}+\frac{3x^2}{2}+2x+c$$

$\therefore$ Curve passes through (0, 1)

$$\Rightarrow \frac{1\times1\times2}{3}=C\Rightarrow C=\frac{2}{3}$$

$$\therefore \frac{y(x+1)^2(x+2)}{(x+3)}=\frac{x^3}{3}+\frac{3x^2}{2}+2x+\frac{2}{3}$$

Put x=1 in above equation, we get

$$y\left(\frac{4\times3}{4}\right)=\frac{1}{3}+\frac{3}{2}+2+\frac{2}{3}$$

$$\Rightarrow y\times3=1+2+\frac{3}{2}=3+\frac{3}{2}\ \Rightarrow 3y=\frac{9}{2}$$

$$\Rightarrow y=\frac{3}{2}\ \therefore y(1)=\frac{3}{2}$$

11. **Option (2) is correct.**

Given differential equation is

$$xdy=\left(\sqrt{x^2+y^2}+y\right)dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{x^2+y^2}+y}{x}=\sqrt{1+\left(\frac{y}{x}\right)^2}+\frac{y}{x}$$

$$\text{Let } y=vx\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$\therefore v+x\frac{dv}{dx}=\sqrt{1+v^2}+v$$

$$\Rightarrow \int\frac{1}{\sqrt{1+v^2}}dv=\int\frac{1}{x}dx$$

$$\Rightarrow \log_e\left|v+\sqrt{1+v^2}\right|=\log_e|x|+\log_e c$$

$$\Rightarrow \frac{y}{x}+\sqrt{1+\frac{x^2}{y^2}}=cx$$

$$\Rightarrow y+\sqrt{x^2+y^2}=cx^2$$

Put x =1 and y = 0 in above equation, we get c = 1

$$\therefore y+x^2+y^2=x^2$$

Put x= 2 and y=α in above equation, we get c=1

$$\therefore y+\sqrt{x^2+y^2}=x^2$$

$$\Rightarrow \alpha + \sqrt{4+\alpha^2} = 4 \;\Rightarrow \sqrt{4+\alpha^2} = 4-\alpha$$

$$\Rightarrow 4+\alpha^2 = 16+\alpha^2 - 8\,\alpha \;\Rightarrow \alpha = \frac{3}{2}$$

12. Option (1) is correct.

Given differential equation is

$$\left(\sin^2 2x\right)\frac{dy}{dx} + \left(8\sin^2 2x + 2\sin 4x\right)y$$

$$= 2e^{-4x}\left(2\sin 2x + \cos 2x\right)$$

$$\Rightarrow \frac{dy}{dx} + \frac{8\sin^2 2x + 2\sin 4x}{\sin^2 2x}y$$

$= \dfrac{2e^{-4x}\left(2\sin 2x + \cos 2x\right)}{\sin^2(2x)}$, which is linear differential equation

$$\therefore \text{P} = 8 + \frac{4\sin 2x.\cos 2x}{\sin^2 2x} = 8 + 4\cot 2x$$

Now, $I.F = e^{\int (8+4\cot 2x)dx} = e^{8x+2\ln(\sin 2x)}$

$\therefore$ Solution is $y.e^{8x+2\ln(\sin 2x)}$

$$= \int 2e^{4x+2\ln(\sin 2x)}.\left(\frac{2\sin 2x + \cos 2x}{\sin^2 2x}\right)dx$$

$$= 2\int e^{4x}\left(2\sin 2x + \cos 2x\right)dx$$

Let $4x = t \Rightarrow dx = \dfrac{1}{4}dt$

$$= \int e^t\left(\sin\frac{t}{2} + \frac{1}{2}\cos\frac{t}{2}\right)dt$$

$\because \dfrac{d\sin\frac{t}{2}}{dt} = \dfrac{1}{2}\cos\dfrac{t}{2}$ and

$$\int e^x\left(f'(x) + f(x)\right)dx = e^x f(x)$$

$$\therefore ye^{8x}.\sin^2 2x = e^{4x}\sin 2x + C$$

Put $x = \dfrac{\pi}{4}$ and $y = e^{-\pi}$ in above equation, we get

$$e^{-\pi}.e^{2\pi} = e^{\pi} + C \Rightarrow C = 0$$

$$\therefore ye^{8x}\sin^2 2x = e^{4x}\sin 2x \;\Rightarrow\; y = e^{-4x}.\text{cos}ec2x$$

$$\therefore y\left(\frac{\pi}{6}\right) = e^{\frac{-2\pi}{3}}.\frac{2}{\sqrt{3}}$$

13. Option (4) is correct.

Given differential equation is

$$\frac{dy}{dx} + \frac{1}{x^2-1}y = \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}, x > 1$$

$$I.F. = e^{\int \frac{1}{x^2-1}dx} = e^{\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|} = \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}$$

Solution:

$$y.\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}} = \int \frac{x-1}{x+1}dx = \int\left(1 - \frac{2}{x+1}\right)dx$$

$$\Rightarrow y\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}} = x - 2\ln|x+1| + C$$

Put x=2 and y= $\sqrt{\dfrac{1}{3}}$

$$\sqrt{\frac{1}{3}}\cdot\sqrt{\frac{1}{3}} = 2 - 2\,ln_3 + \text{C}$$

$$\Rightarrow C = 2\ln 3 - \frac{5}{3}$$

$$\therefore y\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}} = x - 2\ln|x+1| + 2\ln 3 - \frac{5}{3}$$

Put $x = 8$

$$y\left(\frac{7}{9}\right)^{\frac{1}{2}} = 8 - 2\ln 9 + 2\ln 3 - \frac{5}{3}$$

$$\therefore \sqrt{7}y(8) = 19 - 6\,ln3$$

14. Option (1) is correct.

Family of circle passing through the given points (0, 2) and (0, –2) is

$x^2 + (y + 2)(y - 2) + kx = 0, k \in R$

$\Rightarrow x^2 + y^2 - 4 + kx = 0$ ...(i)

Differentiate w.r.t x

$$2x + 2y\frac{dy}{dx} + k = 0$$

$$\Rightarrow k = -\left(2x + 2y\frac{dy}{dx}\right)$$

Putting in (i), we get

$$x^2 + y^2 - 4 - x\left(2x + 2y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow x^2 + y^2 - 4 - 2x^2 - 2xy\frac{dy}{dx} = 0$$

$$\Rightarrow 2xy\frac{dy}{dx} + x^2 - y^2 + 4 = 0$$

15. Option (1) is correct.

Given differential equation is

$$\frac{dy}{dx} = x + y \Rightarrow \frac{dy}{dx} - y = x$$

$\therefore$ IF = $e^{-\int 1dx} = e^{-x}$

$\therefore$ Solution is

$y.e^{-x} = \int xe^{-x}dx$

$= -xe^{-x} + \int e^{-x}dx$

$y.e^{-x} = -xe^{-x} - e^{-x} + C$

$y = -x - 1 + ce^x$

Put $x = 0, y = 0$, we get $c = 1$

$\therefore y_1(x) = -x - 1 + e^x$ ...(i)

Put $x = 0, y = 1$, we get $c = 2$

$\therefore y_2(x) = -x - 1 + 2e^x$ ...(ii)

On solving equations (i) and (ii) we get

$e^x = 0$ which is not possible

So, number of points of intersection

$y_1(x)$ and $y_2(x)$ is 0.

16. Option (1) is correct.

Given differential equation is

$\frac{dy}{dx} + 2y\tan x = \sin x$, which is linear differential equation.

$$\text{Now, I.F} = e^{\int 2\tan x} = e^{2\ln|\sec x|}$$

$$= \sec^2 x$$

So, solution of given differential equation is

$$y \cdot \sec^2 x = \int \sec^2 x . \sin x dx$$

$$= \int \tan x \cdot \sec x dx$$

$$\Rightarrow y\sec^2 x = \sec x + c$$

Put $x = \frac{\pi}{3}$ and $y = 0$ in above equation, we get

$$0 = \sec\frac{\pi}{3} + c \Rightarrow c = -2$$

$$\therefore y\sec^2 x = \sec x - 2 \ \Rightarrow y = \cos x - 2\cos^2 x$$

$$= \frac{1}{8} - 2\left(\cos x - \frac{1}{4}\right)^2$$

$\therefore y$ is maximum when $\cos x - \frac{1}{4} = 0$

$\therefore$ Maximum value of y is $\frac{1}{8}$.

17. Option (2) is correct.

$\frac{dy}{dx} + \frac{x}{x^2-1}y = \frac{x^4+2x}{\sqrt{1-x^2}}$ which is linear differential equation

$$\text{So, } I.F = e^{\frac{1}{2}\int\frac{2x}{x^2-1}dx}$$

$$= e^{-\frac{1}{2}\int\frac{2x}{1-x^2}dx} \quad [\because x \in (-1,1)]$$

$$= e^{\frac{1}{2}\log_e(1-x^2)} = \sqrt{1-x^2}$$

$\therefore$ Solution is,

I.F × $y = \int$ I.F × $Q\,dx$

$$\Rightarrow \quad y\sqrt{1-x^2} = \int\sqrt{1-x^2} \times \frac{x^4+2x}{\sqrt{1-x^2}}dx$$

$$= \int\left(x^4+2x\right)dx$$

$$\Rightarrow \quad y\sqrt{1-x^2} = \frac{x^5}{5} + x^2 + c$$

$\because$ It passes through (0, 0)

$$\therefore 0 = 0 + c \Rightarrow c = 0$$

$$\therefore y = \frac{x^5+5x^2}{5\sqrt{1-x^2}} = f(x)$$

$$\text{Now, } \int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)\,dx$$

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{x^5}{5\sqrt{1-x^2}}dx + \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}dx$$

$$= 0 + 2\int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}dx$$

$\because \frac{x^5}{\sqrt{1-x^2}}$ is an odd function.

and $\frac{x^2}{\sqrt{1-x^2}}$ is an even function.

$$\therefore I = 2\int_0^{\sqrt{3}/2} \frac{1-\left(1-x^2\right)}{\sqrt{1-x^2}}dx$$

$$= 2\int_0^{\sqrt{3}/2}\left(\frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2}\right)dx$$

$$= 2\left[\sin^{-1}x - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x\right]_0^{\sqrt{3}/2}$$

$$= 2\left[\frac{1}{2}\sin^{-1}x - \frac{x}{2}\sqrt{1-x^2}\right]_0^{\sqrt{3}/2}$$

$$= 2\left[\frac{1}{2}\times\frac{\pi}{3} - \frac{\sqrt{3}}{4}\times\frac{1}{2}\right] = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

18. Option (2) is correct.

Given, slope of the curve C : $y = f(x)$

$$m = \frac{dy}{dx} = \frac{2e^{2x} - 6e^{-x} + 9}{2 + 9e^{-2x}}$$

$$\Rightarrow dy = \frac{2e^{2x} - 6e^{-x} + 9}{2 + 9e^{-2x}}dx$$

Since, C : Passes through the points $\left(0, \frac{1}{2} + \frac{\pi}{2\sqrt{2}}\right)$

and $\left(\alpha, \frac{1}{2}e^{2\alpha}\right)$

$$\therefore \int_{\frac{1}{2}+\frac{\pi}{2\sqrt{2}}}^{\frac{1}{2}e^{2\alpha}} dy = \int_0^{\alpha}\frac{2e^{2x} - 6e^{-x} + 9}{2 + 9e^{-2x}}dx$$

$$\Rightarrow [y]_{\frac{1}{2}+\frac{\pi}{2\sqrt{2}}}^{\frac{1}{2}e^{2\alpha}} = \int_0^{\alpha}\frac{(2e^{2x} - 6e^{-x} + 9)e^{2x}}{(2 + 9e^{-2x})e^{2x}}dx$$

$$\Rightarrow \frac{1}{2}e^{2\alpha} - \frac{1}{2} - \frac{\pi}{2\sqrt{2}} = \int_0^{\alpha} \frac{2e^{4x} - 6e^{x} + 9e^{2x}}{2e^{2x} + 9}dx$$

$$= \int_0^{\alpha} \frac{(2e^{3x} + 9e^{x} - 6)e^{x}}{(2e^{2x} + 9)}dx$$

Let $e^x = t$

$\Rightarrow e^x dx = dt$

$$\therefore \frac{1}{2}e^{2\alpha} - \frac{1}{2} - \frac{\pi}{2\sqrt{2}} = \int_1^{e^{\alpha}} \frac{(2t^3 + 9t - 6)dt}{(2t^2 + 9)}$$

$$= \int_1^{e^{\alpha}} \left(t - \frac{6}{2t^2 + 9}\right)dt$$

$$\Rightarrow \frac{1}{2}e^{2\alpha} - \frac{1}{2} - \frac{\pi}{2\sqrt{2}} = \left[\frac{t^2}{2}\right]_1^{e^{\alpha}} - 3\int_1^{e^{\alpha}} \frac{1}{t^2 + \frac{9}{2}}dt$$

$$= \left[\frac{e^{2\alpha} - 1}{2}\right] - 3 \times \frac{\sqrt{2}}{3}\left[\tan^{-1}\left(\frac{t}{\frac{3}{\sqrt{2}}}\right)\right]_1^{e^{\alpha}}$$

$$\Rightarrow \frac{1}{2}e^{2\alpha} - \frac{1}{2} - \frac{\pi}{2\sqrt{2}}$$

$$= \left[\frac{e^{2\alpha} - 1}{2}\right] - \sqrt{2}\left[\tan^{-1}\left(\frac{\sqrt{2}e^{\alpha}}{3}\right) - \tan^{-1}\left(\frac{\sqrt{2}}{3}\right)\right]$$

$$\frac{e^{2\alpha} - 1}{2} - \frac{\pi}{2\sqrt{2}}$$

$$= \left[\frac{e^{2\alpha} - 1}{2}\right] - \sqrt{2}\left[\tan^{-1}\left(\frac{\frac{\sqrt{2}}{3}(e^{\alpha} - 1)}{1 + \frac{2}{9}e^{\alpha}}\right)\right]$$

$$\Rightarrow \frac{-\pi}{2\sqrt{2}} = -\sqrt{2}\tan^{-1}\left(\frac{\frac{\sqrt{2}}{3}(e^{\alpha} - 1)}{\frac{9 + 2e^{\alpha}}{9}}\right)$$

$$\Rightarrow \frac{\pi}{4} = \tan^{-1}\left(\frac{(e^{\alpha} - 1)\sqrt{2}}{3} \times \frac{9}{9 + 2e^{\alpha}}\right)$$

$$\Rightarrow \tan\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}e^{\alpha} - 3\sqrt{2}}{9 + 2e^{\alpha}} \Rightarrow 1 = \frac{3\sqrt{2}e^{\alpha} - 3\sqrt{2}}{9 + 2e^{\alpha}}$$

$$\Rightarrow 9 + 2e^{\alpha} = 3\sqrt{2}e^{\alpha} - 3\sqrt{2}$$

$$\Rightarrow \left(2 - 3\sqrt{2}\right)e^{\alpha} = -3\sqrt{2} - 9$$

$$\Rightarrow e^{\alpha} = \frac{-3\sqrt{2} - 9}{2 - 3\sqrt{2}} = \frac{3\left(3 + \sqrt{2}\right)}{\sqrt{2}\left(3 - \sqrt{2}\right)}$$

19. Option (1) is correct.

Given differential equation

$$(x - y^2)dx + y(5x + y^2)dy = 0 \qquad \text{...(i)}$$

$$\Rightarrow \quad (x - y^2)dx = -y(5x + y^2)dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x - y^2)}{-(5xy + y^3)}$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = \frac{x - y^2}{-(5x + y^2)}$$

Let $y^2 = t$

$$2y\frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore \quad \frac{1}{2}\frac{dt}{dx} = \frac{-(x - t)}{(5x + t)}$$

Which is Homogeneous differential equation of first order.

Put $t = xv$

$$\Rightarrow \quad \frac{dt}{dx} = x\frac{dv}{dx} + v$$

$$\therefore \quad \frac{1}{2}\left(x\frac{dv}{dx} + v\right) = -\left(\frac{x - xv}{5x + xv}\right) = \frac{-x(1 - v)}{x(5 + v)}$$

$$\Rightarrow \quad x\frac{dv}{dx} + v = \frac{-2(1 - v)}{(5 + v)}$$

$$\Rightarrow \frac{xdv}{dx} = \frac{-2(1 - v)}{(5+v)} - v = \frac{-2 + 2v - v(5 + v)}{5+v}$$

$$= \frac{-2 + 2v - 5v - v^2}{5 + v} = \frac{-2 - 3v - v^2}{5 + v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-(v^2 + 3v + 2)}{(v + 5)} = \frac{-(v + 1)(v + 2)}{(v + 5)}$$

$$\therefore \quad \int \frac{v + 5}{(v + 1)(v + 2)}dv = -\frac{dx}{x}$$

$$\Rightarrow \int \frac{4}{(v + 1)} + \frac{3}{(v + 2)}dv = \int -\frac{1}{x}dx$$

$$\Rightarrow 4\log_e(v + 1) + 3\log_e(v + 2) = -\log_e x + \log_e c$$

$$\log_e\left(\frac{(v + 1)^4}{(v + 2)^3}\right) = \log_e\left(\frac{c}{x}\right)$$

$$\Rightarrow \quad \frac{(v + 1)^4}{(v + 2)^3} = \frac{c}{x}$$

$$\Rightarrow \quad x(v + 1)^4 = (v + 2)^3.c$$

Put the value of $v = \frac{t}{x} = \frac{y^2}{x}$

$$x\left(\frac{y^2}{x} + 1\right)^4 = \left(\frac{y^2}{x} + 2\right)^3 c$$

$$\Rightarrow \quad \frac{x(y^2 + x)^4}{x^4} = \frac{(y^2 + 2x)^3 c}{x^3}$$

$$\Rightarrow \quad \frac{(y^2+x)^4}{x^3} = \frac{(y^2+2x)^3}{x^3}c$$

$$\Rightarrow \quad (y^2+x)^4 = c(y^2+2x)^3$$

20. Option (1) is correct.

Given differential equation

$$\left[\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right]x\frac{dy}{dx} = x+\left[\frac{x}{\sqrt{x^2-y^2}}+e^{\frac{y}{x}}\right]y \quad \text{...(i)}$$

$$\Rightarrow \frac{x^2}{\sqrt{x^2-y^2}}dy+e^{\frac{y}{x}}.xdy = xdx+\frac{xy}{\sqrt{x^2-y^2}}dx+ye^{\frac{y}{x}}dx$$

$$\Rightarrow \frac{x^2}{\sqrt{x^2-y^2}}dy-\frac{xy}{\sqrt{x^2-y^2}}dx = xdx + ye^{y/x}dx - xe^{y/x}\,dy$$

$$\Rightarrow \quad \frac{x(xdy-ydx)}{\sqrt{x^2-y^2}} = xdx + e^{y/x}(ydx - xdy)$$

$$= \frac{(xdy-ydx)}{\sqrt{1-\left(\frac{y}{x}\right)^2}}-e^{y/x}(ydx-xdy) = xdx$$

Dividing by x^2 both sides, we get

$$\Rightarrow \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}.\left(\frac{xdy-ydx}{x^2}\right)-e^{y/x}\left(\frac{ydx-xdy}{x^2}\right)$$

$$= \frac{1}{x}dx$$

Integrate both sides

$$\Rightarrow \int\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left(\frac{xdy-ydx}{x^2}\right)+\int e^{y/x}\left(\frac{xdy-ydx}{x^2}\right)$$

$$= \int\frac{1}{x}dx+c$$

$$\Rightarrow \quad \int\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}d\left(\frac{y}{x}\right)+\int e^{y/x}d\left(\frac{y}{x}\right) = +\log_e x + C$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{y}{x}\right)+e^{y/x} = \log_e x + C \quad \text{...(ii)}$$

which passes through the point (1, 0) and (2α , α)

$$\because \begin{cases} \log 1 = 0 \\ \sin^{-1}(0) = 0 \\ e^0 = 1 \end{cases}$$

$\Rightarrow 0 + 1 = C$

$\Rightarrow C = 1$

So, $\quad \sin^{-1}\left(\frac{y}{x}\right)+e^{y/x} = \log_e x + 1 \quad$ ...(iii)

it passes through (2α, α)

$$\Rightarrow \quad \sin^{-1}\left(\frac{\alpha}{2\alpha}\right)+e^{\frac{1}{2}} = \log_e(2\alpha) + 1$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1}{2}\right)+e^{\frac{1}{2}} = \log_e 2\alpha + 1$$

$$\Rightarrow \frac{\pi}{6}+\sqrt{e}-1 = \log_e 2\alpha$$

$$\Rightarrow 2\alpha = e^{\frac{\pi}{6}+\sqrt{e}-1}$$

$$\Rightarrow \alpha = \frac{1}{2}\left[e^{\frac{\pi}{6}+\sqrt{e}-1}\right]$$

21. Option (1) is correct.

Given differential equation

$$x(1-x^2)\frac{dy}{dx}+3x^2y-y-4x^3 = 0$$

$$\Rightarrow x(1-x^2)\frac{dy}{dx}+3x^2y-y = 4x^3$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{(3x^2-1)}{x(1-x^2)}y = \frac{4x^3}{x(1-x^2)} \quad \text{...(i)}$$

which is linear differential equation of first order

So, $P = \dfrac{3x^2-1}{x(1-x^2)}$

and $Q = \dfrac{4x^2}{1-x^2}$

Now, I.F. $= e^{\int pdx} = e^{\int\frac{3x^2-1}{x(1-x^2)}dx} = e^{\int\frac{3x^2-1}{(x-x^3)}dx}$

Let $x - x^3 = t$

$\Rightarrow (1 - 3x^2)dx \qquad = dt$

So, I.F. $= e^{-\int\frac{1}{t}dt} = e^{-\log_e|t|} = \dfrac{1}{|t|} = \dfrac{1}{|x(1-x^2)|}$

$$= \frac{1}{|x-x^3|} = \frac{1}{x^3-x} \quad (\because x > 1)$$

Solution of given differential equation is

y (I.F.)$= \int$(I.F.).Qdx+c ...(ii)

$$\Rightarrow y.\frac{1}{x(-1+x^2)} = \int\frac{4x^2}{1-x^2}\times\frac{1}{x(-1+x^2)}dx+c$$

$$= \int\frac{+4x^2}{(1-x^2).x(x^2-1)}dx+c = -\int\frac{4x}{(x^2-1)^2}dx+c$$

Let $x^2 - 1 = t$

$\Rightarrow 2xdx = dt$

$$= -\int\frac{2}{t^2}dt+c = \frac{2}{t}+c = \frac{2}{x^2-1}+c$$

$$y\frac{1}{x(x^2-1)} = \frac{2}{x^2-1}+c \quad \text{...(ii)}$$

Given, $\qquad y(2) = -2$

$$\therefore \quad -2\times\frac{1}{2(4-1)} = \frac{2}{4-1}+c$$

$$\Rightarrow \frac{-1}{3} = \frac{2}{3}+c \Rightarrow \text{C} = \frac{-1}{2}-\frac{2}{3} = -1$$

$\Rightarrow \text{C} = -1$

$$\Rightarrow y = \frac{2}{x^2-1}\times\frac{(x^2-1)x}{1}+cx(x^2-1)$$

$\Rightarrow \; = 2x - x\,(x^2 - 1)$

$\Rightarrow y = 2x - x^3 + x = 3x - x^3$

$\Rightarrow y(x) = 3x - x^3$

$\therefore y(3) = 9 - 27 = -18$

22. Option (2) is correct.

Given that

$$\frac{dy}{dx} = \frac{ax-by+a}{bx+cy+a}$$

$\Rightarrow bxdy + cydy + ady = axdx - bydx + adx$

$\Rightarrow \quad b(xdy + ydx) = axdx + adx - cydy - ady$

$$\Rightarrow \; b\int d(xy) = \int axdx+\int adx-\int cydy-\int dy$$

$$\Rightarrow \quad bxy = \frac{ax^2}{2}+ax-\frac{cy^2}{2}-ay+k$$

Given that it represent equation of circle

$\therefore \quad b = 0$

and $a = -c$

$$\Rightarrow \; x^2+y^2+2x-2y+\frac{2k}{a} = 0$$

Since it passes through (2, 5)

$$\therefore \quad 4+25+4-10+\frac{2k}{a} = 0$$

$$\Rightarrow \; \frac{2k}{a} = -23$$

$\therefore$ Equation of circle is

$x^2 + y^2 + 2x - 2y - 23 = 0$

which have centre (–1, 1) and radius = 5

So, minimum distance of P from circle will be

$$= \sqrt{(11+1)^2+(6-1)^2}-5 = 13 - 5 = 8$$

> **Hint :** Minimum distance of point $\text{P}(x_1, y_1)$ from circle having centre (g, h) and radius r is
>
> $$\sqrt{(g-x_1)^2+(h-y_1)^2} = r$$

23. Option (4) is correct.

$$\frac{dy}{dx} = \frac{-2^x.2^{-y}(2^y-1)}{2^x-1}$$

$$\Rightarrow \quad \frac{2^y}{2^y-1}dy = -\int\frac{2^x}{2^x-1}dx$$

$$\Rightarrow \quad \int\frac{2^y.\log_e 2}{2^y-1}dy = -\int\frac{2^x.\log_e 2}{2^x-1}dx$$

$$\left[\because \int\frac{f\,(x)}{f(x)}dx = \log_e f(x) + c\right]$$

$$\Rightarrow \quad \log_e|2y-1| = -\log_e|2x-1|+\log c$$

$$\Rightarrow 2^y - 1 = \frac{c}{2^x-1}$$

$\Rightarrow (2^y - 1)\,(2^x - 1) \;\; = c$

Put $x = 1$ and $y = 1$, we get $c = 1$

$\therefore (2^y - 1)\,(2^x - 1) \;\; = 1$

Put $x = 2$ in above equation, we get

$$2^y - = \frac{1}{3} \Rightarrow 2^y = \frac{4}{3} \Rightarrow y = \log_2\left(\frac{4}{3}\right)$$

$$= 2\log_2 2 - \log_2 3 = 2 - \log_2 3$$

24. Option (2) is correct.

Given differential equation is

$$\Rightarrow \quad \frac{dx}{dy} = \frac{(\tan^{-1}y)-x}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy}+\frac{1}{1+y^2}.x = \frac{\tan^{-1}y}{1+y^2}$$

$$\Rightarrow \text{I.F.} = e^{\int \text{P}dy} = e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$\therefore$ Solution is

$$\text{I.F} \times x = \int \text{I.F}\times\text{Q}dy$$

$$\Rightarrow \quad x.e^{\tan-1}y = \int e^{\tan-1}y.\frac{\tan^{-1}y}{1+y^2}dy$$

$$\text{Let } \tan^{-1}y = t \Rightarrow \frac{1}{1+y^2}dy = dt$$

$$\therefore \quad x.e^{\tan^{-1}y} = \int t.e^t dt = t.e^t - \int 1.e^t dt$$

$$= t.e^t - e^t + c$$

$$\Rightarrow \quad x.e^{\tan^{-1}y} = e^{\tan-1}y[\tan^{-1}y - 1] + c$$

$\Rightarrow x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}$

Put $y = 0$ and $x = 1$ in above equation, we get

$1 = 0 - 1 + c.e^{-0}$

$\Rightarrow c = 2$

$\therefore x = \tan^{-1}y - 1 + 2e^{\tan-1y}$

Put $y = \tan(1)$ in above equation, we get

$x = \tan^{-1}(\tan 1) - 1 + 2e^{-\tan^{-1}}(\tan 1)$

$= 1 - 1 + 2e^{-1}$

$$\Rightarrow x = \frac{2}{e}$$

25. Option (4) is correct.

$$\text{Given,} \qquad x\frac{dy}{dx}+2y = xe^x;\; y\,(1) = 0 \qquad \text{...(i)}$$

which is linear differential equation of first order.

$$\frac{dy}{dx}+\frac{2}{x}y = e^x \quad ...(ii)$$

$$\frac{dy}{dx}+\text{P}y = \text{Q} \quad ...(iii)$$

On Comparing $\text{P} = \frac{2}{x}, \text{Q} = e^x$

$$\text{I.F} = e^{\int \text{P}dx} = e^{\int \frac{2}{x}dx} = x^2$$

Solution $y\,(\text{I.F}) = \int (\text{I.F})\text{Q}dx + c_1$ ...(iii)

$yx^2 = \int x^2e^x dx + c$

$yx^2 = e^x(x^2 - 2x + 2) + c$ ...(iv)

$yx^2 = x^2e^x - 2xe^x + 2e^x + c$

Put $x=1, y\,(1) = 0$

$\Rightarrow$ $0 = e(1 - 2 + 2) + c$

$\therefore$ $c = -e$

$\Rightarrow$ $yx^2 = e^x(x^2 - 2x + 2) - e$ ...(v)

Given $z(x) = x^2y(x) - e^x\ x \in R$

Find the local minimum value of $z(x)$

$z(x) = e^x(x^2 - 2x + 2) - e - e^x$

$= e^x(x^2 - 2x + 2 - 1) - e$

$z(x) = e^x(x^2 - 2x + 1) - e$

$z(x) = e^x(x - 1)^2 - e$

Differentiate with respect to x

$z'(x) = e^{x\cdot}2(x - 1) + (x - 1)^2 e^x,$

$z'(x) = 0$

$\Rightarrow e^{x\cdot}2(x - 1) + (x - 1)^2 e^x = 0$

$e^x2(x - 1) = -(x - 1)^2 e^x$

$2 = -(x - 1)$

$\therefore x - 1 = -2$

$x = -1$

Again differentiate

$z'(x) = e^x(x^2 - 1)$

$\Rightarrow z''(x) = e^x2x + (x^2 - 1)\,e^x$

$e^x = (x^2 + 2x - 1)$

$z''(-1) = e^{-1}\,[1 - 2 - 1]$

$= \frac{1}{e}\,[-2] = \frac{-2}{e} < 0$

Then, $z(x)$ has a local maximum at $x = -1$.

Local maximum value at $x = -1$:

$z(-1) = e^{-1}\,(-1 - 1)^2 - e$

$= \frac{4}{e} - e$

26. Option (3) is correct.

$$\frac{dy}{dx}+e^x\,(x^2-2)y = (x^2 - 2x)\,(x^2 - 2)\,e^{2x} \quad ....(i)$$

which is linear differential equation of first order in the form

$$\frac{dy}{dx}+\text{P}y = \text{Q}$$

comparing

$\text{P} = e^x(x^2 - 2)\,dx$

$\text{Q} = (x^2 - 2x)\,(x^2 - 2)e^{2x}$

Find $\text{I.F} = e^{\int \text{P}dx} = e^{\int e^x(x^2-2)dx}$

$= e^{[e^x(x^2 - 2) - (2x)e^x + (2)\,e^x]}$

$\text{I.F} = e^{[e^x(x^2 - 2 - 2x + 2]}$

$= e^{e^x(x^2 - 2x)}$ ...(ii)

Solution for y

$$y(\text{I.F}) = \int (\text{I.F})\text{Q}dx + \text{C}$$

$$y.e^x(x^2 - 2x) = \int e^{ex}(x^2-2x)$$

$$(x^2 - 2x)\,(x^2 - 2)\,e^{2x}\,dx + c$$

$$= \int e^{e^x(x^2-2x)+2x}.(x^2-2x)(x^2-2)dx + c$$

$y_ee^x(x^2 - 2)$

$$= \int e^{e^x(x^2-2x)}e^x.e^x(x^2-2x)(x^2-2)dx + c$$

Let $e^x\,(x^2 - 2x) = t$

$e^x[2x - 2] + e^x[x^2 - 2x]\,dx = dt$

$\Rightarrow e^x\,[2x - 2 + x^2 - 2x]\,dx = dt$

$e^x\,[x^2 - 2]\,dx = dt$

$y_ee^x(x^2 - 2) = \int e^t t\ dt + c$

$y_ee^x(x^2 - 2) = e^tt - e^t + c$

$= e^{e^x(x^2 - 2x)}\,[e^x(x^2 - 2x) - 1] + c$

$y = [e^x(x^2 - 2x - 1] + ce^{-ex}(x^2 - 2x)$

Put $x = 0$ $\quad y(0) = 0$

$0 = [-1] + c \Rightarrow c = 1$

$\therefore y = e^x(x^2 - 2x) - 1 + e^{-e^x(x^2 - 2x)}$

Then $y(2) = e^2(4 - 4) - 1 + e^{ex(4 - 4)}$

$= -1 + 1 = 0$

$\therefore$ $\quad y(2) = 0$

27. Option (2) is correct.

Given, $\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x + 1)$ ...(i)

Comparing with

$$\frac{dy}{dx}+Py = \text{Q}$$

$$\Rightarrow \text{P} = \frac{-1}{x+1}$$

And $\text{Q} = \underline{e}^{3x}(x + 1)$

which is linear differential equations of first order

$$\text{I.F.} = e^{\int \text{P}dx} = e^{\int -\frac{1}{x+1}dx}$$

$$= e^{-\text{in}(x + 1)} = \frac{1}{x+1}$$

$$y(\text{I.F.}) = \int (\text{I.F.}).\text{Q}dx + \text{C}$$

$$\Rightarrow \quad y\frac{1}{(x+1)} = \int e^{3x}(x+1)\times\frac{1}{x+1}dx + \text{C}$$

$$\Rightarrow \quad \frac{y}{x+1} = \int e^{3x}dx + \text{C}$$

$$\Rightarrow \quad \frac{y}{x+1} = \frac{e^{3x}}{3} + \text{C}_2$$

$$\Rightarrow \quad y = \frac{e^{3x}}{3}(x+1)+C(x+1)$$

$$\Rightarrow \quad y = (x+1)\left[\frac{e^{3x}}{3}+C\right] \qquad \text{...(ii)}$$

Put $x = 0$

Given, $y(0) = \frac{1}{3} \Rightarrow y(0) = 1\left[\frac{1}{3}\times 1+C\right]$

$$\frac{1}{3} = \frac{1}{3}+C$$

$\therefore$ $C = 0$

So, $y = \frac{1}{3}(x+1)e^{3x}$ ...(iii)

Now, $\frac{dy}{dx} = \frac{1}{3}\left[(x+1).3e^{3x}+e^{3x}.1\right]$

So, $= \frac{e^{3x}}{3}[3x+4]=0$

For extermum or critical point put $\frac{dy}{dx}=0$

$\Rightarrow 3x + 4 = 0$

$\Rightarrow x = \frac{-4}{3}$

$$\text{Now,} \quad \frac{d^2y}{dx^2} = \frac{1}{3}\left[e^{3c}(3)+(3x+4).3e^{3x}\right]$$

$$\text{Now,} \quad \frac{d^2y}{dx^2} = \frac{e^{3x}\times 3}{3}[1+3x+4]$$

$$\text{Now,} \quad \frac{d^2y}{dx^2} = e^{3x}[3x + 5] > 0,$$

at $x = \frac{-4}{3}$

Therefore, $x = \frac{-4}{3}$ is a point of local minimum.

28. **Option (3) is correct.**

Given differential equation

$y^2dx + (x^2 - xy + y^2)dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y^2}{x^2-xy+y^2} \qquad \text{...(i)}$$

which is Homogeneous differential equation of first order

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = x\frac{dv}{dx}+v$$

$$\text{Now, (i),} \quad \left(x\frac{dv}{dx}+v\right) = \frac{-v^2x^2}{x^2-x^2v+v^2x^2} = \frac{-v^2}{1-v+v^2}$$

$$\Rightarrow \quad \left(x\frac{dv}{dx}+v\right) = \frac{-v^2}{1-v+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^2-v(1-v+v^2)}{1-v+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v^2-v+v^2-v^3}{1-v+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v(1+v^2)}{(1-v+v^2)}$$

$$\Rightarrow \quad \int\frac{1-v+v^2}{v(1+v^2)}dv = \int -\frac{dx}{x}$$

$$\Rightarrow \int\left(\frac{1}{v(1+v^2)}-\frac{1}{1+v^2}+\frac{v}{1+v^2}\right)dv = -\ln x + \ln c$$

$$\Rightarrow \int\left(\frac{1}{v}-\frac{v}{1+v^2}-\frac{1}{1+v^2}+\frac{v}{1+v^2}\right)dv = -\ln x + \ln c$$

$$\Rightarrow \quad \ln v - \tan^{-1}v = \ln\left(\frac{c}{x}\right)$$

Put, $x = 1, y = 1$

$$\Rightarrow \ln\left(\frac{y}{x}\right)-\tan^{-1}\left(\frac{y}{x}\right) = \ln\left(\frac{c}{x}\right)$$

$$\Rightarrow \ln\left(\frac{1}{c}\right) = \tan^{-1}(1)$$

$$\Rightarrow \ln\left(\frac{y}{x}\right)-\ln\left(\frac{c}{x}\right) = \tan^{-1}\left(\frac{y}{x}\right),$$

where $-\ln c - = \frac{\pi}{4}$

$$\Rightarrow \quad \ln\left(\frac{y}{x}\times\frac{x}{c}\right) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow \ln\left(\frac{y}{c}\right) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{or } \ln\left(\frac{y}{x}\right)-\tan^{-1}\left(\frac{y}{x}\right) = -\ln x-\frac{\pi}{4}$$

Put $y = \sqrt{3}x$

$$\ln\left(\frac{\sqrt{3}x}{x}\right)-\tan^{-1}\sqrt{3} = -\ln x-\frac{\pi}{4}$$

$$\Rightarrow \quad +\ln\sqrt{3}-\frac{\pi}{3} = -\ln x-\frac{\pi}{4}$$

$$\Rightarrow \quad \ln(\sqrt{3})+\ln x = \frac{\pi}{3}-\frac{\pi}{4}$$

$$\therefore \quad \ln\left(\sqrt{3}x\right) = \frac{\pi}{4}-\frac{\pi}{3}=\frac{\pi}{12}$$

29. Option (2) is correct.

$$\frac{dy}{dx} = \frac{2xy-3y^2}{2x^2}$$

Which is Homogeneous linear differential equation of first order.

Put $y = x.v$

$$\frac{dy}{dx} = x.\frac{dv}{dx}+v$$

$$x\frac{dv}{dx}+v = \frac{2x.xv-3x^2v^2}{2x^2}$$

$$x\frac{dv}{dx}+v = \frac{2v-3v^2}{2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v-3v^2}{2}-v = \frac{2v-3v^2-2v}{2}$$

$$x\frac{dv}{dx} = \frac{-3v^2}{2}$$

$$\therefore \quad \int\frac{dx}{x} = \int\frac{-2}{3v^2}dv$$

$$\Rightarrow \quad \log x = \frac{-2}{3}\times\frac{-1}{v}+c$$

$$\log x = \frac{2}{3}\frac{x}{y}+c \qquad \text{...(i)}$$

$$\log x - c = \frac{2x}{3y}$$

$$\therefore \quad 3y = \frac{2x}{\log x - c}$$

given $y(e) = \dfrac{e}{3}$

$$3\times\frac{e}{3} = \frac{2e}{\log_e e - c}$$

$$\Rightarrow \quad 1 = \frac{2}{1-c}$$

$$\therefore \quad 1-c = 2$$

$$c = 1-2 = -1$$

$$\log x = \frac{2x}{3y}-1 \quad \therefore 3y = \frac{2x}{\log x+1}$$

$$\Rightarrow \quad y = \frac{1}{3}\left(\frac{2x}{\log x+1}\right)$$

Put $x = 1$

$$y(1) = \frac{1}{3}\times\frac{2\times1}{\log 1+1} = \frac{2}{3}$$

30. Option (1) is correct.

Let r be the radius of spericut balloon.

So, surface area of balloon, $S = 4\pi r^2$

Given, $\dfrac{ds}{dt} = \text{P (constant)}$

$$\Rightarrow \quad 8\pi r\frac{dr}{dt} = \text{P}$$

$$\Rightarrow \quad 8\pi r dr = \text{P}dt$$

$$\Rightarrow \quad 8\pi\int r dr = \int \text{P}dt$$

$$\Rightarrow \quad 8\pi\frac{r^2}{2} = \text{P}t + \text{C}$$

$\because$ At $t = 0, r = 3$

$$\Rightarrow \quad 8\pi\left(\frac{9}{2}\right) = \text{C}$$

$$\because \quad \text{C} = 36\pi$$

Also, at $t = 5, r = 7$

$$\Rightarrow \quad 8\pi\left(\frac{49}{2}\right) = 5\text{P} + 36\pi$$

$$\Rightarrow \quad 5\text{P} = 160\pi$$

$$\Rightarrow \quad \text{P} = 32\pi$$

$$\because \quad 8\pi\frac{r^2}{2} = 32\pi t + 36\pi$$

Put $t = 9$ in above equation, we get

$$8\pi\frac{r^2}{2} = 32\pi\,(9) + 36\pi$$

$$\Rightarrow \quad r = 9$$

Hints:

Surface area of balloon $S = 4\pi r^2$, where r is the radius of spherical ballon and solve further using the concept of application of derivations.

Shortcut method

Let r be the radius of spherical ballon.

So, surface area of ballon, $S = 4\pi r^2$

Given, $\dfrac{ds}{dt} = \text{P (constant)}$

$$\Rightarrow \quad 8\pi r\frac{dr}{dt} = \text{P}$$

$$\Rightarrow \quad 8\pi\frac{r^2}{2} = \text{P}t + \text{C}$$

$\because$ At $t = 0, r = 3$

$\therefore$ $\text{C} = 36\pi$

Also, at $t = 5, r = 7$

$\Rightarrow$ $\text{P} = 32\pi$

$\therefore$ $4\pi r^2 = 32\pi t + 36\pi$

So, at $t = 9, r = 9$

31. Option (4) is correct.

$$x^2dy + ydx = \frac{dx}{x} \Rightarrow \frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^3}$$

$$\text{I.F.} = e^{\int\frac{1}{x^2}dx} = e^{-\frac{1}{x}}$$

$\Rightarrow ye^{-\frac{1}{x}} = \int e^{\frac{1}{x}} \cdot \frac{1}{x^3} dx + C$

Let $\frac{-1}{x} = t \Rightarrow \frac{1}{x^2} dx = dt$

$\Rightarrow ye^{-\frac{1}{x}} = \int -te^t dt + C = -[te^t - e^t] + C$

$\Rightarrow ye^{-\frac{1}{x}} = \frac{1}{x}e^{-\frac{1}{x}} + e^{-\frac{1}{x}} + C$

Put $x = 1$

$\Rightarrow (1)\, e^{-1} = \frac{e^{-1}}{1} + e^{-1} + C$

$\Rightarrow C = e^{-1}$

Equation is $ye^{-\frac{1}{x}} = \frac{1}{x}e^{-\frac{1}{x}} + e^{-\frac{1}{x}} - e^{-1}$

$\Rightarrow y = \frac{1}{x} + 1 - \frac{e^{1/x}}{e}$

At $x = \frac{1}{2} \Rightarrow y\left(\frac{1}{2}\right) = 2 + 1 - \frac{e^2}{e}$

$\Rightarrow y = 3 - e$

32. Option (2) is correct.

$$\frac{dy}{dx} = \frac{2^{x+y} - 2^x}{2^y} = 2^x\left(\frac{2^y - 1}{2^y}\right)$$

$\Rightarrow \int \frac{2^y\, dy}{2^y - 1} = \int 2^x dx$

Let $2^y - 1 = t \Rightarrow 2^y \ln 2 dy = dt$

$\Rightarrow \int \frac{dt}{t}\frac{1}{ln 2} = \int 2^x dx$

$\Rightarrow \frac{ln\, t}{ln 2} = \frac{2^x}{ln 2} + \frac{c}{ln 2}$

$\Rightarrow ln\, t = 2^x + c$

When $x = 0, y = 1, c = -1$

$\therefore ln\,(2^y - 1) = 2^x - 1 \Rightarrow 2^y - 1 = e^{2^x - 1}$

$\Rightarrow 2y = 1 + e^{2^x - 1}$

When, $x = 1 \Rightarrow 2^y = 1 + e^1 \Rightarrow y = log_2(1 + e)$

33. Option (3) is correct.

$$\frac{dy}{dx} = \frac{2^x y + 2^y \cdot 2^x}{2^x + 2^{x+y} \log_e{}^2}$$

$\Rightarrow \frac{dy}{dx} = \frac{2^x(y + 2^y)}{2^x(1 + 2^y \log_e{}^2)}$

$\Rightarrow \int \frac{1 + 2^y \log_e 2}{y + 2^y} dy = \int dx$

$\Rightarrow \ln|y + 2^y| = x + c$

Now, $y(0) = 0 \Rightarrow c = 0$

$\therefore \ln|y + 2^y| = x$

Now, for $y = 1$ we have

$x = \ln(1 + 2) = \ln 3 \in (1, 2)$

34. Option (4) is correct.

$$\frac{y}{x}\frac{dy}{dx} = \left[\frac{y^2}{x^2} + \frac{\phi\left(\frac{y^2}{x^2}\right)}{\phi'\left(\frac{y^2}{x^2}\right)}\right], \text{ let } \frac{y}{x} = t$$

$\therefore \quad y = xt$

$\frac{dy}{dx} = t + x\frac{dt}{dx}$

$\therefore \quad t\left(t + x\frac{dt}{dx}\right) = \left(t^2 + \frac{\phi(t^2)}{\phi'(t^2)}\right)$

$\Rightarrow \quad xt\frac{dt}{dx} = \frac{\phi(t^2)}{\phi'(t^2)}$

$\Rightarrow \quad \frac{t \cdot \phi'(t^2)}{\phi(t^2)} dt = \frac{1}{x} dx$

Integrating both sides

$\int \frac{t\phi'(t^2)}{\phi(t^2)} dt = \int \frac{1}{x} dx$

Let $\varphi(t^2) = p$

$\Rightarrow \quad \varphi'(t^2) \cdot 2t = dp$

$\Rightarrow \quad \frac{1}{2}\int \frac{dp}{p} = \int \frac{1}{x} dx$

$\Rightarrow \quad \frac{1}{2} \ln p = \ln x + C$

$\Rightarrow \quad \frac{1}{2} \ln \varphi(t^2) = \ln + C$

$\Rightarrow \frac{1}{2} \ln \left(\phi\left(\frac{y^2}{x^2}\right)\right) = \ln x + C$

If $x = 1, y = -1$ then $C = \frac{1}{2} \ln(\varphi(1))$

$\Rightarrow \frac{1}{2} \ln\left(\phi\left(\frac{y^2}{x^2}\right)\right) = \ln x + \frac{1}{2} \ln(\varphi(1))$

$\Rightarrow \quad \ln \left(\phi\left(\frac{y^2}{x^2}\right)\right) = \ln x^2 + \ln(\varphi(1))$

$\Rightarrow \quad \ln \left(\phi\left(\frac{y^2}{x^2}\right)\right) = \ln(x^2\varphi(1))$

$\Rightarrow \quad \phi\left(\frac{y^2}{x^2}\right) = x^2\varphi(1)$

If $x = 2$, then $\phi\left(\frac{y^2}{4}\right) = 4\varphi(1)$

35. Option (3) is correct.

Given differential equations is

$$\frac{dy}{dx}=2(y+2\sin x-5)x-2\cos x$$

$$\Rightarrow \frac{dy}{dx}=2xy+4x\sin x-10x-2\cos x$$

Comparing with $\frac{dy}{dx}+P(x)y=Q(x)$

$P(x) = -2x$, $Q(x) = 4x\sin x - 10x - 2\cos x$

So, $\text{I F} = e^{\int P dx}=e^{\int -2x dx}=e^{-x^2}$

$\Rightarrow \quad \text{I F} = e^{-x^2}$

Solution of differntial equation is

$y\,(\text{I F}) = \int Q\,(\text{I F})\,dx + c$

$\Rightarrow y(e^{-x^2}) = \int (4x\sin x - 10x - 2\cos x)\,e^{-x^2}\,dx + c$

$\Rightarrow ye^{-x^2} = \int e^{-x^2}[2x(2\sin x-5)-2\cos x]\,dx$

$$\int \underbrace{e^{-x^2}2x}_{\text{II}}\,\underbrace{(2\sin x-5)}_{\text{II}}\,dx-\int e^{-x^2}\cos x\,dx$$

$$\int\left[-e^{-x^2}(2\sin x-5)\right]+$$

$$\int e^{-x^2}\cos x\,dx-\int e^{-x^2}\cos x\,dx$$

$\Rightarrow ye^{-x^2} = e^{-x^2}(5-2\sin x)+c$

$\Rightarrow \quad y = 5-2\sin x + ce^{x^2}$

Let $F(x) = 5-2\sin x + ce^{x^2}$

$\because \; F(0) = 7$

$\Rightarrow F(0) = 5-2\sin(0)+ce^{0}$

$\Rightarrow \quad 7 = 5-0+c$

$\Rightarrow 7-5 = c$

$\Rightarrow \quad c = 2$

$\therefore F(x) = 5-2\sin x + 2e^{x^2}$

Now at $x = \pi$

$\Rightarrow F(\pi) = 5-2\sin\pi + 2e^{\pi^2}$

$\Rightarrow F(\pi) = 5-2(0)+2e^{\pi^2}$

$\Rightarrow F(\pi) = 5+2e^{\pi^2}$

36. Option (3) is correct.

$(y+1)\tan^2 x\,dx + \tan x\,dy + y\,dx = 0$

So, $\frac{dy}{dx} + (1+y)\tan x = -y\cot x$

$\Rightarrow \frac{dy}{dx} + y(\tan x + \cot x) = -\tan x$

$\text{IF} = e^{\int(\tan x+\cot x)dx} = e^{\int \frac{\tan^2 x+1}{\tan x}dx} = \tan x$

$\therefore \; y\tan x = \int -\tan^2 x\,dx + C$

$\Rightarrow y\tan x = \int (1-\sec^2 x)dx + C$

$\Rightarrow y\tan x = x - \tan x + C$

Now $\lim_{x\to 0^+} xy = 1$

$\Rightarrow \lim_{x\to 0^+}\left(\frac{x}{\tan x}\right)(x-\tan x + C) = 1$

$\Rightarrow 1(0-0+C) = 1 \Rightarrow C = 1$

Then, the function $y\tan x = x - \tan x + 1$ at $x = \frac{\pi}{4}$

$\Rightarrow y\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \tan\frac{\pi}{4} + 1$

$\therefore y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$

37. Option (4) is correct.

$$\frac{dy}{dx} = \frac{-e^{y}}{2x^2} + \frac{1}{x}$$

$$\Rightarrow \quad \frac{e^{-y}dy}{dx} = \frac{e^{-y}}{x} - \frac{1}{2x^2}$$

Let $\quad e^{-y} = t$ …(i)

$\Rightarrow e^{-y}(-1)\frac{dy}{dx} = \frac{dt}{dx}$

(differentiating both sides w.r.t. x)

$\therefore \quad \frac{-dt}{dx} = \frac{t}{x} - \frac{1}{2x^2}$

$\Rightarrow \frac{-dt}{dx} - \frac{t}{x} = \frac{-1}{2x^2} \Rightarrow \frac{dt}{dx} + \frac{t}{x} = \frac{1}{2x^2}$

Now, IF $= e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$\therefore \quad tx = \int \frac{1}{2x^2}\cdot x\,dx + C$

$\Rightarrow e^{-y}x = \frac{1}{2}\ln x + C$

$\because \; y(e) = 1 \Rightarrow e^{-1}\cdot e = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$

$\Rightarrow e^{-y}x = \frac{1}{2}(1+\ln x)$

Put $\; x = 1$ in the above equation, we get

$e^{-y} = \frac{1}{2}$

$\Rightarrow \quad -y = \ln\frac{1}{2}$

$\Rightarrow \quad y = \ln 2$

38. Option (2) is correct.

$$\ln\left(\frac{dy}{dx}\right) = 3x + 4y,\; y(0) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{3x}\cdot e^{4y}$$

$$\Rightarrow \int e^{-4y}dy = \int e^{3x}dx$$

$$\Rightarrow \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

Also, $y(0) = 0 \Rightarrow \frac{-1}{4} = \frac{1}{3} + C$

$$\Rightarrow C = \frac{-7}{12}$$

$$\therefore \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12}$$

Put $x = \frac{-2}{3} \ln 2$

$$\Rightarrow \frac{e^{-4}}{-4} = \frac{e^{3\left(\frac{-2}{3}\right)\log 2}}{3} - \frac{7}{12} = \frac{e^{\log\frac{1}{4}}}{3} - \frac{7}{12}$$

$$\Rightarrow \frac{e^{-4y}}{-4} = \frac{1}{12} - \frac{7}{12} = \frac{-1}{2}$$

$$\Rightarrow e^{-4y} = 2$$

$$\Rightarrow -4y = \ln 2 \Rightarrow y = \frac{-1}{4} \ln 2$$

$$\Rightarrow \alpha = \frac{-1}{4}.$$

39. Option (2) is correct.

Given, $(x - x^3)dy = (y + yx^2 - 3x^4)dx, x > 2$

$$\Rightarrow xdy - ydx = x^2(xdy + ydx) - 3x^4dx$$

$$\Rightarrow \frac{xdy - ydx}{x^2} = xdy + ydx - 3x^4dx$$

Integrating both sides

$$\Rightarrow \frac{y}{x} = xy - x^3 + c$$

Given, $y(3) = 3 \Rightarrow \frac{3}{3} = 9 - 27 + c$

$$\Rightarrow c = 19$$

Now, $\frac{4(4)}{4} = 4y(4) - 6y + 19$

$$\Rightarrow \frac{15}{4}y(4) = 45$$

$$\Rightarrow y(4) = 12$$

40. Option (1) is correct.

$$\frac{dy}{dx} = 1 + xe^{y-x} \quad \text{... (i)}$$

$$\Rightarrow e^{-y}\frac{dy}{dx} = e^{-y} + xe^{-x}$$

Put $e^{-y} = t \Rightarrow e^{-y}\frac{dy}{dx} = -\frac{dt}{dx}$

$$\Rightarrow -\frac{dt}{dx} = t + xe^{-x}$$

$$\Rightarrow \frac{dt}{dx} + t = -xe^{-x} \quad \text{...(ii)}$$

$$\text{IF} = e^{\int 1dx} = e^x$$

Solution of equation (ii) is

$$te^x = \int (-xe^{-x})e^x dx + c$$

$$\Rightarrow te^x = \frac{-x^2}{2} + c$$

$$\Rightarrow e^{x-y} = -\frac{x^2}{2} + c \quad \text{... (iii)}$$

Also, $y(0) = 0 \Rightarrow 1 = c \Rightarrow e^{x-y} = \left(\frac{2-x^2}{2}\right)$

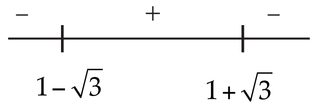
$$\Rightarrow x - y = ln\left(\frac{2-x^2}{2}\right)$$

$$\Rightarrow y = x + ln\left(\frac{2-x^2}{2}\right)$$

Now, $\frac{dy}{dx} = 1 + x\left(\frac{2}{2-x^2}\right)$

$$\Rightarrow \left(\frac{2-x^2+2x}{2-x^2}\right) = 0 \Rightarrow -\left(\frac{x^2-2x-2}{2-x^2}\right) = 0$$

$$\therefore x = 1 \pm \sqrt{3}$$



If $\frac{dy}{dx}$ changes sign from negative to positive at $x = a$, then it will be a point of minima.

$$\Rightarrow y_{\min} \text{ at } x = 1 - \sqrt{3}$$

$$\Rightarrow y_{\min} = \left(1-\sqrt{3}\right) - ln\left(\sqrt{3}-1\right)$$

41. Option (2) is correct.

$$\frac{xdy - ydx}{x^2} = x \cos x \, dx$$

$$\Rightarrow \int d\left(\frac{y}{x}\right) = \int x \cos x \, dx$$

$$\therefore \frac{y}{x} = x \sin x + \cos x + c$$

$$\Rightarrow 0 = 0 - 1 + c \Rightarrow c = 1$$

$$\Rightarrow y = x^2 \sin x + x \cos x + x$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} + 0 + \frac{\pi}{2} = \frac{\pi^2}{4} + \frac{\pi}{2}$$

42. Option (3) is correct.

Given, $\text{cosec}^2 x \, dy + 2dx$

$$= (1 + y \cos 2x)\,\text{cosec}^2 x dx$$

$$\Rightarrow \frac{dy}{dx} + 2\sin^2 x = (1 + y \cos 2x)$$

$$\Rightarrow \frac{dy}{dx} - y\cos 2x = 1 - 2\sin^2 x$$

$$\Rightarrow \quad \frac{dy}{dx} = (\cos 2x)(1+y)$$

$$\Rightarrow \quad \int \frac{dy}{1+y} = \int \cos 2x\, dx$$

$$\Rightarrow \quad ln(1+y) = \frac{\sin 2x}{2} + c$$

Given, $y\left(\frac{\pi}{4}\right) = 0$

$$\Rightarrow ln\left(1+y\left(\frac{\pi}{4}\right)\right) = \frac{\sin\frac{\pi}{2}}{2} + c \Rightarrow c = \frac{-1}{2}$$

Now, $ln\,(1 + y\,(0)) = \frac{\sin 0}{2} - \frac{1}{2}$

So, $(1 + y\,(0)) = e^{-1/2}$

$\therefore$ $(1 + y\,(0))^2 = e^{-1}$

43. Option (3) is correct.

Given differential equation

$$\Rightarrow \quad x\tan\left(\frac{y}{x}\right)dy = y\tan\left(\frac{y}{x}\right)dx - xdx$$

$$\Rightarrow \quad \tan\left(\frac{y}{x}\right)(xdy - ydx) = -xdx$$

$$\Rightarrow \quad \tan\left(\frac{y}{x}\right)\left(\frac{xdy - ydx}{x^2}\right) = \frac{-x}{x^2}dx$$

$$\Rightarrow \quad \int \tan\left(\frac{y}{x}\right)\left(d\left(\frac{y}{x}\right)\right) = \int \frac{-1}{x}dx$$

$$\Rightarrow \quad \ln\left|\sec(y/x)\right| = -\ln x + c$$

$$\Rightarrow \quad \ln\left|x\sec\left(\frac{y}{x}\right)\right| = c$$

Now, apply $y\left(\frac{1}{2}\right) = \frac{\pi}{6}$ in above

$$\therefore \quad \ln\left|\frac{1}{2}\sec\left(\frac{\pi}{3}\right)\right| = c$$

$$\therefore \quad \ln\left|\frac{1}{2}\times 2\right| = c \quad \Rightarrow c = \ln 1 = 0$$

$$\therefore \quad \sec\left(\frac{y}{x}\right) = \frac{1}{x}$$

$$\therefore \quad y = x\sec^{-1}\left(\frac{1}{x}\right)$$

So, required bounded area in upper half,

$$A = \int_0^{1/\sqrt{2}} x\sec^{-1}\left(\frac{1}{x}\right)dx = \int_0^{1/\sqrt{2}} x\cos^{-1}(x)dx$$

Using integration by parts

$$= \left|\left(\frac{x^2}{2}\cos^{-1}x\right)\right|_0^{\frac{1}{\sqrt{2}}} + \int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{2\sqrt{1-x^2}}$$

$$= \left(\frac{1}{4}\cdot\frac{\pi}{4} - 0\right) + \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} \frac{1-(1-x^2)}{\sqrt{1-x^2}}dx$$

$$= \frac{\pi}{16} + \frac{1}{2}\left[\left(\sin^{-1}x\right)_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} \sqrt{1-x^2}\,dx\right]$$

$$= \frac{\pi}{16} + \frac{1}{2}\left[\frac{\pi}{4} - \left\{\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right\}\Big|_0^{\frac{1}{\sqrt{2}}}\right]$$

$$= \frac{\pi}{16} + \frac{1}{2}\left[\frac{\pi}{4} - \left\{\frac{1}{4} + \frac{\pi}{8}\right\}\right]$$

$$\therefore \quad \text{Area} = \frac{\pi - 1}{8}$$

44. Option (3) is correct.

Given differential equation

$$e^x\sqrt{1-y^2}\,dx + \left(\frac{y}{x}\right)dy = 0,\ y(1) = -1$$

$$\Rightarrow e^x\sqrt{1-y^2}\,dx = \frac{-y}{x}dy$$

$$\Rightarrow \frac{ydy}{\sqrt{1-y^2}} = -\int xe^x dx \Rightarrow \int \frac{-ydy}{\sqrt{1-y^2}} = \int xe^x dx$$

$$\Rightarrow \sqrt{1-y^2} = e^x(x-1) + c$$

Given $x = 1, y = -1$

$\Rightarrow 0 = 0 + c \Rightarrow c = 0$

$$\Rightarrow \sqrt{1-y^2} = e^x(x-1)$$

At $x = 3 \Rightarrow 1 - y^2 = (e^3 2)^2 \Rightarrow y^2 = 1 - 4e^6$

45. Option (4) is correct.

Given, $A = \begin{bmatrix} y & \sin x & 1 \\ 0 & -1 & 1 \\ 2 & 0 & \frac{1}{x} \end{bmatrix}$

So, $|A| = y\left(\frac{-1}{x}\right) - \sin x\,(-2) + 1(2)$

$$\Rightarrow \quad |A| = \frac{-y}{x} + 2\sin x + 2$$

Given, $\frac{dy}{dx} - |A| = 0$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{-y}{x} + 2\sin x + 2\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{x}-2\ (\sin x+1)=0$$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = x$$

So, $yx = \int 2(\sin x+1)x\,dx$

So, $yx = 2\left[\int x\,dx+\int x\sin x\,dx\right]$

$$\Rightarrow \quad yx = 2\left[\frac{x^2}{2}+(-x\cos x+\sin x)\right]+c$$

$\Rightarrow \quad yx = x^2 - 2x\cos x + 2\sin x + c$

At $\quad x = \pi,\ y = \pi + 2$

$\Rightarrow \quad (\pi + 2)\pi = \pi^2 + 2\pi + c$

$\Rightarrow \quad \pi^2 + 2\pi = \pi^2 + 2\pi + c$

$\Rightarrow c = 0$

At $x=\frac{\pi}{2} \Rightarrow y.\frac{\pi}{2}=\left(\frac{\pi}{2}\right)^2-0+2+0$

$$\Rightarrow y\left(\frac{\pi}{2}\right)=\left(\frac{\pi}{2}\right)^2+2$$

$$\Rightarrow y\left(\frac{\pi}{2}\right)=\frac{\pi}{2}+\frac{4}{\pi}$$

46. Option (4) is correct.

Given

$y = y(x)$ be the solution of the differential equation

$$\frac{dy}{dx}=(y+1)\left((y+1)e^{\frac{x^2}{2}}-x\right), y(2)=0$$

$$\frac{dy}{dx}=(y+1)\left((y+1)e^{\frac{x^2}{2}}-x\right)$$

Divide both sides by $(y + 1)^2$

$$\frac{1}{(y+1)^2}\frac{dy}{dx}=e^{\frac{x^2}{2}}-\frac{x}{y+1}$$

$$\Rightarrow \quad \frac{-1}{(y+1)^2}\frac{dy}{dx}-\frac{x}{(y+1)}=-e^{\frac{x^2}{2}}$$

Put $\frac{1}{y+1}=z$

Differentiate both sides w.r.t x

$$\Rightarrow \frac{-1}{(y+1)^2}\cdot\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dz}{dx}+z\cdot(-x)=-e^{\frac{x^2}{2}} \qquad \text{(i)}$$

Compare with $\frac{dz}{dx}+\text{P}z=\text{Q}$

Where P and Q are functions of x.

Find integrating factor

$$\text{I.F.} = e^{\int \text{P}dx}$$

$$= e^{\int (-x)dx}$$

$$= e^{-\frac{x^2}{2}} \qquad \left(\int x\,dx=\frac{x^2}{2}\right)$$

Solution of (i) is

$$z\cdot(\text{I.F.})=\int \text{Q}\cdot(\text{I.F.})dx+\text{C}$$

$$\Rightarrow \quad z\cdot e^{-\frac{x^2}{2}}=-\int e^{\frac{x^2}{2}}\cdot e^{-\frac{x^2}{2}}dx+\text{C}$$

$$\Rightarrow \quad z\cdot e^{-\frac{x^2}{2}}=-\int dx+\text{C}$$

$$\Rightarrow \quad z\cdot e^{-\frac{x^2}{2}}=-x+\text{C} \qquad \left(\because \int dx=x\right)$$

Put $\quad z=\frac{1}{y+1} \Rightarrow \frac{e^{-\frac{x^2}{2}}}{y+1}=-x+\text{C}$ ...(ii)

Given, $\quad y = 0$ at $x = 2$

$e^2 = -2 + \text{C}$

$\Rightarrow \quad \text{C} = 2 + e^{-2}$ ...(iii)

From (ii) and (iii)

$$y+1=\frac{e^{-\frac{x^2}{2}}}{e^{-2}+2-x}$$

Again, at $x = 1$

$$\Rightarrow \quad y+1=\frac{e^{\frac{3}{2}}}{e^2+1}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=1}=\frac{e^{\frac{3}{2}}}{e^2+1}\left(\frac{e^{\frac{3}{2}}}{e^2+1}\cdot e^{\frac{1}{2}}-1\right)=-\frac{e^{\frac{3}{2}}}{(e^2+1)^2}$$

Hint :

(i) Rewrite the given differential equation.

(ii) Now, this is in the form of linear differential equation.

(iii) Find integrating factor.

(iv) Write the solution of differential equation.

(v) Find the constant by using given condition.

(vi) Find $\frac{dy}{dx}$ at $x=1$.

Shortcut Method:

$$\frac{dy}{dx}=(y+1)\left((y+1)e^{\frac{x^2}{2}}-x\right)$$

$$\Rightarrow \quad \frac{-1}{(y+1)^2}\frac{dy}{dx}-x\left(\frac{1}{y+1}\right)=-e^{\frac{x^2}{2}}$$

Put, $\frac{1}{y+1} = z$

$$\frac{dz}{dx} + z(-x) = -e^{\frac{x^2}{2}}$$

$$\text{I.F.} = e^{\int -x dx} = e^{-\frac{x^2}{2}}$$

$$z \cdot \left(e^{-\frac{x^2}{2}}\right) = -\int e^{-\frac{x^2}{2}} \cdot e^{\frac{x^2}{2}} dx$$

$$\Rightarrow \quad \frac{e^{-\frac{x^2}{2}}}{y+1} = -x + C \quad \text{...(i)}$$

$y(2) = 0$

$\therefore \quad C = e^{-2} + 2 \quad$ ...(ii)

From (i) and (ii)

$$y + 1 = \frac{e^{-\frac{x^2}{2}}}{e^{-2} + 2 - x}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=1} = \frac{e^{\frac{3}{2}}}{e^2+1}\left(\frac{e^{\frac{3}{2}}}{e^2+1} \times e^{\frac{1}{2}} - 1\right) = -\frac{e^{\frac{3}{2}}}{\left(e^2+1\right)^2}$$

47. Option (4) is correct.

Given

$$\frac{dy}{dx} = xy - 1 + x - y \,;\, y(0) = 0$$

$$\frac{dy}{dx} = xy - 1 + x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = xy - y + x - 1$$

$$\Rightarrow \quad \frac{dy}{dx} = y(x-1) + 1(x-1)$$

$$\Rightarrow \quad \frac{dy}{dx} = (y+1)(x-1)$$

Now, we can use variable separable method to solve this differential equation.

$$\therefore \quad \frac{dy}{y+1} = (x-1)\,dx$$

Integrate both sides.

$$\int \frac{dy}{y+1} = \int (x-1)dx$$

Put $\quad y + 1 = t \Rightarrow \quad dy = dt$

$$\int \frac{dt}{t} = \int (x-1)dx$$

$$\Rightarrow \quad \ln|t| = \frac{x^2}{2} - x + c$$

$\Big[\because \int \frac{dx}{x} = \ln|x| + c_1$

$\int x dx = \frac{x^2}{2} + c_2$

$\int dx = x + c_3$

Take all constant to one side to make single constant$\Big]$

$$\Rightarrow \quad \ln|y+1| = \frac{x^2}{2} - x + c \quad \text{...(i)}$$

$[\because t = y + 1]$

$y = 0$ at $x = 0$ (Given)

$$\therefore \quad \ln|0+1| = \frac{0}{2} - 0 + c$$

$$\Rightarrow \quad \ln 1 = c$$

$$\Rightarrow \quad c = 0 \quad (\because \ln 1 = 0)$$

So, (i) becomes

$$\ln|y+1| = \frac{x^2}{2} - x$$

$$\Rightarrow \quad |y+1| = e^{\frac{x^2}{2} - x}$$

$(\because \log_a b = m \Rightarrow b = a^m)$

Put $\quad x = 1$

$$|y+1| = e^{\frac{1}{2} - 1}$$

$$\Rightarrow \quad y + 1 = e^{\frac{1}{2} - 1}$$

(Taking case of positive)

$$\Rightarrow \quad y = e^{\frac{1}{2} - 1} - 1$$

$$\Rightarrow \quad y = e^{-\frac{1}{2}} - 1$$

So, $\quad y(1) = e^{-\frac{1}{2}} - 1$

Hint :

(i) Rearrange the expression so that differential equation occurs in form $\frac{dy}{dx} = f(x) f(y)$.

(ii) Solve using variable separable method.

(iii) Solve $\int \frac{dy}{f(y)} = \int f(x) dx$

(iv) Use the given initial condition to find constant.

(v) Finally put $x = 1$ to find value of y.

Shortcut Method:

$$\frac{dy}{dx} = xy - 1 + x - y$$

$$\Rightarrow \quad \int \frac{dy}{y+1} = \int (x-1)dx$$

$$\Rightarrow \ln|y+1| = \frac{x^2}{2} - x + c$$

$$y(0) = 0 \Rightarrow c = 0$$

$$y + 1 = e^{\frac{x^2}{2} - x}$$

$$y(1) = e^{-\frac{1}{2}} - 1$$

48. Option (1) is correct.

Given: differential equation is

$\cos x\,(3 \sin x + \cos x + 3) = (1 + y \sin x\,(3 \sin x + \cos x + 3))\,dx$

$\Rightarrow$ $\cos x\,(3 \sin x + \cos x + 3)\,dy = dx + y \sin x\,(3 \sin x + \cos x + 3))\,dx$

$\Rightarrow$ $\cos x\,(3 \sin x + \cos x + 3)\,dy - y \sin x\,(3 \sin x + \cos x + 3) = dx$

$$\Rightarrow \quad \cos x\, dy - y \sin x\, dx = \frac{dx}{3\sin x + \cos x + 3}$$

$$\Rightarrow \quad d(y\cos x) = \frac{dx}{3\sin x + \cos x + 3}$$

Integrating both the sides, we get

$$\int d(y\cos x) = \int \frac{dx}{3\sin x + \cos x + 3}$$

$$\Rightarrow \quad y\cos x = \int \frac{dx}{3\sin x + \cos x + 3}$$

As we know, $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$ and

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\Rightarrow \quad y\cos x = \int \frac{dx}{3\left[\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right] + \left[\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right] + 3}$$

$$\Rightarrow \quad y\cos x = \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{6\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2} + 3\left(1+\tan^2\frac{x}{2}\right)}$$

$$\Rightarrow \quad y\cos x = \int \frac{\sec^2\frac{x}{2}dx}{6\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2} + 3 + 3\tan^2\frac{x}{2}}$$

$$\{\because \sec^2\theta = 1 + \tan^2\theta\}$$

$$\Rightarrow \quad y\cos x = \int \frac{\sec^2\frac{x}{2}dx}{2\tan^2\frac{x}{2} + 6\tan\frac{x}{2} + 4} \qquad \ldots\text{(i)}$$

Put $\tan\dfrac{x}{2} = h$

$$\Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}dx = dh$$

So, equation (i) can be written as,

$$y\cos x = \int \frac{\frac{1}{2}dh}{2h^2 + 6h + 4}$$

$$\Rightarrow \quad y\cos x = \int \frac{dh}{h^2 + 3h + 2}$$

$$\Rightarrow \quad y\cos x = \int \frac{dh}{h^2 + 2h + h + 2}$$

$$\Rightarrow \quad y\cos x = \int \frac{dh}{(h+1)(h+2)}$$

$$\Rightarrow \quad y\cos x = \int \left\{\frac{1}{h+1} - \frac{1}{h+2}\right\} dh$$

As we know, $\int \dfrac{1}{t}dt = \ln|t| + c$

$$\Rightarrow \quad y\cos x = \ln|h+1| - \ln|h+2| + c$$

Also, we know that $\ln(m) - \ln(n) = \ln\left(\dfrac{m}{n}\right)$

$$\Rightarrow \quad y\cos x = \ln\left|\frac{h+1}{h+2}\right| + c$$

$$\Rightarrow \quad y\cos x = \ln\left|\frac{\tan\frac{x}{2}+1}{\tan\frac{x}{2}+2}\right| + c$$

$\because$ Given that $y(0) = 0$.

$$\Rightarrow \quad 0\cdot\cos(0) = \ln\left|\frac{\tan(0)+1}{\tan(0)+2}\right| + c$$

$$\Rightarrow \quad c = -\ln\left|\frac{1}{2}\right|$$

$$\Rightarrow \quad c = \ln(2) \qquad \left\{\because \ln\left(\frac{l}{y}\right) = -\ln(y)\right\}$$

$$\therefore \quad y\cos x = \ln\left|\frac{1+\tan\frac{x}{2}}{2+\tan\frac{x}{2}}\right| + \ln(2)$$

Put $x = \dfrac{\pi}{3}$ in the above equation, we get

$$y\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) = \ln\left|\frac{1+\tan\frac{\pi}{6}}{2+\tan\frac{\pi}{6}}\right| + \ln(2)$$

$$\Rightarrow \quad y\left(\frac{\pi}{3}\right)\left(\frac{1}{2}\right) = \ln\left|\frac{1+\frac{1}{\sqrt{3}}}{2+\frac{1}{\sqrt{3}}}\right| + \ln(2)$$

$$\left\{\because \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}\right\}$$

$$\Rightarrow \quad y\left(\frac{\pi}{3}\right) = 2\left\{\ln\left|\frac{\sqrt{3}+1}{2\sqrt{3}+1}\right| + \ln(2)\right\}$$

$$\Rightarrow \quad y\left(\frac{\pi}{3}\right) = 2\left\{\ln\left|\frac{2\left(\sqrt{3}+1\right)}{2\sqrt{3}+1}\right|\right\}$$

$$\left\{\because \ln(m)+\ln(n) = \ln(mn)\right\}$$

$$\Rightarrow \quad y\left(\frac{\pi}{3}\right) = 2\ln\left|\frac{2\sqrt{3}+2}{2\sqrt{3}+1}\right|$$

Rationalising the term in logarithm, we get

$$y\left(\frac{\pi}{3}\right) = 2\ln\left|\frac{\left(2\sqrt{3}+2\right)\left(-2\sqrt{3}+1\right)}{\left(2\sqrt{3}+1\right)\left(-2\sqrt{3}+1\right)}\right|$$

$$\Rightarrow \quad y\left(\frac{\pi}{3}\right) = 2\ln\left|\frac{-12-4\sqrt{3}+2\sqrt{3}+2}{(1)^2-\left(2\sqrt{3}\right)^2}\right|$$

$$\left\{\because a^2-b^2 = (a+b)(a-b)\right\}$$

$$\Rightarrow \quad y\left(\frac{\pi}{3}\right) = 2\ln\left|\frac{-10-2\sqrt{3}}{1-12}\right|$$

$$\Rightarrow \quad y\left(\frac{\pi}{3}\right) = 2\ln\left(\frac{2\sqrt{3}+10}{11}\right)$$

$$y\left(\frac{\pi}{3}\right) = 2\log_e\left(\frac{2\sqrt{3}+10}{11}\right)$$

Hint :

(i) Simplify the given differential equation.

(ii) Solve the obtained indefinite integral using method of substitution.

49. Option (4) is correct.

Given a differential equation

$$2\left(x^2+x^{5/4}\right)dy - y\left(x+x^{1/4}\right)dx = 2x^{3/4}dx$$

$$\Rightarrow \quad 2\left(x^2+x^{5/4}\right)dy = 2x^{9/4}dx + y\left(x+x^{1/4}\right)dx$$

$$\Rightarrow \quad 2\left(x^2+x^{5/4}\right)dy = \left(2x^{9/4}+y\left(x+x^{1/4}\right)\right)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x^{9/4}+\left(x+x^{1/4}\right)y}{2\left(x^2+x^{5/4}\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x^{9/4}}{2\left(x^2+x^{5/4}\right)} + \frac{\left(x+x^{1/4}\right)}{2\left(x^2+x^{5/4}\right)}y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^{9/4}}{\left(x^2+x^{5/4}\right)} + \frac{\left(x+x^{1/4}\right)}{2x\left(x+x^{1/4}\right)}y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^{9/4}}{x^{5/4}\left(1+x^{3/4}\right)} + \frac{y}{2x}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{2x} = \frac{x}{1+x^{3/4}} \qquad \ldots(\text{i})$$

As we know, linear differential equation is in the form of $\frac{dy}{dx}+\mathrm{P}y=\mathrm{Q}$, where P and Q are functions of x.

So, we can see that equation (i) is a linear differential equation with $\mathrm{P}=\frac{-1}{2x}$ and $\mathrm{Q}=\frac{x}{1+x^{3/4}}$

Now, as we know, the integrating factor,

$$\mathrm{IF} = e^{\int \mathrm{P}\,dx}$$

So, Integrating factor (I.F.) of equation (i) is,

$$\text{I.F.} = e^{\int \frac{-1}{2x}dx}$$

$$\Rightarrow \quad \text{I.F.} = e^{\frac{-1}{2}\int\frac{1}{x}dx} \Rightarrow \text{I.F.} = e^{\frac{-1}{2}\ln x}$$

$$\Rightarrow \quad \text{I.F.} = e^{\ln(x)^{-1/2}} \qquad \left\{\because \ln m^n = n\ln(m)\right\}$$

$$\Rightarrow \quad \text{I.F.} = \frac{1}{\sqrt{x}} \qquad \left\{\because m^{\log_m x} = x\right\}$$

Also, we know that the solution of the linear differential equation is given by

$$y\ (\text{I.F.}) = \int \mathrm{Q}(\text{I.F.})\,dx$$

So, the solution of equation (i) can be written as,

$$y\left(\frac{1}{\sqrt{x}}\right) = \int \frac{x}{1+x^{3/4}}\cdot\frac{1}{\sqrt{x}}\,dx$$

$$\Rightarrow \quad \frac{y}{\sqrt{x}} = \int\frac{x^{1/2}}{1+x^{3/4}}\,dx \qquad \ldots(\text{ii})$$

Put $1+x^{3/4}=h$

$$\Rightarrow \quad \frac{3}{4}x^{\frac{3}{4}-1}dx = dh \Rightarrow \frac{3}{4}x^{\frac{-1}{4}}dx = dh$$

So, equation (ii) can be written as,

$$\frac{y}{\sqrt{x}} = \int\frac{x^{\frac{1}{2}}dh}{\frac{3}{4}x^{\frac{-1}{4}}h} \Rightarrow \frac{y}{\sqrt{x}} = \int\frac{4}{3}\frac{x^{3/4}}{h}\,dh$$

$$\Rightarrow \quad \frac{y}{\sqrt{x}} = \int\frac{4}{3}\frac{h-1}{h}\,dh$$

$$\Rightarrow \quad \frac{y}{\sqrt{x}} = \frac{4}{3}\int\left(1-\frac{1}{h}\right)dh$$

$$\Rightarrow \quad \frac{y}{\sqrt{x}} = \frac{4}{3}(h-\ln h)+c$$

$$\Rightarrow \quad \frac{y}{\sqrt{x}} = \frac{4}{3}\left[\left(1+x^{3/4}\right)-\ln\left(1+x^{3/4}\right)\right]+c$$

This is the solution of differential equation

Now, given that it passes through the point

$$\left(1, 1-\frac{4}{3}\log_e 2\right)$$

$$\Rightarrow \quad \frac{1-\frac{4}{3}\log_e 2}{1} = \frac{4}{3}\left(\left(1+(1)^{3/4}\right)-\ln\left(1+(1)^{3/4}\right)\right)+c$$

$$\Rightarrow \quad 1-\frac{4}{3}\ln 2 = \frac{4}{3}(2-\ln 2)+c$$

$$\Rightarrow \quad 1-\frac{4}{3}\ln 2-\frac{8}{3}+\frac{4}{3}\ln 2 = c \Rightarrow c = \frac{-5}{3}$$

∴ The solution can be written as

$$\frac{y}{\sqrt{x}} = \frac{4}{3}\left(\left(1+x^{3/4}\right)-\ln\left(1+x^{3/4}\right)\right)-\frac{5}{3}$$

Put $x = 16$, we get

$$\frac{y(16)}{\sqrt{16}} = \frac{4}{3}\left(\left(1+6^{3/4}\right)-\ln\left(1+16^{3/4}\right)\right)-\frac{5}{3}$$

$$\Rightarrow \quad y(16) = \frac{16}{3}\left(1+8-\ln(1+8)\right)-\frac{5}{3}$$

$$\Rightarrow \quad y(16) = \frac{16}{3}(9-\ln 9)-\frac{5}{3}$$

$$\Rightarrow \quad y(16) = \frac{16}{3}(9-2\ln 3)-\frac{5}{3}$$

$$\Rightarrow \quad y(16) = 4\left[\frac{31}{3}-\frac{8}{3}\ln 3\right]$$

Hint :

(i) Convert the given equation into linear differential equation,

(ii) Find the integrating factor and then use method of substitution to solve the differential equation.

50. Option (4) is correct.

Given differential equation is

$$\frac{dy}{dx}+2y\tan x = \sin x \qquad \text{...(i)}$$

Solution of this differential equation will be

$$y\,(\text{I.F}) = \int \sin x(\text{I.F})dx \qquad \text{...(ii)}$$

$$\text{I.F} = e^{\int 2\tan x dx}$$

$$\text{I.F} = e^{\int 2\tan x dx}$$

$$\because \int \tan x \; dx = \ln(\sec x)+c$$

$$\text{I.F.} = e^{2ln\sec x}$$

$$\text{I.F} = e^{\ln\sec^2 x}$$

$$\text{I.F} = \sec^2 x \qquad \text{...(iii)}$$

Now using equation (ii) and (iii)

$$y(\sec^2 x) = \int(\sin x)(\sec^2 x)dx$$

$$\Rightarrow y(\sec^2 x) = \int\left(\frac{\sin x}{\cos x}\right)(\sec x)dx$$

$$\Rightarrow y(\sec^2 x) = \int(\tan x\sec x)dx$$

$$\Rightarrow y\left(\sec^2 x\right) = \sec x + \text{C} \qquad \text{...(iv)}$$

Given that $y\left(\frac{\pi}{3}\right)=0$ i.e., when $x = \frac{\pi}{3}, y = 0$

$$\Rightarrow (0)\left(\sec^2\frac{\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right)+\text{C} \Rightarrow \text{C} = -\sec\frac{\pi}{3}$$

$$\Rightarrow \text{C} = -2 \qquad \text{...(v)}$$

Using equation (iv) and (v)

$$y(\sec^2 x) = \sec x - 2$$

$$\Rightarrow y = \frac{\sec x - 2}{\sec^2 x} \quad \Rightarrow y = \frac{\sec x}{\sec^2 x}-\frac{2}{\sec^2 x}$$

$$\Rightarrow y = \frac{1}{\sec x}-\frac{2}{\sec^2 x} \quad \Rightarrow y = \cos x - 2\cos^2 x$$

$$\Rightarrow y = -2\{\cos^2 x - \frac{1}{2}\cos x\}$$

$$\Rightarrow y = -2\left\{\cos^2 x - \frac{1}{2}\cos x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right\}$$

$$\Rightarrow y = \frac{1}{8} - 2\left(\cos x - \frac{1}{2}\right)^2$$

For $y_{\max}$ put $\cos x = \frac{1}{2} \Rightarrow y_{\max} = \frac{1}{8}$

Hint:

(i) Solution of D.E will be

$$y(\text{I.F}) = \int(\sin x)(\text{I.F})dx$$

(ii) $\text{I.F} = e^{\int 2\tan x dx}$

(iii) $\int \tan x dx = \ln(\sec x)+\text{C}$

(iv) $-1 \leq \cos x \leq 1$

Shortcut Method:

$$\text{I. F.} = e^{2\int \tan x dx} = \sec^2 x$$

$$y.\sec^2 x = \int \sin x.\sec^2 x dx$$

$\Rightarrow y\sec^2 x = \sec x + \mathrm{C}$

As $y\left(\frac{\pi}{3}\right) = 0 \Rightarrow \mathrm{C} = -2$

$\Rightarrow y\left(\sec^2 x\right) = \sec x - 2$

$\Rightarrow y = \cos x - 2\cos^2 x$

$\Rightarrow y = t - 2t^2$; $t = \cos x$

$\Rightarrow \frac{dy}{dt} = 1 - 4t$, when $\frac{dy}{dt} = 0 \Rightarrow t = \frac{1}{4}$

$y_{max} = \frac{1}{8}$

51. Option (3) is correct.

Given, $y(x)$ is a solution of

$$\frac{dy}{dx} + (\tan x)y = \sin x,$$

$$\text{where } x \in \left[0, \frac{\pi}{3}\right] \text{ and } y(0) = 0$$

- The given equation is a liner differential equation as it is of the form

$$\frac{dy}{dx} + y\mathrm{P}(x) = \mathrm{Q}(x)$$

Integration factor (I.F.) = $e^{\int \mathrm{P}(x)dx}$

Here, $\mathrm{P}(x) = \tan x$

So, I.F. $= e^{\int \tan x\, dx}$

$\Rightarrow$ I.F. $= e^{\ln(\sec x)}$

$\Rightarrow$ I.F. $= \sec x$

So, solution of differential equation $\frac{dy}{dx} + y(\tan x) = \sin x$ is given by

$$\Rightarrow \quad y(\sec x) = \int (\sin x)(\sec x)\, dx$$

$$\Rightarrow \quad y\sec x = \int \frac{\sin x}{\cos x}\, dx$$

$$\Rightarrow \quad y\sec x = \int \tan x\, dx$$

$$\Rightarrow \quad y\sec x = \ln(\sec x) + \mathrm{C}$$

Also, $y(0) = 0$

$\Rightarrow$ $0 \sec(0) = \ln(\sec(0)) + \mathrm{C}$

$\Rightarrow$ $0 = \ln(1) + \mathrm{C}$

$\Rightarrow$ $\mathrm{C} = 0$

So, $y \sec x = \ln(\sec x)$

Now, $y\left(\frac{\pi}{4}\right)$ is given by

$$\sec\left(\frac{\pi}{4}\right) \cdot y = \ln\left(\sec\frac{\pi}{4}\right)$$

$$\Rightarrow \quad y = \frac{\ln\sqrt{2}}{\sqrt{2}}$$

$$\Rightarrow \quad y = \frac{1}{2\sqrt{2}}\ln 2 \text{ or } \frac{1}{2\sqrt{2}}\log_e 2$$

Hint :

(i) Solution of linear differential equation

$$\frac{dy}{dx} + y\mathrm{P}(x) = \mathrm{Q}(x)$$

is $ye^{\int \mathrm{P}(x)dx} = e^{\int \mathrm{P}(x)dx}.\mathrm{Q}(x)dx$

(ii) Use property of logarithm,

$$\log_{(b)}\left(a^m\right) = m\log_b a$$

Shortcut Method:

$$\sec x \frac{dy}{dx} + \sec x \tan x\, y = \sec x \sin x$$

$\Rightarrow \frac{d}{dx}(y\sec x) = \tan x$

$\Rightarrow y \sec x = \ln(\sec x) + \mathrm{C}$

$x = 0, y = 0 \Rightarrow 0 = 0 + \mathrm{C}$

$\Rightarrow \mathrm{C} = 0$

So, $y = \frac{\ln(\sec x)}{\sec x}$

At, $x = \frac{\pi}{4}$, $\quad y = \frac{\ln\left(\sec\frac{\pi}{4}\right)}{\sec\left(\frac{\pi}{4}\right)} \Rightarrow y = \frac{1}{2\sqrt{2}}\ln 2$

52. Option (1) is correct.

For curve $C_1 : 2xy\frac{dy}{dx} = y^2 - x^2, x > 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad \text{...(i)}$$

This a homogeneous differential equation.

$\Rightarrow$ Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\text{(i)} \Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left(\frac{v^2 - 1}{2v}\right)$$

$$\Rightarrow \quad \frac{2v}{v^2+1} = -\frac{dx}{x}$$

$$\Rightarrow \quad \frac{d\left(v^2+1\right)}{v^2+1} = -\frac{dx}{x}$$

Integrating both sides

$$\Rightarrow \quad \ln (v^2 + 1) = -\ln x + \ln c$$

$$\Rightarrow \quad v^2 + 1 = \frac{c}{x}$$

$$\Rightarrow \quad \frac{y^2}{x^2} + 1 = \frac{c}{x}$$

$$\Rightarrow \quad x^2 + y^2 = cx$$

$$\Rightarrow \quad x^2 + y^2 - cx = 0$$

It passes through (1, 1)

$$\Rightarrow \quad 1 + 1 - c = 0$$

$$\Rightarrow \quad c = 2$$

So, C_1 is $x^2 + y^2 - 2x = 0$

$$\Rightarrow \quad (x-1)^2 + (y-0)^2 = 1$$

For curve C_2, $\frac{2xy}{x^2-y^2} = \frac{dy}{dx}$ ...(ii)

$$\Rightarrow \frac{dx}{dy} = \frac{x^2-y^2}{2xy}$$

Put $x = py$

$$\Rightarrow \frac{dx}{dy} = p + y\,\frac{dp}{dy}$$

$$\text{(ii)} \Rightarrow p + y\,\frac{dp}{dy} = \frac{p^2y^2-y^2}{2py^2}$$

$$\Rightarrow p + y\,\frac{dp}{dy} = \frac{p^2-1}{2p}$$

$$\Rightarrow y\,\frac{dp}{dy} = \frac{p^2-1}{2p} - p$$

$$\Rightarrow y\,\frac{dp}{dy} = \frac{p^2-1-2p^2}{2p}$$

$$\Rightarrow y\,\frac{dp}{dy} = -\left(\frac{p^2+1}{2p}\right)$$

$$\Rightarrow \left(\frac{2p}{p^2+1}\right)dp = -\frac{dy}{y}$$

$$\Rightarrow \frac{d\left(p^2+1\right)}{p^2+1} = -\frac{dy}{y}$$

Integrating both sides, we get

$$\Rightarrow \quad \ln (p^2 + 1) = -\ln y + \ln k$$

$$\Rightarrow \quad p^2 + 1 = \frac{k}{y}$$

$$\Rightarrow \quad \frac{x^2}{y^2} + 1 = \frac{k}{y}$$

$$\Rightarrow \quad x^2 + y^2 = ky$$

$$\Rightarrow \quad x^2 + y^2 - ky = 0$$

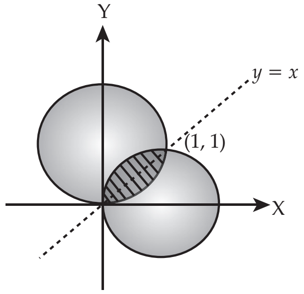
It passes through (1, 1)

$$\Rightarrow 1 + 1 - k = 0$$

$$\Rightarrow k = 2$$

So, C_2 is $x^2 + y^2 - 2y = 0$

$$\Rightarrow \quad (x-0)^2 + (y-1)^2 = 1$$



$$\text{Required area} = 2\int_0^1\left(\sqrt{2x-x^2} - x\right)dx$$

$$= 2\int_0^1 \sqrt{2x-x^2}\,dx - 2\int_0^1 x\,dx$$

$$= 2I_1 - 2\left[\frac{x^2}{2}\right]_0^1$$

$$= 2I_1 - \frac{2}{2}\,(1-0)$$

$$= 2I_1 - 1$$

Now, $$I_1 = \int_0^1 \sqrt{2x-x^2}\,dx$$

$$\Rightarrow \quad I_1 = \int_0^1 \sqrt{1-(x-1)^2}\,dx$$

Put $x - 1 = \sin\theta$

$$\Rightarrow \quad dx = \cos\theta\, d\theta$$

$$I_1 = \int_{-\frac{\pi}{2}}^{0} \cos\theta\cdot\cos\theta\, d\theta$$

$$\Rightarrow \quad I_1 = \int_{-\frac{\pi}{2}}^{0} \cos^2\theta\, d\theta$$

$$\Rightarrow \quad I_1 = \frac{1}{2}\int_{-\frac{\pi}{2}}^{0} (1+\cos 2\theta)\, d\theta$$

$$\Rightarrow \quad I_1 = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{0}$$

$$\Rightarrow \qquad I_1 = \frac{1}{2}\left[\left(0-\left(\frac{-\pi}{2}\right)\right)+\frac{1}{2}(0-0)\right]$$

$$\Rightarrow \qquad I_1 = \frac{\pi}{4}$$

So, $\qquad \text{area} = 2I_1 - 1$

$$= 2\left(\frac{\pi}{4}\right) - 1 = \frac{\pi}{2} - 1$$

Hint:

(i) For homogeneous differential equation put $y = vx$ or $x = py$ to obtain its solution.

(ii) Both curves are passing through (1, 1), use this to find arbitrary of integration.

(iii) Use the concept of symmetric curves *w.r.t.* line $y = x$ for simplification of area integral.

Shortcut Method:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}, \text{ Put } y = vx$$

$$\Rightarrow \qquad x\frac{dv}{dx} = -\left(\frac{v^2+1}{2v}\right)$$

$$\Rightarrow \qquad \ln(v^2+1) = -\ln x + \ln c$$

Curve passes through (1, 1) and $y = vx$

$\Rightarrow \qquad x^2 + y^2 - 2x = 0$, curve C_1

Similarly, curve C_2 is $x^2 + y^2 - 2y = 0$ as differential equation is given as $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - y2}$

$$\text{Area} = 2\int_0^1\left(\sqrt{2x - x^2} - x\right)dx = \frac{\pi}{2}$$

53. Option (1) is correct.

Given : Rate of growth of bacteria in a culture is proportional to the number of bacteria present and the bacteria count is 1000 at initial time $t = 0$.

Number of bacteria is increased by 20% in 2 hours.

Population of bacteria is 2000 after

$\dfrac{k}{\log_e\left(\dfrac{6}{5}\right)}$ hours.

$\therefore \quad \dfrac{dx}{dt} \propto x$ where x is number of bacteria at time t.

$\Rightarrow \quad \dfrac{dx}{dt} = \lambda x$ where λ is a constant.

$$\Rightarrow \qquad \frac{dx}{x} = \lambda dt$$

$x = 1000$ at $t = 0$

and $x = x$ at $t = t$

Integrating both sides, we get

$$\int_{1000}^{x}\frac{dx}{x} = \lambda\int_0^t dt$$

$$\Rightarrow \qquad \ln x \Big|_{1000}^{x} = \lambda.t\Big|_0^t$$

$$\Rightarrow \qquad \ln x - \ln 1000 = \lambda\, t$$

$$\Rightarrow \qquad \ln\left(\frac{x}{1000}\right) = \lambda t$$

$$\Rightarrow \qquad \frac{x}{1000} = e^{\lambda t} \qquad (\because \ln a = b \Rightarrow a = e^b)$$

$$\Rightarrow \qquad x = 1000e^{\lambda t} \qquad \text{...(i)}$$

Number of bacteria after 2 hours grows 20% more,

$$= 1000 + \frac{20}{100}\times 1000$$

$$= 1200$$

$$1200 = 1000e^{2\lambda}$$

$$\Rightarrow \qquad e^{2\lambda} = \frac{6}{5}$$

$$\Rightarrow \qquad 2\lambda = \ln\left(\frac{6}{5}\right)$$

(Taking log both sides)

$$\Rightarrow \qquad \lambda = \frac{1}{2}\ln\left(\frac{6}{5}\right) \qquad \text{...(ii)}$$

Given,

$$x = 2000 \text{ at } t = \frac{k}{\ln\left(\frac{6}{5}\right)} \qquad \text{...(iii)}$$

From (i), (ii), and (iii)

$$2000 = 1000e^{\frac{1}{2}\ln\left(\frac{6}{5}\right)\cdot\frac{k}{\ln\left(\frac{6}{5}\right)}}$$

$$\Rightarrow \qquad 2 = e^{k/2}$$

Taking log both sides

$$\ln 2 = \frac{k}{2} \Rightarrow \frac{k}{\ln 2} = 2 \Rightarrow \left(\frac{k}{\ln 2}\right)^2 = 4$$

Hint:

(i) $\displaystyle\int\frac{dx}{x} = \ln|x| + c$

(ii) $\displaystyle\int dt = t + c$

(iii) $a^{\log_a x} = x,$ whenever defined

(iv) $\ln x = b \Rightarrow x = e^b$

Shortcut Method:

$$\frac{dB}{dt} = \lambda B \Rightarrow \int_{1000}^{1200} \frac{dB}{B} = \lambda \int_{0}^{2} dt$$

$$\Rightarrow \qquad \lambda = \frac{1}{2} \ln\left(\frac{6}{5}\right)$$

$$\int_{1000}^{2000} \frac{dB}{B} = \frac{1}{2} \ln\left(\frac{6}{5}\right) \int_{0}^{T} dt$$

$$\Rightarrow \qquad T = \frac{2 \ln 2}{\ln\left(\frac{6}{5}\right)}$$

$$\Rightarrow \qquad k = 2 \ln 2$$

$$\Rightarrow \qquad \boxed{\left(\frac{k}{\ln 2}\right)^2 = 4}$$

54. Option (2) is correct.

Given:

Population (P) = P (t) at time 't' of a certain species follows the differential equation.

$$\frac{dP}{dt} = 0.5\,P - 450 = \frac{P}{2} - \frac{900}{2} \Rightarrow \frac{dP}{dt} = \frac{P - 900}{2}$$

It is in variable-separable form

$$\therefore \frac{dP}{P - 900} = \frac{1}{2} dt$$

Integrate both sides

P = 850 at t = 0

P = 0 at t = t

$$\int_{850}^{0} \frac{dP}{P - 900} = \frac{1}{2} \int_{0}^{t} dt$$

$$\Rightarrow \qquad \ln|P - 900|\Big|_{850}^{0} = \frac{1}{2}[t]_0^t \qquad \left(\because \int \frac{dx}{x} = ln|x| + c \quad \int dx = x + c\right)$$

$$\Rightarrow \qquad \ln 900 - \ln 50 = \frac{1}{2} t \Rightarrow \ln\left(\frac{900}{50}\right) = \frac{t}{2}$$

$$\left[\text{Using } \ln x - \ln y = \ln \frac{x}{y}\right]$$

$$\Rightarrow \qquad \ln 18 = \frac{t}{2}$$

$$\Rightarrow \qquad t = 2 \ln 18$$

Hint:

(i) $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ is variable separable form that can be solved as

$$\int g(y)\,dy = \int f(x)\,dx$$

(ii) $\int \frac{dx}{x} = ln|x| + c$

(iii) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ where $n \neq -1$.

Shortcut Method:

$$\frac{dP}{dt} = \frac{P - 900}{2}$$

$$\Rightarrow \qquad \int_{850}^{0} \frac{dP}{P - 900} = \frac{1}{2} \int_{0}^{t} dt$$

$$\Rightarrow \qquad \ln\left(\frac{900}{50}\right) = \frac{t}{2}$$

$$\Rightarrow \qquad t = 2 \ln 18$$

55. Option (2) is correct.

Given : $f(x)$ is a twice differentiable function defined on R such that $f(0) = 1, f'(0) = 2$ and $f'(x) \neq 0$. For all $x \in$ R

$$\begin{vmatrix} f(x) & f(x) \\ f(x) & f(x) \end{vmatrix} = 0 \quad \forall x \in R$$

$$\Rightarrow f(x) f''(x) - (f'(x))^2 = 0$$

$$\Rightarrow f(x) f''(x) = (f'(x))^2$$

$$\Rightarrow \frac{f''(x)}{f'(x)} = \frac{f'(x)}{f(x)}$$

Integrate both sides

$$\int \frac{f''(x)}{f'(x)} dx = \int \frac{f'(x)}{f(x)} dx$$

$$\Rightarrow \ln|f'(x)| = \ln|f(x)| + \ln c$$

$$\Rightarrow \ln|f'(x)| = \ln|cf(x)| \Rightarrow f'(x) = cf(x)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = c$$

Integrate both sides

$$\int \frac{f'(x)}{f(x)} dx = \int c\,dx$$

$$\Rightarrow \ln|f(x)| = cx + d$$

$$\Rightarrow |f(x)| = e^{(cx+d)} \qquad (\because \ln x = a \Rightarrow x = e^a)$$

$$f(0) = 1$$

$$\Rightarrow 1 = e^d$$

$$\Rightarrow d = \ln 1 \qquad (\because a^b = c \Rightarrow \log_a c = b)$$

$$\Rightarrow d = 0$$

$$|f(x)| = e^{cx}$$

$$\Rightarrow f(x) = \pm e^{cx}$$

$$f'(x) = \pm ce^{cx}$$

$\because f'(0) = 2$

$\Rightarrow 2 = \pm c$

$\Rightarrow c = \pm 2$

So, $f(x) = \pm e^{\pm 2x}$

$$f(x) = e^{2x} \qquad (\because f(0) > 0, f'(0) > 0)$$

$f(1) = e^2$ which lies in interval (6, 9) according to given options as $e^2 \approx 7.39$.

Hint:

(i) $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c_1$

(ii) $\int \frac{f''(x)}{f'(x)} dx = \ln|f'(x)| + c_2$

Shortcut Method:

$$f(x)f''(x) - (f'(x))^2 = 0$$

$$\Rightarrow \quad \frac{f''(x)}{f'(x)} = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \quad f'(x) = cf(x)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = c$$

$$\Rightarrow \quad \ln f(x) = cx + k_1$$

$$\Rightarrow \quad f(x) = ke^{cx}$$

$$f(0) = 1 = k$$

$$f'(0) = c = 2$$

$$f(x) = e^{ex}$$

$$f(1) = e^2 \approx 7.39 \in (6,9)$$

56. Option (4) is correct.

Given: Curve $y = f(x)$ passes through the point (1, 2) and satisfies

$$x\frac{dy}{dx} + y = bx^4$$

Divide by x

$$\frac{dy}{dx} + \frac{y}{x} = bx^3$$

which is of the form $\frac{dy}{dx} + \text{P}y = \text{Q}$

where $\text{P} = \frac{1}{x}$, $\text{Q} = bx^3$

Integrating Factor (I.F.)

$$\text{I.F.} = e^{\int p dx} = e^{\int \frac{dx}{x}} = e^{lnx} = x \qquad \left(\because a^{\log_a x} = x\right)$$

$$\frac{dy}{dx} + \frac{y}{x} = bx^3$$

Multiply both sides by x

$$x\frac{dy}{dx} + y = bx^4 \Rightarrow \frac{d}{dx}(xy) = bx^4$$

$$\Rightarrow \quad d(xy) = bx^4 dx$$

Integrate both sides

$$\int d(xy) = b\int x^4 dx$$

$$\Rightarrow \quad xy = \frac{bx^5}{5} + c$$

$$\left(\because \int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1\right)$$

$$\Rightarrow \quad y = \frac{bx^4}{5} + \frac{c}{x}$$

Curve passes through (1, 2)

$$\therefore \quad 2 = \frac{b}{5} + c \qquad \text{...(i)}$$

$$\int_1^2 f(x) dx = \frac{62}{5}$$

$$\Rightarrow \quad \int_1^2 \left(\frac{bx^4}{5} + \frac{c}{x}\right) dx = \frac{62}{5}$$

$$\Rightarrow \quad \left[\frac{bx^5}{25} + clnx\right]_1^2 = \frac{62}{5}$$

$$\Rightarrow \quad \frac{b}{25} \times 32 + cln2 - \frac{b}{25} = \frac{62}{5}$$

$$\Rightarrow \quad \frac{31b}{25} + cln2 = \frac{62}{5}$$

Compare both sides

$$c = 0, b = 10$$

Hint:

(i) $\frac{dy}{dx} + \text{P}y = \text{Q}$ where P and Q are functions of x is Linear Differential equation.

(ii) I.F. = $e^{\int \text{P}dx}$

(iii) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$

(iv) $\int \frac{dx}{x} = ln|x| + c$

Shortcut Method:

$$\frac{dy}{dx} + \frac{y}{x} = bx^3$$

I.F. = $e^{\int \frac{dx}{x}} = e^{ln x} = x$

So, solution of D.E. is

$$y.\, x = \int bx^4 dx + c$$

$$\Rightarrow \quad y = \frac{c}{x} + \frac{bx^4}{5}$$

passes through (1, 2)

$$2 = c + \frac{b}{5} \quad \text{...(i)}$$

$$\int_1^2 f(x)dx = \frac{62}{5}$$

$$\Rightarrow \quad \left[c \ln x + \frac{bx^5}{25}\right]_1^2 = \frac{62}{5}$$

$$\Rightarrow \quad c \ln 2 + \frac{31b}{25} = \frac{62}{5}$$

$$\Rightarrow \quad c = 0 \text{ and } b = 10.$$

Integer Type Questions (Chapter Based)

1. If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{4x}{(x^2-1)}y = \frac{x+2}{(x^2-1)^{\frac{1}{2}}}, x > 1$

such that $y(2) = \frac{2}{9}\log_e(2+\sqrt{3})$ and

$y(\sqrt{2}) = \alpha \log_e\ (\sqrt{\alpha} + \beta) + \beta - \sqrt{\gamma}$, $\alpha, \beta, \gamma, \in$ N, then $\alpha\beta\gamma$ is equal to _________.

[JEE (Main) – 13th April 2023 - Shift-2]

Sol. The correct answer is (6).

Given that

$$\frac{dy}{dx} + \frac{4x}{x^2-1}y = \frac{x+2}{(x^2-1)^{\frac{5}{2}}}$$

which is LDE, where $P = \frac{4x}{x^2-1}$ and $Q = \frac{x+2}{(x^2-1)^{5/2}}$

So I.F. = $e^{\int P dx} = e^{\int \frac{4x}{x^2-1}dx} = e^{2\log(x^2-1)} = (x^2-1)^2$

So, solution of DE is given by

$$y \times (x^2-1)^2 = \int \frac{x+2}{(x^2-1)^{\frac{5}{2}}} \times (x^2-1)^2 dx + C$$

$$= \int \frac{x+2}{(x^2-1)^{\frac{1}{2}}} dx + C$$

$$= \int \frac{x}{\sqrt{x^2-1}} + 2\int \frac{1}{\sqrt{x^2-1}} dx + C$$

$$y(x^2-1)^2 = \sqrt{x^2-1} + 2\log\left|x + \sqrt{x^2-1}\right| + C$$

Putting $y(2) = \frac{2}{9}\log(2+\sqrt{3})$

$$\frac{2}{9}\log(2+\sqrt{3}) \times 9 = \sqrt{3} + 2\log|2+\sqrt{3}| + C$$

$$\Rightarrow C = -\sqrt{3}$$

putting $x = \sqrt{2}$

$$y(1)^2 = \sqrt{1} + 2\log|\sqrt{2}+1| - \sqrt{3}$$

$$\Rightarrow y = 1 - \sqrt{3} + 2\log|1+\sqrt{2}| = \alpha\log(\sqrt{\alpha}+\beta) + \beta - \sqrt{\gamma}$$

On comparing ⇒ $\alpha = 1, \beta = 1, \gamma = 3$

and $\alpha\,.\,\beta\,.\,\gamma = 2 \times 1 \times 3 = 6$

2. Let $y = y(x)$ be a solution of the differential $(x \cos x)dy + (xy \sin x + y \cos x - 1)\, dx = 0, 0 < x < \frac{\pi}{2}$.

If $\frac{\pi}{3}y\left(\frac{\pi}{3}\right) = \sqrt{3}$, then $\left|\frac{\pi}{6}y''\left(\frac{\pi}{6}\right) + 2y'\left(\frac{\pi}{6}\right)\right|$ is equal to ____.

[JEE (Main) – 6th April 2023 - Shift-1]

Sol. The correct answer is (2).

$(x \cos x)\, dy + (xy \sin x + y \cos x - 1)\, dx$

$= 0, 0 < x < \frac{\pi}{2}$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{x\sin x + \cos x}{x\cos x}\right)y = \frac{1}{x\cos x}$$

I.F. $= e^{\int \frac{x\sin x + \cos x}{x\cos x}dx}$ I.F. = $x \sec x$

General solution y (IF) = $\int Q(\text{I.F.})dx$

$$y.x\sec x = \int \frac{x\sec x}{x\cos x}dx = \tan x + C$$

Since $y\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{\pi}$ Hence $C = \sqrt{3}$

Hence $\left|\frac{\pi}{6}y''\left(\frac{\pi}{6}\right) + y'\left(\frac{\pi}{6}\right)\right| = |-2| = 2$

3. Let $y = y(x)$ be the solution curve of the differential equation $\sin(2x^2)\ log_e\ (\tan\ x^2)\ dy + \left(4xy - 4\sqrt{2x}\sin\left(x^2 - \frac{\pi}{4}\right)\right) dx = 0$, $0 < x < \sqrt{\frac{\pi}{2}}$, which passes through the point $\left(\sqrt{\frac{\pi}{6}}, 1\right)$. Then, $\left|y\left(\sqrt{\frac{\pi}{3}}\right)\right|$ is equal to_____.

[JEE (Main) – 27th July 2022 - Shift-1]

Sol. The correct answer is (1).

Given differential equation is

$$\sin\left(2x^2\right)\log_e\left(\tan x^2\right)dy$$

$$+(4xy-4\sqrt{2}x\sin\left(x^2-\frac{\pi}{4}\right)dx=0$$

$$\Rightarrow \log_e(\tan x^2)\,dy+\frac{4xy}{\sin\left(2x^2\right)}dx$$

$$-\frac{4\sqrt{2}x\sin\left(x^2-\frac{\pi}{4}\right)}{\sin(2x^2)}dx=0$$

$$\Rightarrow \log_e(\tan x^2)\,dy+\frac{2xy}{\tan(x^2)}\sec^2(x^2)\,dx$$

$$=\frac{4\sqrt{2}x\left(\sin(x^2)\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\cos(x^2)\right)}{2\sin x^2\cos(x^2)}dx$$

$$\Rightarrow \int d\left(y.\log_e(\tan x^2\right)dy$$

$$=\int 2x\left[\sec(x^2)-\operatorname{cosec}x^2\right]dx$$

$$\Rightarrow y.\log_e(\tan x^2)=\log_e(\sec x^2+\tan x^2)$$

$$-\log_e(\operatorname{cosec}x^2-\cot x^2)+C$$

$$\Rightarrow y.\log_e(\tan x^2)=\log\left|\frac{\sec x^2+\tan x^2}{\operatorname{cosec}x^2-\cot x^2}\right|+C$$

Put $x=\sqrt{\frac{\pi}{6}}$ and $y=1$

$$\Rightarrow \log_e\tan\frac{\pi}{6}=\log_e\left|\frac{\sec\frac{\pi}{6}+\tan\frac{\pi}{6}}{\operatorname{cosec}\frac{\pi}{6}-\cot\frac{\pi}{6}}\right|+C$$

$$\Rightarrow \log_e\left(\frac{1}{\sqrt{3}}\right)=\log_e\left|\frac{\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}}{2-\sqrt{3}}\right|+C$$

$$=C=\log_e\left(\frac{2-\sqrt{3}}{3}\right)$$

Put $x=\sqrt{\frac{\pi}{3}}$

$$\Rightarrow y.\log_e\left(\sqrt{3}\right)=\log_e\left|\frac{2+\sqrt{3}}{\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}}\right|+\log_e\left(\frac{2-\sqrt{3}}{3}\right)$$

$$=\log_e\left(\sqrt{3}\left(2+\sqrt{3}\right)\times\frac{\left(2-\sqrt{3}\right)}{3}\right)$$

$$\Rightarrow y.\log_e\sqrt{3}=-\log_e\sqrt{3}$$

So, $y=-1\Rightarrow|y|=1$

4. Suppose $y=y(x)$ be the solution curve to the differential equation $\frac{dy}{dx}-y=2-e^{-x}$ such that $\lim\limits_{x\to\infty}y(x)$ is finite. If a and b are respectively the x – and y – intercepts of the tangent to the curve at $x=0$, then the value of $a-4b$ is equal to _________.

[JEE (Main) – 26th July 2022 - Shift-2]

Sol. The correct answer is (3).

Given differential equation is

$\frac{dy}{dx}-y=2-e^{-x}$ which is linear differential equation.

So, I.F $=e^{-\int dx}=e^{-x}$

$\therefore$ Solution is

$$\Rightarrow y.e^{-x}=\int(2e^{-x}-e^{-2x})dx$$

$$\Rightarrow y.e^{-x}=-2e^{-x}+\frac{e^{-2x}}{2}+C$$

$$y=-2+\frac{e^{-x}}{2}+Ce^{x}$$

$\because \lim\limits_{x\to\infty}y(x)$ is infinite

$\therefore \lim\limits_{x\to\infty}(-2+\frac{e^{-x}}{2}+Ce^{x})$ is infinite

So, it is possible only $C=0$

$$\therefore y=-2+\frac{e^{-x}}{2}$$

When $x=0\Rightarrow y=-\frac{3}{2}$

$$\left(\frac{dy}{dx}\right)=-\frac{e^{-x}}{2}=-\frac{1}{2}$$

$x=0$

$\therefore$ Equation of tangent to the curve at $x=0$ is

$$\left(y+\frac{3}{2}\right)=-\frac{1}{2}(x-0)$$

$\Rightarrow x+2y+3=0$

x and y – axis intercept is -3 and $\frac{-3}{2}$

$\therefore a=-3$ and $b=\frac{-3}{2}$

Now, $a-4b=-3+6=3$

5. Let $y=y(x)$ be the solution of the differential equation $\frac{dy}{dx}=\frac{4y^3+2yx^2}{3xy^2+x^3}$, $y(1)=1$. If for some $n\in\mathbb{N}$, $y(2)\in[n-1,n)$, then n is equal to ;

[JEE (Main) – 25th July 2022 - Shift-2]

Sol. The correct answer is (3).

$$\frac{dy}{dx}=\frac{4y^3+2yx^2}{3xy^2+x^3}$$

Which is a Homogeneous differential equation of first order.

let $y = xv \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$

$$\left(x\frac{dv}{dx}+v\right) = \frac{4x^3v^3+2x^3v}{3x^3v^2+x^3} = \frac{4v^3+2v}{3v^2+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{4v^3+2v}{3v^2+1} - v$$

$$= \frac{4v^3+2v-3v^3-v}{3v^2+1} = \frac{v^3+v}{3v^2+1}$$

$$\Rightarrow \int \frac{3v^2+1}{v\left(v^2+1\right)}\,dv = \int \frac{1}{x}dx$$

Let $v^3 + v = t$

$\Rightarrow (3v^2 + 1)dv = dt$

$\therefore \int \frac{1}{t}dt = \int \frac{1}{x}dx$

$\Rightarrow \log_e t = \log_e x + \log_e c \Rightarrow t = cx$

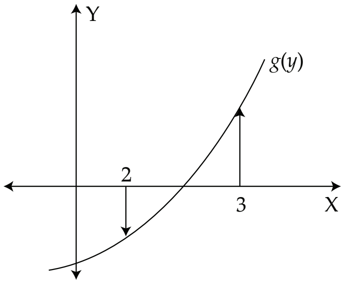
$\therefore \quad (v^3 + v) = cx$

$\Rightarrow \frac{y^3}{x^3} + \frac{y}{x} = cx$

$\because y(1) = 1$

$\Rightarrow C = 2$

$\therefore y^3 + yx^2 = 2x^4;\ y(2) \in [n-1, n)$



$(2, y(2))$ satisfies $y^3 + 4y = 32$

Let $g(y) = y^3 + 4y - 32$

$\because \quad g'(y) = 3y^2 + 4 > 0$

$\therefore n = 3$

6. Let $y = y(x)$ be the solution of the differential equation

$$(1-x^2)dy = (xy+(x^3+2)(\sqrt{1-x^2}))\,dx, -1 < x < 1$$

and $y(0) = 0$. If

$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1-x^2}\,y(x)dx = k$, then k^{-1} is equal to ….

[JEE (Main) – 27th June 2022 - Shift-2]

Sol. The correct answer is (320).

Given differential equation is :

$$(1-x^2)dy = \left(xy+(x^3+2)\sqrt{1-x^2}\right)dx$$

$$\Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{(x^3+2)\sqrt{1-x^2}}{1-x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{x^3+2}{\sqrt{1-x^2}}$$

Now, I.F.= $e^{\frac{1}{2}\int\frac{-2x}{1-x^2}dx} = e^{\frac{1}{2}\log e(1-x^2)}$

$= \sqrt{1-x^2}$

$\therefore$ Solution is $y\left(\sqrt{1-x^2}\right) = \int f(x^3+2)dx$

$$\Rightarrow \quad y\left(\sqrt{1-x^2}\right) = \frac{x^4}{4} + 2x + C$$

$\because y(0) = 0$

$\Rightarrow 0 = 0 + 0 + C \Rightarrow C = 0$

$$\therefore y\sqrt{1-x^2} = \frac{x^4}{4} + 2x$$

$$\because \quad \int_{\frac{-1}{2}}^{\frac{1}{2}} \sqrt{1-x^2}\,y(x)dx = k$$

$$\Rightarrow \int_{\frac{-1}{2}}^{\frac{1}{2}} \sqrt{1-x^4} \times \frac{x^4+8x}{4} \times \frac{1}{\sqrt{1-x^2}}dx = k$$

$$\Rightarrow k = \frac{1}{4}\int_{\frac{-1}{2}}^{\frac{1}{2}} (x^4+8x)dx \quad \Rightarrow k = \frac{1}{2}\int_{0}^{\frac{1}{2}} x^4\,dx$$

$$\left[\because \int_{\frac{-1}{2}}^{\frac{1}{2}} x\,dx = 0,\ x \text{ is odd function}\right]$$

$$\Rightarrow k = \frac{1}{2} \times \left[\frac{x^5}{5}\right]_0^{1/2} \Rightarrow k = \frac{1}{320} \Rightarrow k^{-1} = 320$$

7. Let the solution curve $y = y(x)$ of the differential equation $(4 + x^2)dy - 2x(x^2 + 3y + 4)dx = 0$ pass through the origin. Then $y(2)$ is equal to ______.

[JEE (Main) – 26th June 2022 - Shift-1]

Sol. The correct answer is (12).

Given differential equation

$(4 + x^2)dy - 2x(x^2 + 3y + 4)dx = 0$ ...(i)

$\Rightarrow \quad (4 + x^2)dy = 2x(x^2 + 3y + 4)dx$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x(x^2+4+3y)}{(x^2+4)}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x + \frac{6x}{x^2+4}y$$

$$\Rightarrow \frac{dy}{dx} - \frac{6x}{x^2+4}y = 2x$$

Which is Linear Differential Equation of first order.

Now, $\text{I.F.} = e^{\int \frac{-6x}{x^2+4}dx}$

Let $x^2 + 4 = t$

$2xdx = dt$

$\therefore \quad \text{I.F.} = e^{-3\int \frac{1}{t}dt} = e^{-3}\log e^{t}$

$= e^{-3\log_e(x^2+4)}$

$$= \frac{1}{(x^2+4)^3}$$

Solution of given differential equation is

$$y(\text{I.F}) = \int (\text{I.F.})Q\,dx + \text{C}$$

$$y\left(\frac{1}{(x^2+4)^3}\right) = \int \frac{1}{(x^2+4)^3}.2x\,dx + C$$

Let $x^2 + 4 = u$

$2x\,dx = du$

$$= \int \frac{1}{u^3}du + \text{C} = \frac{1}{-2u^2} + C = \frac{-1}{2(x^2+4)^2} + C$$

$$y\left(\frac{1}{(x^2+4)^3}\right) = \frac{-1}{2(x^2+4)^2} + \text{C}$$

Which passes through the origin $x = 0, y = 0$

$$0 = \frac{-1}{2\times 16} + \text{C} \quad \therefore \text{C} = \frac{1}{32}$$

So, $y\left(\frac{1}{(4+x^2)^3}\right) = \frac{-1}{2(x^2+4)^2} + \frac{1}{32}$

$\therefore y = -\frac{1}{2}(x^2+4) + \frac{1}{32}(x^2+4)^3$

Put $x = 2$

$\therefore y(2) = -\frac{1}{2}[4+4] + \frac{1}{32}[4+4]^3$

$= -\frac{8}{2} + \frac{1}{32}\times 8\times 8\times 8$

$y(2) = -4 + 16 = 12$

$y(2) = 12$

8. Let $S = (0, 2\pi) - \left\{\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\right\}$. Let $y = y(x)$, $x \in S$, be the solution curve of the differential equation $\frac{dy}{dx} = \frac{1}{1+\sin 2x}$, $y\left(\frac{\pi}{4}\right) = \frac{1}{2}$. If the sum of abscissas of all the points of intersection of the curve $y = y(x)$ with the curve $y = \sqrt{2}\sin x$ is $\frac{k\pi}{12}$, then k is equal to ______.

[JEE (Main) – 26th June 2022 - Shift-1]

Sol. The correct answer is (42).

Given,

$$\frac{dy}{dx} = \frac{1}{1+\sin 2x} \qquad \text{...(i)}$$

which is first order differential equation.

By separation of variables

$$dy = \frac{1}{1+\sin 2x}dx$$

$$\int dy = \int \frac{1}{(\cos^2 x + \sin^2 x + 2\sin x\cos x)}dx$$

$$y = \int \frac{1}{(\sin x + \cos x)^2}dx$$

$$= \int \frac{\sec^2 x}{(1+\tan x)^2}dx$$

Let $1 + \tan x = t$

$\sec^2 x dx = dt$

$$\therefore \quad y = \int \frac{dt}{t^2} = \frac{-1}{t} + \text{C} = \frac{-1}{1+\tan x} + \text{C}$$

$$\Rightarrow \quad y = \frac{-1}{1+\tan x} + \text{C}$$

$$\because \quad y\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{-1}{1+\tan\frac{\pi}{4}} + \text{C} \Rightarrow \frac{1}{2} = \frac{-1}{2} + \text{C}$$

$$\Rightarrow \quad \text{C} = 1$$

$$\therefore \quad y = \frac{-1}{1+\tan x} + 1$$

$$\Rightarrow \quad y = \frac{-1+1+\tan x}{1+\tan x} = \frac{\tan x}{1+\tan x}$$

$$y = \frac{\tan x}{1+\tan x} \qquad \text{...(ii)}$$

Given, $\quad y = \sqrt{2}\sin x \qquad$ ...(iii)

Equating (ii) and (iii)

$$\sqrt{2}\sin x = \frac{\tan x}{1+\tan x}$$

$$\Rightarrow \sqrt{2}\sin x = \frac{\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}} = \frac{\sin x}{\cos x + \sin x}$$

$$\Rightarrow \sqrt{2}\sin x = \frac{\sin x}{\cos x + \sin x}$$

$$\Rightarrow \sqrt{2}\sin x - \frac{\sin x}{\cos x + \sin x} = 0$$

$$\Rightarrow \sqrt{2} = \frac{1}{\cos x + \sin x}$$

⇒ If sin$x$ = 0 ⇒ $x = \pi$

$$\text{If } \cos x + \sin x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \frac{1}{2}$$

$$\Rightarrow \sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x = \sin\frac{\pi}{6}$$

$$\Rightarrow \sin\left(\frac{\pi}{4} + x\right) = \sin\frac{\pi}{6}$$

$$\Rightarrow \sin\left(\frac{\pi}{4} + x\right) = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\left(2\pi + \frac{\pi}{6}\right)$$

(1) (2) (3)

From (1) and (2)

$$\frac{\pi}{4} + x = \pi - \frac{\pi}{6}$$

$$\therefore x = \frac{5\pi}{6} - \frac{\pi}{4}$$

From (2) and (3)

$$\frac{\pi}{4} + x = 2\pi + \frac{\pi}{6}$$

$$\therefore x = \frac{13\pi}{6} - \frac{\pi}{4}$$

Sum of solution

$$= \pi + \frac{7\pi}{12} + \frac{23\pi}{12} = \frac{12\pi + 7\pi + 23\pi}{12} = \frac{42\pi}{12} = \frac{k\pi}{12}$$

$$\therefore k = 42$$

9 If $x\phi(x) = \int_5^x \left(3t^2 - 2\phi'(t)\right) dt$, $x > -2$, and $\phi(0) = 4$, then $\phi(2)$ is ______.

[JEE (Main) – 31st Aug 2021 - Shift-1]

Sol. The correct answer is (4)

$$\text{Given, } x\phi(x) = \int_5^x (3t^2 - 2\phi'(t))dt$$

Also, $\phi(0) = 4$, $x > -2$

Now, $x\phi'(x) + \phi(x) = 3x^2 - 2\phi'(x)$

$\Rightarrow (x + 2)\,\phi'(x) = \phi(x) = 3x^2$

$$\Rightarrow \phi'(x) + \frac{\phi(x)}{x+2} = \frac{3x^2}{x+2}$$

$$\text{So, IF} = e^{\int dx/x+2} = x + 2$$

$$\Rightarrow \phi(x)\cdot(x + 2) = \int (x+2)\left(\frac{3x^2}{x+2}\right)dx + C$$

$\Rightarrow \phi(x)(x + 2) = x^3 + C$

Put $x = 0$

$\Rightarrow \phi(0) \times 2 = 0 + C$

$\Rightarrow C = 8$

$$\text{Now, } \phi(2) = \frac{8+8}{4} = 4$$

10. If $y = y(x)$, $y \in \left[0, \frac{\pi}{2}\right)$ is the solution of the differential equation $\sec y \frac{dy}{dx} - \sin(x + y) - \sin(x - y) = 0$, with $y(0) = 0$, then $5y'$ is equal to_____. **[JEE (Main) – 27th July 2021 - Shift-1]**

Sol. The correct answer is (2).

$$\sec y \frac{dy}{dx} = 2\sin x \cos y \quad \ldots\text{(i)}$$

$$\int \sec^2 y\, dy = \int 2\sin x\, dx$$

$\Rightarrow \tan y = -2\cos x + c$

When $x = 0$, $y = 0 \Rightarrow c = 2$

So, $m \tan y = -2\cos x + 2$ …(ii)

$$\text{Also, } \sec^2 y \frac{dy}{dx} = 2\sin x$$

$$\Rightarrow (1 + \tan^2 y)\frac{dy}{dx} = 2\sin x \quad \ldots\text{(iii)}$$

By equation (ii) and (iii)

$$\frac{dy}{dx} = \frac{2\sin x}{1 + (2 - 2\cos x)^2}$$

$$\text{So, } f'\left(\frac{\pi}{2}\right) = \frac{2}{1+2^2} = \frac{2}{5} \Rightarrow 5f'\left(\frac{\pi}{2}\right) = 2$$

Q. 11. Let $y = y(x)$ be the solution of the differential equation $dy = e^{\alpha x + y}dx$; $\alpha \in \mathbb{N}$. If $y(\log_e 2) = \log_e 2$ and $y(0) = \log_e\left(\frac{1}{2}\right)$, then the value of α is equal to ________.

[JEE (Main) – 27th July 2021 - Shift-2]

Sol. The correct answer is (2)

$$dy = e^{\alpha x + y}dx$$

$$\Rightarrow e^{-y}\,dy = e^{\alpha x}\,dx$$

$$\Rightarrow -e^{-y} = \frac{e^{\alpha x}}{\alpha} + C$$

Now, $y(\ln 2) = \ln 2$ and $y(0) = -\ln 2$

$$\Rightarrow -e^{-\ln 2} = \frac{e^{\alpha \ln 2}}{\alpha} + C$$

$$= \frac{2^\alpha}{\alpha} + C = \frac{-1}{2} \quad \ldots\text{(i)}$$

$$\text{And } -e^{\ln 2} = \frac{1}{\alpha} + C \Rightarrow \frac{1}{\alpha} + C = -2 \quad \ldots\text{(ii)}$$

Solving (i) and (ii)

$$\frac{2^{\alpha}-1}{\alpha} = \frac{3}{2} \Rightarrow \alpha = 2 \text{ (as } \alpha \in N)$$

12. Let $y = y(x)$ be solution of the following differential equation $e^{y}\frac{dy}{dx} - 2e^{y}\sin x + \sin x \cos^2 x = 0, y\left(\frac{\pi}{2}\right) = 0$. If $y(0) = \log_e(\alpha + \beta e^{-2})$, then $4(\alpha + \beta)$ is equal to____.

[JEE (Main) – 25th July 2021 - Shift-1]

Sol. The correct answer is (4)

$$e^{y}\frac{dy}{dx} - 2e^{y}\sin x = -\sin x\cos^2 x$$

Put $e^{y} = t$

$$e^{y}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dt}{dx} - 2t\sin x = -\sin x\cos^2 x$$

$$\text{I.F} = e^{-\int 2\sin x\,dx} = e^{2\cos x}$$

$$\therefore e^{y}.e^{2\cos x} = \int e^{2\cos x}\left(-\sin x.\cos^2 x\right)dx$$

Put $\cos x = z$

$$\Rightarrow \frac{e^{2z}}{2}\cdot z^2 - \int e^{2z}\cdot z\,dz = \frac{e^{2z}}{2}\cdot z^2 - \left[\frac{e^{2z}}{2}\cdot z - \frac{e^{2z}}{4}\right] + c$$

$$\Rightarrow e^{y}e^{2\cos x} = \frac{e^{2z}}{4}(2z^2 - 2z + 1) + c$$

$$\Rightarrow e^{y}e^{2\cos x} = \frac{e^{2\cos x}}{4}(2\cos^2 x - 2\cos x + 1) + c$$

At $x = \frac{\pi}{2}, y = 0 \Rightarrow 1 = \frac{1}{4} + c \Rightarrow c = \frac{3}{4}$

$$\Rightarrow e^{y} = \frac{1}{4}(2\cos^2 x - 2\cos x + 1) + \frac{3}{4}e^{-2\cos x}$$

$$\Rightarrow y(0) = \ln\left(\frac{1}{4} + \frac{3}{4}e^{-2}\right)$$

$$\Rightarrow y(0) = \ln(\alpha + \beta e^{-2})$$

$$\Rightarrow \alpha = \frac{1}{4}, \beta = \frac{3}{4} \Rightarrow 4(\alpha + \beta) = 4$$

13. Let $y = y(x)$ be the solution of the differential equation

$$\left((x+2)e^{\left(\frac{y+1}{x+2}\right)} + (y+1)\right)dx = (x+2)\,dy,$$

$y(1) = 1$.

If the domain of $y = y(x)$ is an open interval (α, β), then $|\alpha + \beta|$ is equal to ________.

[JEE (Main) – 22nd July 2021 - Shift-2]

Sol. The correct answer is (4).

$$\left((x+2)e^{\frac{y+1}{x+2}} + (y+1)\right)dx = (x+2)\,dy$$

Let $x + 2 = X, \Rightarrow dx = dX,$

And $y + 1 = Y$

$\Rightarrow dy = dY$

So, $\left(Xe^{\frac{Y}{X}} + Y\right)dX = X\,dY$

$$\Rightarrow \frac{dY}{dX} = e^{\frac{Y}{X}} + \frac{Y}{X}$$

Put $Y = tX$

$$\Rightarrow \frac{dY}{dX} = t + X\frac{dt}{dX}$$

So, $t + X\frac{dt}{dx} = e^{t} + t$

$$\Rightarrow \int e^{-t}dt = \int\frac{dX}{X}$$

$\Rightarrow e^{-t} = \ln|X| + \ln|c|$

$\Rightarrow \ln|cX| = -e^{-t}$

$\Rightarrow -\ln|cX|) = -t$

$\therefore y + 1 = -(x+2)\ln(-\ln|c(x+2)|)$

$\ln|c(x+2)| < 0$

$\Rightarrow |c(x+2)| < 1 \Rightarrow -1 < c(x+2) < 1$

Case 1: $c > 0$

$$\therefore \frac{-1}{c} < x + 2 < \frac{1}{c}$$

$$\Rightarrow \frac{-1}{c} - 2 < x < \frac{1}{c} - 2$$

Domain: $\left(\frac{1}{c} - 2, \frac{-1}{c} - 2\right) = (\alpha, \beta)$

$\Rightarrow |\alpha + \beta| = 4$

Case 2: $c < 0$

$$\frac{-1}{c} - 2 < x < \frac{1}{c} - 2$$

Domain: $\left(\frac{1}{c} - 2, \frac{-1}{c} - 2\right) = (\alpha, \beta)$

$\Rightarrow |\alpha + \beta| = 4$

Hence, $|\alpha + \beta| = 4$

14. Let a curve $y = y(x)$ be given by the solution of the differential equation

$$\cos\left(\frac{1}{2}\cos^{-1}\left(e^{-x}\right)\right)dx = \sqrt{e^{2x} - 1}\,dy$$

If it intersects y–axis at $y = -1$, and the intersection point of the curve with x– axis is $(\alpha, 0)$, then e^{α} is equal to…………….

[JEE (Main) – 20th July 2021 - Shift-2]

Sol. The correct answer is [2].

$$\cos\left(\frac{1}{2}\cos^{-1}\left(e^{-x}\right)\right)dx = \sqrt{e^{2x}-1}dy$$

$$\Rightarrow \quad \int \cos\frac{\left(\frac{1}{2}\cos^{-1}\left(e^{-x}\right)\right)}{\sqrt{e^{2x}-1}}dx = \int dy$$

Put, $\cos^{-1}(e^{-x}) = t$

$$\therefore \quad \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx = dt \quad \therefore \int \cos\left(\frac{t}{2}\right)dt = y+c$$

$$\therefore \quad 2\sin\left(\frac{1}{2}\cos^{-1}\left(e^{-x}\right)\right) = y+c$$

So, $x = 0 \Rightarrow y = -1$

$\Rightarrow c = 1$

$$\therefore \quad y = 2\sin\left(\frac{1}{2}\cos^{-1}\left(e^{-x}\right)\right)$$

If $y = 0$, then $x = \alpha$

$$\Rightarrow \quad 2\sin\left(\frac{1}{2}\cos^{-1}\left(e^{-x}\right)\right) = 1$$

$$\Rightarrow \quad e^{-x} = \frac{1}{2} \Rightarrow x = \ln 2 = \alpha$$

So, the value of $e^{\alpha} = 2$

15. Let $y = y(x)$ be the solution of the differential equation $xdy - ydx = \sqrt{\left(x^2-y^2\right)}dx$, $x > 1$, with $y(1) = 0$. If the area bounded by the line $x = 1$, $x = e^{\pi}$, $y = 0$ and $y = y(x)$ is $\lambda e^{2\pi} + \beta$, then the value of $10(\alpha + \beta)$ is equal to ______.

[JEE (Main) – 18th March 2021 - Shift-2]

Sol. The correct answer is (4).

Given

$y = y(x)$ is solution of differential equation

$x\,dy - y\,dx = \sqrt{x^2-y^2}\,dx$, $x \geq 1$, $y(1) = 0$

Area bounded by line $x = 1$, $x = e^{\pi}$, $y = 0$ and $y = y(x)$ is $\alpha e^{2\pi} + \beta$.

$x\,dy - y\,dx = \sqrt{x^2-y^2}\,dx$

Divide both sides by x^2

$$\Rightarrow \quad \frac{xdy-ydx}{x^2} = \frac{1}{x}\sqrt{1-\frac{y^2}{x^2}}dx$$

$$\because d\left(\frac{y}{x}\right) = \frac{xdy-ydx}{x^2}$$

$$\int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1-\left(\frac{y}{x}\right)^2}} = \int\frac{dx}{x}$$

(on integrating both sides)

$$\because \quad \int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C_1$$

$$\text{and } \int\frac{dx}{x} = \ln|x| + C_2$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{y}{x}\right) = \ln|x| + C \qquad \text{...(i)}$$

$(C = C_2 - C_1)$

At $x = 1, y = 0$

$\therefore \quad \sin^{-1}(0) = \ln|1| + C$

$0 = 0 + C$

$C = 0$

Equation (i) is

$$\sin^{-1}\left(\frac{y}{x}\right) = \ln|x|$$

$$\frac{y}{x} = \sin(\ln|x|)$$

$$y = x\sin(\ln|x|)$$

Required Area

$$= \int_1^{e^{\pi}} x\sin(\ln x)dx$$

Put $x = e^t$ $x \to 1 \Rightarrow t \to 0$

$dx = e^t\,dt$ $x \to e^{\pi} \Rightarrow t \to \pi$

$$\text{Area} = \int_0^{\pi} e^{2t}\sin t\,dt$$

Compare with given area

$$\alpha e^{2\pi} + \beta = \left(\frac{e^{2t}}{5}(2\sin t - \cos t)\right)_0^{\pi}$$

$$\alpha e^{2\pi} + \beta = \frac{1+e^{2\pi}}{5}$$

$$\alpha e^{2\pi} + \beta = \frac{1}{5}e^{2\pi} + \frac{1}{5}$$

$$\alpha = \frac{1}{5} \text{ and } \beta = \frac{1}{5}$$

$$\therefore \quad 10(\alpha + \beta) = 10\left(\frac{1}{5}+\frac{1}{5}\right)$$

$$= 10 \times \frac{2}{5}$$

$$10(\alpha + \beta) = 4$$

Hint :

(i) Apply inspection method to solve given differential equation.

(ii) Find constant using initial conditions.

(iii) Use integration to find area and compare with given area to find α and β.

Shortcut Method:

$$\frac{xdy - ydx}{x^2} = \frac{1}{x}\sqrt{1-\frac{y^2}{x^2}}dx$$

$$\Rightarrow \quad \int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1-\left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x}$$

$$\sin^{-1}\left(\frac{y}{x}\right) = \ln x + c$$

$$y(1) = 0 \Rightarrow c = 0$$

$$y = x\sin(\ln x)$$

$$\text{Area} = \int_1^{e^{\pi}} x\sin(\ln x)dx$$

$$\alpha e^{2\pi} + \beta = \frac{e^{2\pi}+1}{5}$$

$$\alpha = \beta = \frac{1}{5}$$

$$10(\alpha + \beta) = 4$$

16. If $y = y(x)$ is the solution of the equation $e^{\sin y} \cos y \frac{dy}{dx} + e^{\sin y} \cos x = \cos x$, $y(0) = 0$; then $1+y\left(\frac{\pi}{6}\right)+\frac{\sqrt{3}}{2}y\left(\frac{\pi}{3}\right)+\frac{1}{\sqrt{2}}y\left(\frac{\pi}{4}\right)$ is equal to ______. **[JEE (Main) – 26th Feb 2021 - Shift-1]**

Sol. The correct answer is (1).

Given: $y = y(x)$ is the solution of the equation

$$e^{\sin y}\cos y\frac{dy}{dx} + e^{\sin y}\cos x = \cos x, y(0) = 0$$

$$e^{\sin y}\cos y\frac{dy}{dx} + e^{\sin y}\cos x = \cos x$$

Substitute $\sin y = t$

$$\Rightarrow \cos y\frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore\ e^t\frac{dt}{dx} + e^t\cos x = \cos x$$

Substitute $e^t = z$

$$\Rightarrow e^t\frac{dt}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dz} + z\cos x = \cos x \qquad \text{...(i)}$$

Integrating factor = $e^{\int \cos x dx} = e^{\sin x}$

Multiply both sides of (i) by $e^{\sin x}$

$$e^{\sin x}.\frac{dz}{dx} + z\cos x.e^{\sin x} = e^{\sin x}\cos x \qquad \text{...(ii)}$$

Product rule of differentiation:

$$\frac{d}{dx}(u.v) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\therefore \quad \frac{d}{dx}\left(z.e^{\sin x}\right) = \frac{d}{dx}\left(e^{\sin x}\right)$$

Integrating both sides

$$\int \frac{d}{dx}\left(z.e^{\sin x}\right) = \int \frac{d}{dx}\left(e^{\sin x}\right)$$

$$\Rightarrow \quad z.e^{\sin x} = e^{\sin x} + c$$

$$\Rightarrow \quad e^t.e^{\sin x} = e^{\sin x} + c \qquad (\because z = e^t)$$

$$\Rightarrow \quad e^{\sin y}.e^{\sin x} = e^{\sin x} + c \qquad (\because t = \sin y)$$

$$\Rightarrow \quad e^{\sin y + \sin x} = e^{\sin x} + c \qquad (\because e^a.e^b = e^{a+b})$$

Given, $y = 0$ at $x = 0$

$$e^0 = e^0 + c$$

$$\Rightarrow \quad c = 0$$

$$\Rightarrow \quad e^{\sin y + \sin x} = e^{\sin x}$$

$$\Rightarrow \sin y + \sin x = \sin x$$

$$\Rightarrow \quad \sin y = 0$$

$$\Rightarrow \quad y = 0$$

$$1+y\left(\frac{\pi}{6}\right)+\frac{\sqrt{3}}{2}y\left(\frac{\pi}{3}\right)+\frac{1}{\sqrt{2}}y\left(\frac{\pi}{4}\right)$$

$$= 1 + 0 + \frac{\sqrt{3}}{2}(0) + \frac{1}{\sqrt{2}}(0) = 1$$

Hint:

(i) For linear differential equation of the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x,

I.F. = $e^{\int Pdx}$.

(ii) Solution is

$$y.(\text{I.F.}) = \int Q(\text{I.F.})dx + c$$

Shortcut Method:

Put $e^{\sin y} = t$

$$\Rightarrow \quad e^{\sin y}\cos y\frac{dy}{dx} = \frac{dt}{dx}$$

Differential equation is

$$\frac{dt}{dx} + t\cos x = \cos x$$

Solution is

$$e^{\sin x}.t = \int \cos x.e^{\sin x}dx + c$$

$$\Rightarrow \quad e^{\sin y + \sin x} = e^{\sin x} + c$$

$y(0) = 0$

$$\Rightarrow \quad e^{\sin y + \sin x} = e^{\sin x}$$

$$\Rightarrow \quad \sin y + \sin x = \sin x$$

$$\Rightarrow \quad y = 0$$

$$1+y\left(\frac{\pi}{6}\right)+\frac{\sqrt{3}}{2}y\left(\frac{\pi}{3}\right)+\frac{1}{\sqrt{2}}y\left(\frac{\pi}{4}\right) = 1.$$

17. The difference betweeen degree and order of differential equation that represents the family of curves given by $y^2 = a\left(x+\frac{\sqrt{a}}{2}\right)$, $a > 0$ is ______.

[JEE (Main) – 26th Feb 2021 - Shift-1]

Sol. Correct answer is [2].

Given: $y^2 = a\left(x+\left(\frac{\sqrt{a}}{2}\right)\right), a>0$ ...(i)

Since, there is only one arbitrary constant 'a', we need to differentiate once and eliminate 'a' using that constant.

Differentiate (i) w.r.t. x

$2yy' = a$

Put in (i)

$$y^2 = 2yy'\left(x+\frac{\sqrt{2yy'}}{2}\right)$$

$$\Rightarrow \quad y = 2y'x + y'\sqrt{2yy'}$$

$$\Rightarrow \left(y-2x\frac{dy}{dx}\right)^2 = \left(y'\sqrt{2yy'}\right)^2$$

$$\Rightarrow \quad y^2 + 4x^2\left(\frac{dy}{dx}\right)^2 - 4xy\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2 .2y\frac{dy}{dx}$$

$$\left(\because y' = \frac{dy}{dx}\right)$$

$$\Rightarrow \quad 2y\left(\frac{dy}{dx}\right)^3 - 4x^2\left(\frac{dy}{dx}\right)^2 + 4xy\frac{dy}{dx} - y^2 = 0$$

Maximum order of $\frac{dy}{dx} = 1$

$\therefore$ Order = 1

Maximum power of $\frac{dy}{dx} = 3$

$\therefore$ Degree = 3.

Degree – Order = 3 – 1 = 2.

Hint:

(i) Differentiate the given curve as much times as the number of arbitrary constants.

(ii) Use these equations to eliminate number of constants.

(iii) Order = Number of arbitrary constants

(iv) Degree = Highest degree of the differential coefficient providing the order when expressed in polynomial form.

Shortcut Method:

$$y^2 = ax + \frac{a^{3/2}}{2} \qquad \text{...(i)}$$

$\Rightarrow 2yy' = a$

Put in (i) $\quad (y^2 - 2xyy') = \frac{(2yy')^{3/2}}{2}$

Squaring $\quad (y^2 - 2xyy')^2 = \frac{y^3(y')^3}{2}$

$\therefore$ Order = 1

Degree = 3

Degree – Order = 3 – 1 = 2.

18. If the curve $y = y(x)$ represented by the solution of the differential equation $(2xy^2 - y)dx + xdy = 0$, passes through the intersection of the lines, $2x - 3y=1$ and $3x+2y=8$, then $|y(1)|$ is equal to ______.

[JEE (Main) – 25th Feb 2021 - Shift-2]

Sol. The correct answer is (1).

Given: Curve $y = y(x)$ represented by the solution of differential equation $(2xy^2-y)\,dx + xdy = 0$, passes through the intersection of the lines, $2x - 3y = 1$ and $3x + 2y = 8$.

$(2xy^2-y)\,dx + xdy = 0$

$\Rightarrow (2xy^2-y)\,dx = -xdy$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy^2-y}{-x} \Rightarrow \frac{dy}{dx} = \frac{-2xy^2}{x} + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = -2y^2$$

Divide by $-y^2$

$$\Rightarrow \frac{-1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 2 \qquad \text{...(i)}$$

Let $\frac{1}{y} = t$

$$\Rightarrow \frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$\therefore$ (i) becomes

$$\frac{dt}{dx} + \frac{t}{x} = 2 \qquad \text{...(ii)}$$

Which is a linear differential equation.

Integrating factor, I.F. $= e^{\int \frac{1}{x}dx}$

$= e^{\ln x}$

$= x \qquad (\because a^{\log_a x} = x)$

Multiply both sides of (ii) by I.F.

$$x\cdot\frac{dt}{dx} + x\cdot\frac{t}{x} = 2x$$

$$\Rightarrow \quad x\frac{dt}{dx} + t = 2x \qquad \text{...(iii)}$$

Product rule of differentiation

$$\frac{d}{dx}(u.v) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\therefore \quad x\frac{dt}{dx} + t.\frac{d}{dx}(x) = \frac{d}{dx}(x.t)$$

∴ (iii) becomes

$$\frac{d}{dx}(tx) = 2x$$

$\Rightarrow d(tx) = 2xdx$

Integrating both sides

$\int d\,(tx) = \int 2xdx$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow tx = \frac{2x^2}{2} + c$$

$$\Rightarrow tx = x^2 + c \qquad \left(\because t = \frac{1}{y}\right)$$

$$\Rightarrow \frac{x}{y} = x^2 + c \qquad \text{...(iv)}$$

Solve $2x - 3y = 1$ and $3x + 2y = 8$ simultaneously

$$2x - 3y = 1 \qquad \text{...(v)}$$

$$3x + 2y = 8 \qquad \text{...(vi)}$$

Multiply (v) by 3 and then subtract from (vi) multiplied by 2

$$\therefore \quad \begin{array}{r} 6x + 4y = 16 \\ 6x - 9y = 3 \\ \underline{\quad + \quad -} \\ 13y = 13 \end{array}$$

$$\Rightarrow \quad y = 1 \quad \text{From (v)}$$

$$2x = 4$$

$$\Rightarrow \quad x = 2$$

So, the curve (iv) passes through (2, 1)

$$\Rightarrow \quad \frac{2}{1} = 2^2 + c$$

$$\Rightarrow \quad c = -2$$

So, curve is $\frac{x}{y} = x^2 - 2$

Put $x = 1$ to find $|y(1)|$

$$\frac{1}{y(1)} = 1 - 2$$

$$\Rightarrow \quad y(1) = -1$$

$$\Rightarrow \quad |y(1)| = 1$$

Hint:

(i) Find Integrating factor $= e^{\int Pdx}$ for the liner differential equation of the form

$$\frac{dy}{dx} + py = Q$$

(ii) Solution of such D.E. is

$y.(\text{I.F.}) = \int Q.(\text{I.F.})\, dx + c$

Shortcut Method:

$$(2xy^2 - y)\, dx + xdy = 0$$

$$\Rightarrow \quad 2xy^2dx - ydx + xdy = 0$$

$$\Rightarrow \quad 2xdx = \frac{ydx - xdy}{y^2}$$

$$\Rightarrow \quad 2xdx = d\left(\frac{x}{y}\right)$$

Now, integrate

$$x^2 = \frac{x}{y} + c$$

Point of intersection of lines are (2, 1).

$$4 = \frac{2}{1} + c$$

$$\Rightarrow \quad c = 2$$

$$\therefore \quad \text{Curve is } x^2 = \frac{x}{y} + 2$$

Now, $\quad y(1) = -1$

$$\Rightarrow \quad |y(1)| = 1$$

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