2024 UPDATED SYLLABUS

# MATHS NCERT - 11


- Useful for CBSE, JEE exams
- Each topic contains Detailed Theory with images
- Every topic contains Exercises and Detailed solutions

# MATHS

# BINOMIAL THEOREM

## 1 BINOMIAL EXPRESSION

Any algebraic expression consisting of only two terms is known as binomial expression. The terms may consist of variables *x*, *y* etc. or constants or their mixed combinations. For example: $2x + 3y$, $4xy + 5$ etc.

## 2 BINOMIAL THEOREM FOR POSITIVE INDEX

Binomial theorem gives a formula for the expansion of a binomial expression raised to any positive integral power.

In general for a positive integer *n*

$(x + y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \ldots + {}^nC_n x^0 y^n$, where ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$

for $r = 0, 1, 2, \ldots, n$ is called binomial coefficient.

### 2.1 PROOF OF BINOMIAL THEOREM

The Binomial theorem can be proved by mathematical induction

Let $P(n)$ stands for the mathematical statement

$$(x+a)^n = x^n + {}^nC_1\, x^{n-1} a + {}^nC_2\, x^{n-2} a^2 + \ldots + {}^nC_r\, x^{n-r} a^r + \ldots + a^n \qquad \ldots(i)$$

Note that there are $(n+1)$ terms in R.H.S. and all the terms are of the same degree in *x* and a together.

When $n = 1$, L.H.S. $= x + a$ and R.H.S. $= x + a$ (there are only 2 terms)

$\therefore P$ (1) is verified to be true

Assume $P(m)$ to be true

i.e., $$(x+a)^m = x^m + {}^mC_1\, x^{m-1} a + {}^mC_2\, x^{m-2} a^2 + \ldots + {}^mC_r\, x^{m-1} a^r + \ldots + a^m \qquad \ldots(ii)$$

Multiplying equation (ii) by $(x + a)$, we have

$$(x+a)^m (x+a) = (x+a)\{x^m + {}^mC_1\, x^{m-1} a + {}^mC_2\, x^{m-2} a^2 + \ldots + {}^mC_r\, x^{m-r} a^r + \ldots + a^m\}$$

i.e., $$(x+a)^{m+1} = x^{m+1} + \left({}^mC_1 + 1\right) x^m a + \left({}^mC_2 + {}^mC_1\right) x^{m-1} a^2 + \ldots + \left({}^mC_r + {}^mC_{r-1}\right) x^{m-r+1} a^r + \ldots + a^{m+1}$$

$$= x^{m+1} + {}^{(m+1)}C_1\, x^m a + {}^{(m+1)}C_2\, x^{m-1} a^2 + \ldots + {}^{m+1}C_r\, x^{m+1-r} a^r + \ldots a^{m+1} \qquad \ldots(iii)$$

(using the formula ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$)

Equation (iii) implies that $P(m+1)$ is true and hence by induction $P(n)$ is true.

***Alternative method***

By choosing *x* from all the brackets we get the term $x^n$. Choosing *x* from $(n-1)$ factors and '*a*' from the remaining factor we get $x^{n-1}a$. But the number of ways of doing this is equal to the number of ways of choosing one factor from *n* factors (i.e.,) ${}^nC_1$. Choosing *x* from $(n-2)$

factor and a from the remaining two factors, we get $x^{n-2}\, a^2$. But the number of ways of doing this is equal to the number of ways of choosing two factors from $n$ factors. i.e., ${}^nC_2$. Finally choosing '$a$' from all the factors we get the term $a^n$.

$$\therefore (x+a)^n = x^n + {}^nC_1\, x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + a^n$$

**Illustration 1**

**Question:** **Expand $\left(x - \frac{1}{x}\right)^6$**

**Solution:**

$$\left(x-\frac{1}{x}\right)^6 = {}^6C_0x^6 + {}^6C_1x^5\left(\frac{-1}{x}\right) + {}^6C_2x^4\left(\frac{-1}{x}\right)^2 + {}^6C_3x^3\left(\frac{-1}{x}\right)^3 + {}^6C_4x^2\left(\frac{-1}{x}\right)^4 + {}^6C_5x\left(\frac{-1}{x}\right)^5 + {}^6C_6x^0\left(\frac{-1}{x}\right)^6$$

$$= x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

## 3 BINOMIAL COEFFICIENTS

In the binomial expansion of $(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$

- The binomial coefficients of the expansion equidistant from the beginning and the end are equal. In other words ${}^nC_r = {}^nC_{n-r}$.
- The greatest binomial coefficient in the expansion is always the binomial coefficient of middle term/terms.

**Illustration 7**

**Question:** **If the coefficient of $(2r+4)^{th}$ term and $(r-2)^{th}$ term in the expansion of $(1+x)^{18}$ are equal, find $r$.**

**Solution:** Since coefficient of $(2r+4)^{th}$ term in $(1+x)^{18} = {}^{18}C_{2r+3}$.

Coefficient of $(r-2)^{th}$ term $= {}^{18}C_{r-3}$

$\Rightarrow \quad {}^{18}C_{2r+3} = {}^{18}C_{r-3} \quad \Rightarrow \quad 2r + 3 + r - 3 = 18$

$\Rightarrow \quad 3r = 18 \quad \Rightarrow r = 6.$

**Illustration 8**

**Question:** **Find the coefficient of $x^5$ in the expansion of $(1+x+x^3)^9$.**

**Solution:** $(1+x+x^3)^9 = [(1+x)+x^3]^9$

$= (1+x)^9 + {}^9C_1\,(1+x)^8\, x^3 + {}^9C_2\,(1+x)^7\, x^6 + \ldots \qquad \ldots(i)$

The coefficient of $x^5$ in $(1+x)^9$ is ${}^9C_5$ i.e. ${}^9C_4$

The coefficient of $x^5$ in ${}^9C_1\,(1+x)^8\, x^3$ = coefficient of $x^2$ in ${}^9C_1\,(1+x)^8 = 9.\ {}^8C_2$

The remaining terms in (i) all contain powers of $x$ higher than the fifth.

$\therefore$ the required coefficient is ${}^9C_4 + 9\times{}^8C_2 = \frac{9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4} + \frac{9\cdot8\cdot7}{1\cdot2} = 378$

**Alternatively**

$[1+x(1+x^2)]^9 = 1 + {}^9C_1\, x\,(1+x^2) + {}^9C_2\, x^2\,(1+x^2)^2 + {}^9C_3\, x^3\,(1+x^2)^3 + {}^9C_4\, x^4\,(1+x^2)^4 + \ldots$

$x^5$ occurs in 4th and 6th terms only and it is equal to $= 3\times{}^9C_3 + {}^9C_5$

= 252 + 126 = 378

**Illustration 9**

**Question:** **Show that $11^{n+2} + 12^{2n+1}$ is divisible by 133.**

**Solution:** $11^{n+2} + 12^{2n+1} = 11^2\cdot 11^n + 12\,(144)^n$

Now 144 and 121 should be expressed in terms of 133; 144 as (133 + 11) or 121 as (133 – 12)

$= 121 \cdot 11^n + 12\,(11 + 133)^n$

$= 11^n\,(121 + 12)$ + terms containing 133 as a factor

$= 11^n \cdot 133$ + terms containing 133 as a factor

Hence the expression is divisible by 133.

- ***The ratio of (r + 1)$^{th}$ coefficient to r$^{th}$ coefficient is $\dfrac{n-r+1}{r}$ as***

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{\dfrac{n!}{(n-r)!\,r!}}{\dfrac{n!}{(n-r+1)!\,(r-1)!}} = \frac{n-r+1}{r}$$

**Illustration 10**

**Question:** **If $a_r$ be the coefficient of $x^r$ in the expansion of $(1 + x)^n$, then prove that**

$$(a_0 + a_1)(a_1 + a_2)\ldots(a_{n-1} + a_n) = a_0 \cdot a_1 \cdot a_2 \ldots a_{n-1}\,\frac{(n+1)^n}{n!}$$

**Solution:** $a_r = {}^nC_r$ and $a_{r-1} = {}^nC_{r-1}$

$$\therefore \quad \frac{a_r}{a_{r-1}} = \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n!}{r!\,(n-r)!}\,\frac{(r-1)!\,(n-r+1)!}{n!} = \frac{n-r+1}{r} = \frac{n+1}{r} - 1$$

$$\therefore \quad \frac{a_r + a_{r-1}}{a_{r-1}} = \frac{n+1}{r} \qquad \ldots(i)$$

Putting $r$ = 1, 2, ….. $n$ and multiplying together,

$$\left(\frac{a_0 + a_1}{a_0}\right)\left(\frac{a_1 + a_2}{a_1}\right)\ldots\left(\frac{a_{n-1} + a_n}{a_{n-1}}\right) = \frac{(n+1)^n}{1.2\ldots n}$$

$$\therefore \quad (a_0 + a_1)(a_1 + a_2)\ldots(a_{n-1} + a_n) = \frac{(a_0\, a_1 \ldots a_{n-1})(n+1)^n}{n!}$$

## 4 GREATEST TERM IN THE EXPANSION

The numerically greatest term (absolute value) in the expansion of $(1 + x)^n$ is determined by the following process:

Let $t_{r+1}$ be the numerically greatest term, then $|t_r| \le |t_{r+1}| \ge |t_{r+2}|$

Considering $\left|\dfrac{t_{r+1}}{t_r}\right| \ge 1$

$\Rightarrow \quad \left|\dfrac{n-r+1}{r}x\right| \ge 1 \qquad \Rightarrow \quad \dfrac{n-r+1}{r}\,|x| \ge 1$

$\Rightarrow \quad (n-r+1)\,|x| \ge r \qquad \Rightarrow \quad r \le (n+1)\dfrac{|x|}{1+|x|} \qquad \ldots(i)$

Also $\left|\dfrac{t_{r+1}}{t_{r+2}}\right| \ge 1$

$\Rightarrow \quad \left|\dfrac{r+1}{n-r}\,\dfrac{1}{x}\right| \ge 1 \qquad \Rightarrow \quad r + 1 \ge (n-r)\,|x|$

$\Rightarrow \quad r \ge \dfrac{n|x| - 1}{1+|x|}$

$\Rightarrow \quad r \ge (n+1)\dfrac{|x|}{1+|x|} - 1 \qquad \ldots(ii)$

Thus from (i) and (ii) for $t_{r+1}$ to be the greatest term,

$r \le m$ and $r \ge m-1$ where $m = (n+1)\dfrac{|x|}{1+|x|}$

- *If m is an integer, then there are two terms with numerically greatest magnitude, for $r = m$ and $r = m-1$ i.e. $t_{m+1}$ and $t_m$.*
- *If m is not an integer, then there is only one numerically greatest term, for $r = [m]$ i.e. $t_{[m]+1}$, where [m] is the greatest integer function of m.*

*In general, for the expansion $(a + x)^n$ or $a^n\left(1+\dfrac{x}{a}\right)^n$ we can consider* $m = (n+1)\dfrac{|x/a|}{1+|x/a|}$.

**Illustration 11**

**Question:** **Find the greatest term in the expansion of $(4 + 3x)^7$ when $x = \dfrac{2}{3}$.**

**Solution:** Here the greatest term means the numerically greatest term.

$$\left|\frac{t_{r+1}}{t_r}\right| = \frac{{}^7C_r\, 4^{7-r}\,(3x)^r}{{}^7C_{r-1}\, 4^{8-r}\,(3x)^{r-1}} = \frac{8-r}{r}\,\frac{3x}{4} = \frac{8-r}{2r} \text{ since } x = 2/3$$

Now $|t_{r+1}| \ge |t_r|$ if $8 - r \ge 2r$ or $\dfrac{8}{3} \ge r$

This inequality is valid only for $r$ = 1 or 2

Thus for $r = 1, 2$; $|t_{r+1}| > |t_r|$ and

for $r = 3, 4$; $|t_{r+1}| < |t_r|$

$\therefore \quad |t_1| < |t_2| < |t_3| > |t_4| > |t_5| > \ldots\ldots$

greatest term = $|t_3| = {}^7C_2\, 4^5 \cdot (3x)^2$, where $x = \dfrac{2}{3}$

$= 21 \times 4^5 \times 2^2 = 86016$

## 5 SUMMATION OF SERIES INCLUDING BINOMIAL COEFFICIENTS

In the binomial expansion of $(1 + x)^n$, let us denote the coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ….. , ${}^nC_r$, ….. , ${}^nC_n$ by $C_0$, $C_1$, $C_2$, …… , $C_r$, …… , $C_n$ respectively.

- *The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is $2^n$ :*

$\because \quad (1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_n x^n$

Putting $x = 1$

$\therefore \quad 2^n = C_0 + C_1 + C_2 + \ldots\ldots + C_n \qquad \ldots$(i)

or $\quad \displaystyle\sum_{r=0}^{n} C_r = 2^n$.

**Illustration 12**

**Question:** **Prove that the sum of the coefficients in the expansion of $(1 + x - 3x^2)^{2163}$ is –1.**

**Solution:** Putting $x = 1$ in $(1 + x - 3x^2)^{2163}$, the required sum of coefficients

$= (1 + 1 - 3)^{2163} = (-1)^{2163} = -1$.

**Illustration 13**

**Question:** **If the sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$ is equal to the sum of the coefficients in the expansion of $(x - \alpha y)^{35}$, then find the value of $\alpha$.**

**Solution:** Sum of the coefficients in the expansion of $(\alpha x^2 - 2x + 1)^{35}$

= Sum of the coefficients in the expansion of $(x - \alpha y)^{35}$

Putting $x = y = 1$

$\therefore \quad (\alpha - 1)^{35} = (1 - \alpha)^{35}$

$\Rightarrow$ $(\alpha - 1)^{35} = -(\alpha - 1)^{35}$

$\Rightarrow$ $2(\alpha - 1)^{35} = 0$

$\therefore$ $\alpha - 1 = 0$

$\therefore$ $\alpha = 1$

- ***The sum of the coefficients of the odd terms in the expansion of $(1 + x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to $2^{n-1}$***

Since $(1 + x)^n = C_0 + C_1 + C_2x^2 + C_3x^3 + \ldots\ldots. + C_nx^n$

Putting $x = -1$,

$0 = C_0 - C_1 + C_2 - C_3 + \ldots\ldots.. + (-1)^n C_n$

and $2^n = C_0 + C_1 + C_2 + C_3 + \ldots\ldots. + C_n$ {from (i)}

Adding and subtracting these two equations, we get

$2^n = 2(C_0 + C_2 + C_4 + \ldots\ldots)$ and $2^n = 2(C_1 + C_3 + C_5 + \ldots..)$

$\therefore$ $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots. = 2^{n-1}$

sum of coefficients of odd terms = sum of coefficients of even terms = $2^{n-1}$

**Illustration 14**

**Question:** **Evaluate the sum : $^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$.**

**Solution:** Since $^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$ = sum of even terms coefficients in the expansion of $(1 + x)^8$

$= 2^{8-1} = 2^7 = 128$.

- ***Differentiation can be used to solve series in which each term is a product of an integer and a binomial coefficient i.e. in the form k . $^nC_r$.***

**Illustration 15**

**Question:** **Show that $3 . C_0 + 7 . C_1 + 11 . C_2 + \ldots.. + (4n + 3) C_n = (2n + 3) 2^n$.**

**Solution:** This problem can be done be differentiating the expansion of $x^3(1 + x^4)^n$ and putting $x = 1$.

$x^3(1 + x^4)^n = x^3(C_0 + C_1x^4 + C_2x^8 + \ldots..+ C_nx^{4n})$

$= C_0x^3 + C_1x^7 + C_2x^{11} + \ldots..+ C_nx^{4n+3}$

Differentiating we get,

$\Rightarrow$ $3x^2(1 + x^4)^n + x^3n(1 + x^4)^{n-1}4x^3 = 3x^2C_0 + 7x^6C_1 + 11x^{10}C_2 + \ldots. + (4n + 3)x^{4n+2}C_n$

Now substituting $x = 1$ in both sides.

$\Rightarrow$ $3C_0 + 7C_1 + 11C_2 + \ldots\ldots+ (4n + 3) C_n$

$= 3(2^n) + 4n(2)^{n-1}$

$= (3 + 2n) 2^n$

**Alternative method**

Let $S = 3 \cdot C_0 + 7 \cdot C_1 + 11 \cdot C_2 + \ldots.. + (4n - 1)\, {}^nC_{n-1} + (4n + 3)\, C_n$

$S = (4n + 3)\, C_0 + (4n - 1)\, C_1 + \ldots + 3\, C_0$

Adding

$2S = (4n + 6)\{C_0 + C_1 + \ldots.. + C_n\} = (4n + 6)\, 2^n$

$\therefore$ $S = (2n + 3)\, 2^n$

**Illustration 16**

**Question:** **Show that $C_1 + 2 . C_2 + 3 . C_3 + \ldots\ldots+ n . C_n = n . 2^{n-1}$ .**

**Solution:** The problem can be done by differentiating the expansion of $(1 + x)^n$ and then putting $x = 1$

**Alternative method**

The numbers multiplying binomial coefficients are 1, 2, 3, …., $n$ and these are in arithmetic progression.

Let $S = C_1 + 2 \cdot C_2 + 3 \cdot C_3 + \ldots. + (n - 1)\, C_{n-1} + n \cdot C_n$

$S = n \cdot C_0 + (n - 1)\, C_1 + (n - 2)\, C_2 + (n - 3) \cdot C_3 + \ldots. + 1 \cdot C_{n-1}$

(Writing the terms in the reverse order and remembering that $C_r = C_{n-r}$), adding

$2S = n \cdot C_0 + n . C_1 + n \cdot C_2 + \ldots.. + n \cdot C_{n-1} + n \cdot C_n$

$= n \cdot [C_0 + C_1 + C_2 + \ldots\ldots. C_n] = n \cdot 2^n$

$\therefore \quad S = n \cdot 2^{n-1}$

**Alternative method**

$$S = {}^nC_1 + 2\cdot {}^nC_2 + 3\cdot {}^nC_3 + \ldots\ldots + n\cdot {}^nC_n$$

$$= n + \frac{2\cdot n(n-1)}{1.2} + 3\cdot\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \ldots. + n$$

$$= n\left\{1 + \frac{n-1}{1} + \frac{(n-1)(n-2)}{1.2} + \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3} + \ldots. + 1\right\}$$

$$= n\left\{{}^{(n-1)}C_0 + {}^{(n-1)}C_1 + {}^{(n-1)}C_2 + \ldots. + {}^{(n-1)}C_{n-1}\right\}$$

$$= n\cdot(1+1)^{n-1} = n\cdot 2^{n-1}$$

**Illustration 17**

**Question:** **Show that $C_1 - 2\,.\,C_2 + 3\,.\,C_3 - 4\,.\,C_4 + \ldots.. + (-1)^{n-1} n\,.\,C_n = 0$.**

**Solution:** The problem can be done by differentiating the expansion of $(1+x)^n$ and then putting $x$ =–1.

**Alternative method**

L.H.S. $= {}^nC_1 - 2\cdot {}^nC_2 + 3\cdot {}^nC_3 - \ldots. + (-1)^{n-1}\cdot n\cdot {}^nC_n$

$$= n\left\{1 - {}^{(n-1)}C_1 + {}^{(n-1)}C_2 - {}^{n-1}C_3 + \ldots. + (-1)^{n-1}\,{}^{(n-1)}C_{n-1}\right\}$$

$$= n(1-1)^{n-1} = n\times 0 = 0$$

- *Integration can be used to solve series in which each term is a binomial coefficient divided by an integer i.e. in the form $\frac{{}^nC_r}{k}$.*

**Illustration 18**

**Question:** **Show that $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots. + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$**

**Solution:** Integrating the expansion of $(1 + x)^n$ between the limits 0 to 1.

$$\int_0^1 (1+x)^n\,dx = \int_0^1 (C_0 + C_1x + \ldots.. + C_nx^n)\,dx$$

$$\Rightarrow \quad \left.\frac{(1+x)^{n+1}}{n+1}\right|_0^1 = \left. C_0x + C_1\frac{x^2}{2} + \ldots + C_n\frac{x^{n+1}}{n+1}\right|_0^1$$

$$\Rightarrow \quad C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots. + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

**Alternative method**

L.H.S. $= 1 + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \frac{{}^nC_3}{4} + \ldots. + \frac{{}^nC_n}{n+1}$

$$= 1 + \frac{n}{2} + \frac{n(n-1)}{1\cdot 2\cdot 3} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} + \ldots + \frac{n(n-1)(n-2)\ldots 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\ldots n\cdot(n+1)}$$

$$= \frac{1}{n+1}\left\{(n+1) + \frac{(n+1)n}{1\cdot 2} + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3} + \ldots + \frac{(n+1)n(n-1)\ldots 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\ldots(n+1)}\right\}$$

$$= \frac{1}{(n+1)}\left\{{}^{(n+1)}C_1 + {}^{(n+1)}C_2 + {}^{(n+1)}C_3 + \ldots + {}^{(n+1)}C_{n+1}\right\}$$

$$= \frac{1}{(n+1)}\{2^{n+1} - 1\} \text{ (from the expansion of } (1+1)^{n+1})$$

**Illustration 19**

**Question:** Show that

$$2\cdot C_0 + 2^2\cdot\frac{C_1}{2} + 2^3\cdot\frac{C_2}{3} + \dots + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{(n+1)}-1}{n+1}$$

**Solution:** Integrating the expansion of $(1 + x)^n$ between the limits 0 to 2.

$$\int_0^2 (1+x)^n dx = \int_0^2 (C_0 + C_1 x + \dots + C_n x^n)\,dx$$

$$\Rightarrow \quad \left.\frac{(1+x)^{n+1}}{n+1}\right|_0^2 = \left. C_0 x + C_1\frac{x^2}{2} + \dots + C_n\frac{x^{n+1}}{n+1}\right|_0^2$$

$$\Rightarrow \quad 2\cdot C_0 + 2^2\cdot\frac{C_1}{2} + 2^3\cdot\frac{C_2}{3} + \dots + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{(n+1)}-1}{n+1}$$

**Alternative method**

L.H.S. $= \frac{1}{n+1}\left\{{}^{(n+1)}C_1\cdot 2 + {}^{(n+1)}C_2\cdot 2^2 + {}^{(n+1)}C_3\cdot 2^3 + \dots + {}^{(n+1)}C_{n+1}\cdot 2^{(n+1)}\right\}$

$= \frac{1}{n+1}\left\{1 + {}^{(n+1)}C_1\cdot 2 + {}^{(n+1)}C_2\cdot 2^2 + \dots + {}^{(n+1)}C_{n+1}\cdot 2^{(n+1)} - 1\right\}$

$= \frac{1}{(n+1)}\left\{(1+2)^{n+1} - 1\right\} = \frac{3^{n+1}-1}{n+1}$

- ***Product of two expansions can be used to solve some problems related to series of binomial coefficients in which each term is a product of two binomial coefficients.***

**Illustration 20**

**Question:** **Show that** $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \frac{(2n)!}{n!\,n!}$

**Solution:** This example can be solved by considering two binomial expansions $(1+x)^n$ and $\left(1+\frac{1}{x}\right)^n$ in which the coefficients of $x^n$ and $\frac{1}{x^n}$ are equal and in the product of these expansions the constant term will contain the square of binomial coefficients.

Consider $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_n x^n$

$$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots + \frac{C_n}{x^n}$$

Taking the product of these two expansions and collecting the constant term in the product,

Constant term in R.H.S. $= C_0^2 + C_1^2 + C_2^2 + \dots C_n^2$

= constant term in L.H.S. = constant term in $(1+x)^n\left(1+\frac{1}{x}\right)^n$

= constant term in $\frac{(1+x)^{2n}}{x^n}$ = coefficient of $x^n$ in $(1+x)^{2n}$

$= {}^{2n}C_n = \frac{(2n)!}{(n!)\,(n!)}$

**Illustration 21**

**Question:** **Show that** $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \dots + (-1)^n C_n^2$

$$= \begin{cases} 0, \text{if } n \text{ is odd} \\ \dfrac{(-1)^{n/2}\, n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}, \text{if } n \text{ is even.} \end{cases}$$

**Solution:** Consider the product of the expansion of $(1+x)^n$ and $\left(1-\frac{1}{x}\right)^n$ and compare the constant term.

$C_0^2 - C_1^2 + C_2^2 + ... + (-1)^n C_n^2$ = constant term in $(1+x)^n \left(1-\frac{1}{x}\right)^n$

= constant term in $\dfrac{(1+x)^n\,(x-1)^n}{x^n}$

= constant term in $\dfrac{(-1)^n (1-x^2)^n}{x^n}$

= coefficient of $x^n$ in $(-1)^n\,(1-x^2)^n$

= 0, if $n$ is odd since all the terms in $(1-x^2)^n$ contain only even power of $x$

= coefficient of $x^{2m}$ in $(-1)^{2m}(1-x^2)^{2m}$, if $n$ is even = 2m

$= (-1)^m\ {}^{2m}C_m = (-1)^m\ {}^{2m}C_m = (-1)^m \dfrac{(2m)!}{m!\,m!} = \dfrac{n!\,(-1)^{n/2}}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$

**Illustration 22**

**Question:** **Show that**

$$C_0\,C_r + C_1\,C_{r+1} + C_2\,C_{r+2} + ... + C_{n-r}\,C_n = \frac{(2n)!}{(n-r)!\,(n+r)!}$$

**Solution:** Consider $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_rx^r + C_{r+1}x^{r+1} + ... + C_nx^n$

and $\left(1+\frac{1}{x}\right)^n = C_0 + \dfrac{C_1}{x} + \dfrac{C_2}{x^2} + ... + \dfrac{C_r}{x^r} + \dfrac{C_{r+1}}{x^{r+1}} + ... + \dfrac{C_n}{x^n}$

In the product of these two expansions, collecting the coefficient of $x^r$

$C_0\,C_r + C_1\,C_{r+1} + C_2\,C_{r+2} + .... + C_{n-r}\,C_n$ = coefficient of $x^r$ in $\dfrac{(1+x)^{2n}}{x^n}$

= coefficient of $x^{n+r}$ in $(1+x)^{2n}$

$= {}^{2n}C_{n+r} = \dfrac{(2n)!}{(n+r)!\,(n-r)!}$

## 6 IMPORTANT RESULTS

8.1 ***If $(\sqrt{P}+Q)^n = I + f$ where I and n are positive integers, n being odd, and $0 \le f < 1$, then show that $(I + f)\, f = k^n$ where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.***

**Proof.** Given $\sqrt{P} - Q < 1$

$\therefore \quad 0 < (\sqrt{P} - Q)^n < 1$

Now let $(\sqrt{P} - Q)^n = f'$ where $0 < f' < 1$

$\therefore \quad I + f - f' = (\sqrt{P} + Q)^n - (\sqrt{P} - Q)^n$

$\because$ RHS contains even powers of $\sqrt{P}$ ($\because n$ is odd)

$\Rightarrow$ RHS is an integer

Since RHS and I are integers,

$\therefore$ f – f′ is also integer.

$\therefore$ $\Rightarrow$ f – f′ = 0 $\because -1 < f - f' < 1$

or f = f′

$$\therefore \quad (I+f)f = (I+f)f' = (\sqrt{P}+Q)^n(\sqrt{P}-Q)^n = (P-Q^2)^n = k^n.$$

**8.2** ***If $(\sqrt{P}+Q)^n = I+f$ where I and n are positive integer, n being even, and $0 \le f < 1$, then show that (I + f) (1 – f) = $k^n$ where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.***

**Proof :** If *n* is an even integer then

$$(\sqrt{P}+Q)^n + (\sqrt{P}-Q)^n = I + f + f'$$

Hence LHS and I are integer.

$\therefore$ f + f′ is also integer.

$\Rightarrow$ f + f′ = 1 $\because 0 < f + f' < 2$

$\therefore$ f′ = (1 – f)

Hence $(I+f)(1-f) = (I+f)f' = (\sqrt{P}+Q)^n(\sqrt{P}-Q)^n$

$= (P - Q^2)^n = k^n$.

**Illustration 23**

**Question:** **If $(2+\sqrt{3})^n = I + f$ where I and *n* are positive integers and $0 < f < 1$, show that (i) I is an odd integer and (ii) $(I+f)(1-f) = 1$.**

**Solution:** (i) Now $0 < 2-\sqrt{3} < 1$, since $2 - \sqrt{3} = 0.268$ (approx.)

$\therefore 0 < (2-\sqrt{3})^n < 1$; we can take $(2-\sqrt{3})^n$ as $f'$.

Now $(2+\sqrt{3})^n + (2-\sqrt{3})^n = I + f + f'$

But L.*H*.S. $= 2\left\{2^n + {}^nC_2 2^{n-2}(\sqrt{3})^2 + {}^nC_4 2^{n-4}(\sqrt{3})^4 + .....\right\}$ = an integer

(in fact an even integer)

$\therefore$ RHS. = I + f + f′ = an even integer

Also $f + f' = 1$, since f and f ’ are both positive proper fractions.

$\therefore$ I = an even integer –1 = an odd integer.

(ii) $(I+f)(1-f) = (I+f)(f') = (2+\sqrt{3})^n \cdot (2-\sqrt{3})^n$

$= (4-3)^n = 1^n = 1$

**Illustration 24**

**Question:** **Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$, where [ ] denotes the greatest integer function. Prove that $Rf = 4^{2n+1}$.**

**Solution:** Greatest integer function is defined as follows:

[*x* ] = greatest integer $\le$ x

In the case of positive number, *x*

[*x* ] =integral part of *x*

$\therefore f = R - [R]$ implies that f is the fractional part of *R*

$\therefore$ 0 < *f* < 1

Since $144 > 125 > 121$, $\sqrt{125} = 5\sqrt{5}$ lies between 11 and 12.

$\therefore 0 < 5\sqrt{5} - 11 < 1$ and hence $(5\sqrt{5}-11)^{2n+1}$ will also be a proper fraction.

Let $g = (5\sqrt{5}-11)^{2n+1}$

Now $[R] + f - g = R - g$

$= (5\sqrt{5}+11)^{2n+1} - (5\sqrt{5}-11)^{2n+1}$

$= 2\left\{ {}^{(2n+1)}C_1 (5\sqrt{5})^{2n} \cdot 11^1 + {}^{(2n+1)}C_3 (5\sqrt{5})^{2n-2} \cdot 11^2 + ....\right\}$

$= \text{an even integer}$

Since [R] is an integer, the above implies $f - g = 0$ (*i.e.*) $f = g$.

Hence $Rf = Rg = (5\sqrt{5} + 11)^{2n+1} \cdot (5\sqrt{5} - 11)^{2n+1}$

$= (125 - 121)^{2n+1} = 4^{2n+1}$

## 7 MULTINOMIAL EXPANSION

If $n \in N$, then the general term of the multinomial expansion $(x_1 + x_2 + x_3 + \ldots.. + x_k)^n$ is

$$\frac{n!}{a_1!\, a_2!\, a_3! \ldots.. a_k!} x_1^{a_1} \cdot x_2^{a_2} \cdot x_3^{a_3} \ldots\ldots x_k^{a_k},$$

where $a_1 + a_2 + a_3 + \ldots.. + a_k = n$

and $0 \le a_i \le n$, $i = 1, 2, 3, \ldots\ldots. k$.

and the number of terms in the expansion are ${}^{n+k-1}C_{k-1}$.

**Illustration 25**

**Question:** **Find the total number of terms in the expansion of $(x + y + z + w)^n$, $n \in N$.**

**Solution:** The number of terms in the expansion of $(x + y + z + w)^n$ is ${}^{n+4-1}C_{4-1}$.

$= {}^{n+3}C_3$

$$= \frac{(n+3)(n+2)(n+1)}{6}$$

**Alternative method**

We know that

$$(x+y+z+w)^n = \{(x+y)+(z+w)\}^n$$

$$= (x+y)^n + {}^nC_1((x+y)^{n-1}(z+w) + {}^nC_2(x+y)^{n-2}(z+w)^2 + \ldots\ldots. + {}^nC_n(z+w)^n$$

$\therefore$ Number of terms in RHS.

$$= (n+1) + n.2 + (n-1).3 + \ldots.. + 1.(n+1)$$

$$= \sum_{r=0}^{n} (n-r+1)(r+1)$$

$$= \sum_{r=0}^{n} \left((n+1) + nr - r^2\right)$$

$$= (n+1)\sum_{r=0}^{n} 1 + n\sum_{r=0}^{n} r - \sum_{r=0}^{n} r^2$$

$$= (n+1).(n) + n.\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(n+2)(n+3)}{6}$$

## 8 BINOMIAL THEOREM FOR ANY INDEX

If $n \in R$, $-1 < x < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots.. + \frac{n(n-1)(n-2)\ldots..(n-r+1)}{r!}x^r + \ldots..\infty$$

**Note:**

(i) In the expansion if $n \in \boldsymbol{R}$ and $n > 0$ then $-1 < x < 1$.

(ii) ${}^nC_r$ can not be used because it is defined only for natural number.

(iii) If $x$ be so small then its square and higher powers may be neglected, then approximate value of $(1 + x)^n = 1 + nx$.

**Important results:**

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots + (-1)^r x^r + \ldots\ldots\ldots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots + x^r + \ldots\ldots\ldots\ldots$
- $(1+x)^{-2} = 1 - {}^2C_1 x + {}^3C_2 x^2 - {}^4C_3 x^3 + \ldots\ldots\ldots + (-1)^r\ {}^{r+1}C_r x^r + \ldots\ldots\ldots$
- $(1-x)^{-2} = 1 + {}^2C_1 x + {}^3C_2 x^2 + {}^4C_3 x^3 + \ldots\ldots + {}^{r+1}C_r x^r + \ldots\ldots\ldots$
- $(1+x)^{-3} = 1 - {}^3C_1 x + {}^4C_2 x^2 - {}^5C_3 x^3 + \ldots\ldots + (-1)^r\ {}^{r+2}C_r x^r + \ldots\ldots$
- $(1-x)^{-3} = 1 + {}^3C_1 x + {}^4C_2 x^2 + {}^5C_3 x^3 + \ldots\ldots + {}^{r+2}C_r x^r + \ldots\ldots$

In general, the coefficient of $x^n$ in $(1-x)^{-k}$ is ${}^{n+k-1}C_{k-1}$

## PROFICIENCY TEST

***The following questions deal with the basic concepts of this section. Answer the following briefly. Do not consult the Study Material while attempting the questions.***

**1.** Find the sum of coefficients in the expansion of binomial $(5p - 4q)^n$, where $n$ is a positive integer.

**2.** Show that $a\,{}^nC_0 - (a-1)\,{}^nC_1 + (a-2)\,{}^nC_2 - (a-3)\,{}^nC_3 + \ldots\ldots (-1)^n (a-n)\,{}^nC_n = 0$, where $a$ and $n$ are positive integers.

**3.** How many terms are there in the expansion of $((x + y)^4 - (x-y)^4)^6$.

**4.** Find the term independent of $x$ in the expansion of $\left(x^2 + \frac{1}{x^2}\right)^{2n}$.

**5.** Find the constant term in the expansion of $\left(x^3 - \frac{1}{x^2}\right)^{15}$.

**6.** Find the numerically greatest term in the expansion of $(3 + x)^6$, when $x = 3$ ?

**7.** Evaluate the sum of ${}^8C_2 + {}^8C_4 + {}^8C_6 + {}^8C_8$.

**8.** Show that $3\,{}^nC_0 - 7\,{}^nC_1 + 11\,{}^nC_2 + \ldots\ldots\ldots + (4n + 3)\,{}^nC_n (-1)^n = 0$.

**9.** Show that $$\frac{C_0}{2} + \frac{C_1}{3} + \frac{C_2}{4} + \ldots. + \frac{C_n}{n+2} = \frac{(1 + n \cdot 2^{(n+1)})}{(n+1)(n+2)}$$

**10.** Show that $C_0C_1 + C_1C_2 + \ldots\ldots. + C_{n-1}C_n = {}^{2n}C_{n-1}$ or ${}^{2n}C_{n+1}$.

**11.** If $(\sqrt{2} + 1)^{11} = I + f$ where $I$ is a positive integer and $0 \le f < 1$ then show that $(I + f)\, f = 1$.

**12.** Find the total number of terms in the expansion of $(1 + x + x^2 + x^3)^4$.

## ANSWERS TO PROFICIENCY TEST

**1.** 1

**3.** 7

**4.** $\frac{1.3.5.....(2n-1)}{n!}2^n$

**5.** $-{}^{15}C_6$

**6.** $20(3^6)$

**7.** 127

**12.** 13

# SOLVED OBJECTIVE EXAMPLES

**Example 1:**

**The digit at units place in the number $17^{1995} + 11^{1995} - 7^{1995}$ is**

**(a) 0 (b) 1 (c) 2 (d) 3**

**Solution:**

We have

$17^{1995} + 11^{1995} - 7^{1995} = (7 + 10)^{1995} + (1 + 10)^{1995} - 7^{1995}$

$= [7^{1995} + {}^{1995}C_1 7^{1994} . 10^1 + {}^{1995}C_2 7^{1993} . 10^2 + ..... + {}^{1995}C_{1995} . 10^{1995}]$

$+ [{}^{1995}C_0 + {}^{1995}C_1 10^1 + {}^{1995}C_2 10^2 + ......+ {}^{1995}C_{1995} 10^{1995}] - 7^{1995}$

$= [{}^{1995}C_1 7^{1994} . 10^1 + .....+ 10^{1995}] + [{}^{1995}C_1 10^1 + .....+ {}^{1995}C_{1995}\ 10^{1995}] + {}^{1995}C_0$

= (a multiple of 10) + 1

Thus, the units place digits is 1.

Hence (b) is the correct answer.

**Example 2:**

**If the sum of the coefficients in the expansion of $(1 + 2x)^n$ is 6561, the greatest term in the expansion at $x = 1/2$ is**

**(a) 4th (b) 5th (c) 6th (d) none of these**

**Solution:**

Sum of the coefficients in the expansion of $(1 + 2x)^n = 6561$

$\Rightarrow$ $(1 + 2x)^n = 6561$, when $x = 1$

$\Rightarrow$ $3^n = 6561$

$\Rightarrow$ $3^n = 3^8 \Rightarrow n = 8$

Now, $m = (n+1)\dfrac{|2x|}{1+|2x|} = \dfrac{9}{2} = 4.5$

Hence, 5th term is the greatest term.

Hence (b) is the correct answer.

**Example 3:**

**The number of terms in the expansion of $(a + b + c)^n$, where $n \in N$, is**

**(a) $\dfrac{(n+1)(n+2)}{2}$ (b) $n + 1$ (c) $n + 2$ (d) $(n + 1)n$**

**Solution:**

$(a + (b + c))^n = a^n + {}^nC_1 a^{n-1} (b + c)^1 + {}^nC_2 a^{n-2}(b + c)^2 + ...+ {}^nC_n (b + c)^n$

Further expanding each term of R.H.S.

First term on expansion gives one term

Second term on expansion gives two terms and so on.

$\therefore$ total number of terms $= 1 + 2 + 3 + .....+ (n + 1) = \dfrac{(n+1)(n+2)}{2}$

Hence (a) is the correct answer.

**Example 4:**

**The number of terms which are free from fractional powers in the expansion of $(a^{1/5} + b^{2/3})^{45}$, $a \neq b$ is**

**(a) 9 (b) 15 (c) 4 (d) none of these**

**Solution:**

The general term in the expansion of $(a^{1/5} + b^{2/3})^{45}$ is

$T_{r+1} = {}^{45}C_r (a^{1/5})^{45-r} (b^{2/3})^r$

$= {}^{45}C_r a^{9-(r/5)} b^{2r/3}$ .

This will be free from fractional powers if both $r/5$ and $2r/3$ are whole numbers i.e. if $r$ = 0, 15, 30, 45.
Hence, there are only four terms which are free from fractional powers.
Hence (c) is the correct answer.

**Example 5:**

**If $n$ is an odd natural number, then $\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r}$ equals**

**(a) 0** **(b) $1/n$** **(c) $n/2^n$** **(d) none of these**

**Solution:**

Now, $\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r}$

$= \sum_{r=0}^{(n+1)/2} \left\{ \frac{(-1)^r}{{}^nC_r} + \frac{(-1)^{n-r}}{{}^nC_{n-r}} \right\}$ (collecting the terms equidistant from the beginning and end in pairs)

$= \sum_{r=0}^{(n+1)/2} (-1)^r \left\{ \frac{1}{{}^nC_r} + \frac{-1}{{}^nC_r} \right\}$ ($\because$ $(-1)^n = -1$ as $n$ is odd)

= 0.
Hence (a) is the correct answer.

**Example 6:**

**The value of $\sum_{k=0}^{n} k^2 \left({}^nC_k\right)$ is equal to**

**(a) $n(n+1)2^{n-2}$** **(b) $n(n+1)2^{n-1}$** **(c) $n(n+1)2^n$** **(d) none of these**

**Solution:**

General term $= k^2 \; {}^nC_k$

$$= k^2 \frac{n(n-1)(n-2)\ldots(n-k+1)}{1\cdot 2\cdot 3 \ldots\ldots (k-1)\,k}$$

$$= k.\frac{n(n-1)(n-2)\ldots(n-k+1)}{1\cdot 2\cdot 3 \ldots\ldots (k-1)}$$

$$= (k-1+1)\frac{n(n-1)(n-2)\ldots(n-k+1)}{1\cdot 2\cdot 3 \ldots\ldots (k-2)(k-1)}$$

$$= \frac{n(n-1)(n-2)\ldots(n-k+1)}{1\cdot 2\cdot 3 \ldots\ldots (k-2)} + \frac{n(n-1)(n-2)\ldots(n-k+1)}{1\cdot 2 \ldots (k-2)(k-1)}$$

$$= n(n-1)\; {}^{(n-2)}C_{k-2} + n.{}^{n-1}C_{k-1}$$

$$\therefore \quad \sum_{k=0}^{n} k^2 \; {}^nC_k = n(n-1)\sum_{k=0}^{n} {}^{(n-2)}C_{k-2} + n\sum_{k=0}^{n-1} {}^{(n-1)}C_{k-1}$$

$$= n(n-1)\sum_{k=2}^{n} {}^{(n-2)}C_{k-2} + n\sum_{k=1}^{n-1} {}^{(n-1)}C_{k-1}$$

$$= n(n-1)\cdot 2^{(n-2)} + n\cdot 2^{n-1}$$

$$= 2^{n-2}\{n(n-1) + 2n\} = n(n+1)\,2^{n-2}$$

Hence (a) is the correct answer.

**Example 7:**

**The term independent of $x$ in the expansion of $\left\{\frac{x+1}{(x^{2/3}-x^{1/3}+1)}-\frac{x-1}{x-\sqrt{x}}\right\}^{15}$ is**

**(a) $^{15}C_7$** **(b) $^{15}C_9$** **(c) $^{15}C_5$** **(d) none of these**

***Solution:***

Now $\frac{x+1}{(x^{2/3}-x^{1/3}+1)}$

$$= \frac{(x^{1/3})^3+1^3}{(x^{1/3})^2-x^{1/3}+1^2} = x^{1/3}+1 \qquad (\because a^3+b^3=(a+b)(a^2-ab+b^2))$$

and $\frac{x-1}{x-\sqrt{x}} = \frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}(\sqrt{x}-1)} = \frac{\sqrt{x}+1}{\sqrt{x}} = 1+x^{-1/2}$

Hence $\left\{\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-\sqrt{x}}\right\}^{15} = \{x^{1/3}+1-(1+x^{-1/2})\}^{15} = (x^{1/3}-x^{-1/2})^{15}$.

General term

$$T_{r+1} = {}^{15}C_r\,(x^{1/3})^{15-r}(-x^{-1/2})^r$$
$$= {}^{15}C_r\,x^{(15-r)/3}(-1)^r x^{-r/2}$$
$$= {}^{15}C_r(-1)^r x^{(15-r)/3-(r/2)})$$

This will be independent of $x$ if $\frac{15-r}{3}-\frac{r}{2}=0$ i.e. if $30-2r-3r=0$. i.e. if $r=6$

∴ The term independent of $x$ is $T_{6+1}$ i.e. $T_7 = {}^{15}C_6(-1)^6x^0 = {}^{15}C_6 = {}^{15}C_9$.

Hence (b) is the correct answer.

## *Example* 8:

**The coefficient of $x^m$ in $(1+x)^r+(1+x)^{r+1}+(1+x)^{r+2}+\ldots\ldots+(1+x)^n$, $r\le m\le n$ is**

**(a) $^{n+1}C_{m+1}$** **(b) $^{n-1}C_{m-1}$** **(c) $^nC_m$** **(d) $^nC_{m+1}$**

***Solution:***

Required coefficient = coefficient of $x^m$ in $\frac{(1+x)^r\ \{(1+x)^{n-r+1}-1\}}{1+x-1}$

(∵ given series is a G.P. of $n-(r-1)$ terms with common ration $1+x$)

= coefficient of $x^{m+1}$ in $(1+x)^{n+1}-(1+x)^r$

$= {}^{n+1}C_{m+1}$ (note than $m+1>r$)

Hence (a) is the correct answer.

## *Example* 9:

**If $p(x)=x^n$, then the value of $p(1)+\frac{p'(1)}{1!}+\frac{p''(1)}{2!}+\ldots..+\frac{p^{(n)}(1)}{n!}$, where $p^{(r)}(x)$ stands for $r^{th}$ derivative of $p(x)$ with respect to $x$, is**

**(a) $2^n$** **(b) $n$** **(c) $2^{n-1}$** **(d) none of these**

***Solution:***

Here, $p(x)=x^n$, so

$p'(x) = nx^{n-1}$,

$p''(x) = n(n-1)x^{n-2}$,

$p'''(x) = n(n-1)(n-2)x^{n-3}$, …… ,

$p^{(r)}(x) = n(n-1)(n-2)\ldots\ldots(n-r+1)x^{n-r} = \frac{n!}{(n-r)!}x^{n-r} = r!\,{}^nC_rx^{n-r}$

∴ $p(1)+\frac{p'(1)}{1!}+\frac{p''(1)}{2!}+\ldots..+\frac{p^n(1)}{1!}$

$= 1+{}^nC_1+{}^nC_2+\ldots..+{}^nC_n = 2^n$

Hence (a) is the correct answer.

**Example 10:**

**If $\frac{1}{\sqrt{2x+1}} \times \left\{\left(\frac{1+\sqrt{2x+1}}{2}\right)^n - \left(\frac{1-\sqrt{2x+1}}{2}\right)^n\right\}$ is a polynomial of degree 5, then $n$ =**

**(a) 9** **(b) 10** **(c) 11** **(d) none of these**

**Solution:**

Now $\frac{1}{\sqrt{2x+1}} \times \left\{\left(\frac{1+\sqrt{2x+1}}{2}\right)^n - \left(\frac{1-\sqrt{2x+1}}{2}\right)^n\right\}$

$= \frac{1}{2^n} \frac{2}{\sqrt{2x+1}} \times \left\{ {}^nC_1\sqrt{2x+1} + {}^nC_3(\sqrt{2x+1})^3 + {}^nC_5(\sqrt{2x+1})^5 + \ldots.. \right\}$

$=$

$\frac{1}{2^{n-1}} \left\{ {}^nC_1 + {}^nC_3(2x+1) + {}^nC_5(2x+1)^2 + {}^nC_7(2x+1)^3 + {}^nC_9(2x+1)^4 + {}^nC_{11}(2x+1)^5 + \ldots.. \right\}$

Since, this polynomial is given to be of degree 5, therefore, $n$ can be 11 or 12.

Hence (c) is the correct answer.

## SOLVED SUBJECTIVE EXAMPLES

**Example 1:**

**Find the coefficient of (i) $x^7$ in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ (ii) and $x^{-7}$ in $\left(ax - \frac{1}{bx^2}\right)^{11}$. Find the relation between $a$ and $b$ if these coefficients are equal.**

**Solution:**

The general term in $\left(ax^2 + \frac{1}{bx}\right)^{11} = {}^{11}C_r\,(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r\,\frac{a^{11-r}}{b^r}\,x^{22-3r}$

If in this term, power of $x$ is 7, then $22 - 3r = 7 \Rightarrow r = 5$

$\therefore$ coefficient of $x^7 = {}^{11}C_5\,\frac{a^6}{b^5}$ … (i)

The general term in $\left(ax - \frac{1}{bx^2}\right)^{11} = (-1)^r\,{}^{11}C_r\,(ax)^{11-r}\left(\frac{1}{bx^2}\right)^r$

$= (-1)^r\,{}^{11}C_r\,\frac{a^{11-r}}{b^r}\,x^{11-3r}$

If in this term power of $x$ is $-7$, then $11 - 3r = -7 \Rightarrow r = 6$

$\therefore$ coefficient of $x^{-7} = (-1)^6\,{}^{11}C_6\,\frac{a^{11-6}}{b^6} = {}^{11}C_5\,\frac{a^5}{b^6}$ … (ii)

If these two coefficients are equal, then $= {}^{11}C_5\,\frac{a^6}{b^5} = {}^{11}C_5\,\frac{a^5}{b^6}$

$\Rightarrow$ $a^6\,b^6 = a^5\,b^5 \Rightarrow a^5\,b^5\,(ab-1) = 0 \Rightarrow ab = 1\,(a \neq 0, b \neq 0)$

**Example 2:**

**If $a$, $b$, $c$, $d$ are any four consecutive coefficients of a binomial expansion, prove that $\frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}$ or $\frac{a+b}{a}, \frac{b+c}{b}, \frac{c+d}{c}$ are in H.P.**

**Solution:**

Let a, b, c, d be the coefficient of $(r+1)^{th}$, $(r+2)^{th}$, $(r+3)^{th}$ and $(r+4)^{th}$ terms of $(1+x)^n$.

$\therefore$ $a = {}^nC_r,\ b = {}^nC_{r+1},\ c = {}^nC_{r+2},\ d = {}^nC_{r+3}$

$$\frac{a}{a+b} = \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} = \frac{{}^nC_r}{{}^{(n+1)}C_{r+1}} = \frac{r+1}{n+1}$$

$$\frac{b}{b+c} = \frac{{}^nC_{r+1}}{{}^nC_{r+1} + {}^nC_{r+2}} = \frac{{}^nC_{r+1}}{{}^{(n+1)}C_{r+2}} = \frac{r+2}{n+1}$$

Similarly $\frac{c}{c+d} = \frac{r+3}{n+1}$

Hence $\frac{a}{a+b} + \frac{c}{c+d} = \frac{r+1}{n+1} + \frac{r+3}{n+1} = \frac{2(r+2)}{n+1} = 2\cdot\frac{b}{b+c}$

$\therefore$ $\frac{a}{a+b}, \frac{b}{b+c}, \frac{c}{c+d}$ are in A.P.

**Example 3:**

**Find the coefficient of $x^4$ in the expansion of $(1+x+x^2+x^3)^{11}$.**

**Solution:**

$$1+x+x^2+x^3=(1+x)+x^2(1+x)=(1+x)(1+x^2)$$

$$\therefore \quad (1+x+x^2+x^3)^{11}=(1+x)^{11}(1+x^2)^{11}$$

$$=\left[\sum_{r=0}^{11} {}^{11}C_r\, x^r\right]\times\left[\sum_{s=0}^{11} {}^{11}C_s\,(x^2)^s\right]$$

The general term in the product of these two series is

$${}^{11}C_r \times {}^{11}C_s\, x^{r+2s}$$

Now $r$ + 2s must be equal to 4 for values of $r$, s, $0 \le r, s \le 11$.

The possible values of $r$ and $s$ are $r$ = 0, $s$ = 2; $r$ = 2, $s$ = 1; $r$ = 4, $s$ = 0

$\therefore$ coefficient of $x^4 = {}^{11}C_0 \times {}^{11}C_2 + {}^{11}C_2 \times {}^{11}C_1 + {}^{11}C_4 \times {}^{11}C_0$

$= 55 + 605 + 330$

$= 990$

**Alternative method**

Coefficient of $x^4$ in $(1 + x + x^2 + x^3)^{11}$

= Coefficient of $x^4$ in $(1 - x^4)^{11}(1 - x)^{-11}$

= Coefficient of $x^4$ in $(1 - 11x^4)(1 - x)^{-11}$

$= {}^{11+4-1}C_{10} - 11 = {}^{14}C_{10} - 11 = 1001 - 11 = 990.$

**Example 4:**

**For what value of $x$ is the fourth term in the expansion of**

$$\left\{\left(\sqrt{x}\right)^{\frac{1}{\log x+1}}+\sqrt[12]{x}\right\}^6$$ **equal to 200; log $x$ = $\log_{10}x$**

**Solution:**

The fourth term of the expansion $= {}^6C_3\,(\sqrt{x})^{\frac{3}{\log x+1}}\,(x^{1/12})^3$

$$=20\,x^{\frac{1}{4}+\frac{3}{2(1+\log x)}}=200$$

$$\therefore x^{\frac{1}{4}+\frac{3}{2(1+\log x)}}=10$$

Taking logarithm on both sides,

$$\left\{\frac{1}{4}+\frac{3}{2(1+\log x)}\right\}\log x=1$$

i.e. $\{(1+\log x)+6\}\log x = 4(1+\log x)$

i.e. $(\log x)^2 + 3\log x - 4 = 0 \Rightarrow (\log x + 4)(\log x - 1) = 0$

Either $\log_{10}x = -4$ or $\log_{10}x = 1 \Rightarrow x = 10^{-4}$ or 10

**Example 5:**

**Find the coefficient of $x^{50}$ in the expansion of**

$$(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \ldots\ldots + 1001x^{1000}$$

**Solution:**

Take $(1+x)^{1000}$ common, and let $\frac{x}{1+x} = r$

$$S = (1+x)^{1000}\left[1 + 2\frac{x}{1+x} + \frac{3x^2}{(1+x)^2} + \ldots.. + 1001\left(\frac{x}{1+x}\right)^{1000}\right]$$

$$= (1+x)^{1000}[1 + 2r + 3r^2 + \ldots. + 1001r^{1000}]$$

$$= (1+x)^{1000}\left\{\left(\frac{1-r^{1001}}{(1-r)^2}\right) - \frac{1001r^{1000}}{1-r}\right\}$$ , using the formula of A.G.P.

$$= (1+x)^{1000}\left\{\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{\left(\frac{1}{1+x}\right)^2} - \frac{1001\left(\frac{x}{1+x}\right)^{1001}}{\frac{1}{1+x}}\right\} = (1+x)^{1002} - x^{1001}(1+x) - 1001x^{1001}$$

$\therefore$ coefficient of $x^{50}$ in S = coefficient of $x^{50}$ in $(1+x)^{1002}$

$= {}^{1002}C_{50}$

**Example 6:**

*Find the sum of the series*

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \ldots \text{upto m terms}\right].$$

**Solution:**

We have

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \ldots \text{upto m terms}\right]$$

$$= \sum_{r=0}^{n}(-1)^r\,{}^nC_r\cdot\left(\frac{1}{2}\right)^r + \sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{3}{4}\right)^r + \sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{7}{8}\right)^r + \ldots \text{upto m terms}$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \ldots\ldots.. \text{upto m terms}$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \ldots\ldots.. \text{upto m terms}$$

$$= \frac{1}{2^n} + \frac{1}{(2^2)^n} + \frac{1}{(2^3)^n} + \ldots\ldots.. \text{upto m terms}$$

$$= \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2^n}\right)^m}{1-\frac{1}{2^n}}\right)$$ being the sum of $m$ terms of a G.P. with $r = \frac{1}{2^n}$

$$= \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

**Example 7:**

**Prove that the coefficient of $x^r$ in the expansion**

**$(x+3)^{n-1} + (x+3)^{n-2}\ (x+2) + (x+3)^{n-3}\ (x+2)^2 + .... + (x+2)^{n-1}$ is $\left(3^{n-r} - 2^{n-r}\right)^nC_r$.**

**Solution:**

The expression $= (x+3)^{n-1}\left\{1 + r + r^2 + .... + r^{n-1}\right\}$ where $r = \dfrac{x+2}{x+3}$

$$= (x+3)^{n-1}\left(\frac{1-r^n}{1-r}\right)$$ being the sum of a G.P.

$$= (x+3)^{n-1}\left(\frac{1-\left(\frac{x+2}{x+3}\right)^n}{\left(1-\frac{x+2}{x+3}\right)}\right) = (x+3)^n - (x+2)^n$$

$$= (3+x)^n - (2+x)^n$$

$\therefore$ coefficient of $x^r = {}^nC_r\ 3^{n-r} - {}^nC_r\ 2^{n-r} = {}^nC_r\left(3^{n-r} - 2^{n-r}\right)$

**Example 8:**

**If $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$ and $a_k = 1$ for all $k \geq n$ then show that $b_n = {}^{(2n+1)}C_{n+1}$.**

**Solution:**

$\sum_{r=0}^{2n} a_r\ (x-2)^r$ is a polynomial in $x$ of degree 2n expressed in powers of $x-2$ whereas $\sum_{r=0}^{2n} b_r\ (x-3)^r$ is an equivalent polynomial in $x$ of the same degree expressed in powers of $(x-3)$ in which the coefficient $(x-3)^n$ is $b_n$. Compare the coefficient of $(x-3)^n$.

Now L.H.S. $= \sum_{r=0}^{2n} a_r\ (x-2)^r$

$$= \sum_{r=0}^{2n} a_r \left\{(x-3)+1\right\}^r$$

Expanding this summation fully, we get

$$a_0 + a_1 (x-3+1)^1 + a_2\left\{(x-3)+1\right\}^2 + a_3\left\{(x-3)+1\right\}^3 + a_{n-1}\left\{(x-3)+1\right\}^{n-1}$$

$$+ a_n\left\{(x-3)+1\right\}^n + a_{n+1}\left\{(x-3)+1\right\}^{n+1} + .... + a_{2n}\left\{(x-3)+1\right\}^{2n}$$

In this, collecting the coefficient of $(x-3)^n$ and remembering $a_k = 1$ for $k \geq n$,

The coefficient of $(x-3)^n = a_n + a_{n+1}\,{}^{(n+1)}C_1 + a_{n+2}\,{}^{(n+2)}C_2 + .... + a_{2n}\,{}^{2n}C_n$

$$= \underbrace{1 + {}^{(n+1)}C_1}_{} + {}^{(n+2)}C_2 + ... + {}^{2n}C_n$$

$$= {}^{(n+2)}C_1 + {}^{(n+2)}C_2 + {}^{(n+3)}C_3 + ...\ {}^{2n}C_n$$

$$= {}^{(n+3)}C_2 + {}^{(n+3)}C_3 + ....\ {}^{2n}C_n$$

........................................................

........................................................

$$= {}^{2n}C_{n-1} + {}^{2n}C_n$$

$= {}^{(2n+1)}C_n$ which is also equal to ${}^{(2n+1)}C_{n+1}$

**Example 9:**

**Show that the sum of the product of the $C_i$'s taken two at a time and represented by $\sum\sum_{1\le i<j\le n} C_i C_j$ is equal to $2^{2n-1} - \frac{(2n)!}{2(n!)^2}$.**

**Solution:**

We know that $C_0 + C_1 + C_2 + .... + C_n = 2^n$ squaring,

$(C_0 + C_1 + C_2 + .... + C_n)^2 = 2^{2n}$

i.e. $\left(C_0^2 + C_1^2 + ... + C_n^2\right) + 2\sum\sum C_i C_j = 2^{2n}$

$\therefore 2\sum\sum C_i C_j = 2^{2n} - \left\{C_0^2 + C_1^2 + ... + C_n^2\right\}$

$\therefore \sum\sum C_i C_j = 2^{2n-1} - \frac{(2n)!}{2\,(n!)\,(n!)} = 2^{2n-1} - \frac{(2n)!}{2\,(n!)^2}$

**Example 10:**

**Show that $C_1^2 - 2\,C_2^2 + 3\cdot C_3^2 - ... - (2n)\,C_{2n}^2 = (-1)^{n-1} n \cdot C_n$ where $C_r = {}^{2n}C_r$**

**Solution:**

Let $S = C_1^2 - 2\cdot C_2^2 + 3\cdot C_3^2 - .... - (2n)\,C_{2n}^2$

$= -(2n)\,C_0^2 + (2n-1)\,C_1^2 - (2n-2)\,C_2^2 - .... + C_1^2,$

since $C_r = {}^{2n}C_r = {}^{2n}C_{2n-r} = C_{2n-r}$

adding

$2S = (-2n)\left\{C_0^2 - C_2^2 - ... + C_{2n}^2\right\}$

$= (-2n)\,(-1)^n \;\; {}^{2n}C_n$

$\therefore S = (-1)^{n+1} \cdot n \cdot {}^{(2n)}C_n = (-1)^{n-1} \cdot nC_n$.

# MIND MAP

## BINOMIAL THEOREM

### SUMMATION OF SERIES OF BINOMIAL COEFFICIENTS

- The sum of the binomial coefficients in the expansion of $(1 + x)^n$ is $2^n$
- The sum of the coefficients of the odd terms in the expansion of $(1 + x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to $2^{n-1}$
- Differentiation can be used to solve series in which each term is a product of an integer and a binomial coefficient i.e. in the form $k\,{}^nC_r$.
- Integration can be used to solve series in which each term is a binomial coefficient divided by an integer i.e. in the form $\frac{{}^nC_r}{k}$.
- Product of two expansions can be used to solve some problems related to series of binomial coefficients in which each term is a product of two binomial coefficients.

### MIDDLE TERM/TERMS

In the binomial expansion of $(x + y)^n$.

**WHEN $n$ IS ODD**

- There are $(n + 1)$ i.e. even terms in the expansion and hence two middle terms given by

$$t_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}}\, x^{\frac{n+1}{2}}\, y^{\frac{n-1}{2}} \quad \text{for} \quad r = \frac{n-1}{2}$$

and $t_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}}\, x^{\frac{n-1}{2}}\, y^{\frac{n+1}{2}}$ for $r = \frac{n+1}{2}$

**WHEN $n$ IS EVEN**

- There are odd terms in the expansion and hence only one middle term given by

$$t_{\frac{n}{2}+1} = {}^nC_{n/2}\, x^{n/2}\, y^{n/2} \quad \text{for} \quad r = \frac{n}{2}$$

### NUMERICALLY GREATEST TERM

- In the expansion of $(1 + x)^n$, calculate

$$m = (n+1)\frac{|x|}{1+|x|}$$

- If $m$ is an integer, then there are two terms with numerically greatest magnitude, for $r = m$ and $r = m - 1$ i.e. $t_{m+1}$ and $t_m$.
- If $m$ is not an integer, then there is only one numerically greatest tern, for $r = [m]$ i.e. $t_{[m]+1}$, where $[m]$ is the greatest integer function of $m$.
- In general for the expansion

$$(a + x)^n \text{ or } a^n\left(1+\frac{x}{a}\right)^n$$

we can consider $m = (n+1)\frac{|x/a|}{1+|x/a|}$.

### BINOMIAL THEOREM FOR POSITIVE INDEX

- For a positive integer $n$, $(x + y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \ldots + {}^nC_n x^0 y^n$ where ${}^nC_r = \frac{n!}{(n-r)!\,r!}$ for $r = 0, 1, 2, \ldots, n$ are called binomial coefficient.

**GENERAL TERM**

- The general term in the expansion of $(x + y)^n$ is $(r + 1)^{th}$ term, given by $t_{r+1} = {}^nC_r x^{n-r} y^r$ where $r = 0, 1, 2 \ldots n$.
- Every term in the expansion is of nth degree in variables $x$ and $y$.
- The total number of terms in the expansion is $n + 1$.
- Binomial theorem can also be expressed as

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r.$$

- The binomial coefficients of the expansion equidistant from the beginning and the end are equal. In other words ${}^nC_r = {}^nC_{n-r}$.
- The ratio of $(r + 1)^{th}$ coefficient to $r^{th}$ coefficient is $\frac{n-r+1}{r}$

### BINOMIAL THEOREM FOR ANY INDEX

- If $n \in \mathbf{R}$, $-1 < x < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots\infty$$

- In the expansion if $n \in \mathbf{R}$ and $n > 0$ then $-1 \le x < 1$.
- ${}^nC_r$ can not be used because it is defined only for natural number.
- If $x$ be so small then its square and higher powers may be neglected, then approximate value of $(1 + x)^n = 1 + nx$.

### MULTINOMIAL EXPANSION

- If $n \in \mathbf{N}$, then the general term of the multinomial expansion $(x_1 + x_2 + x_3 + \ldots + x_k)^n$ is

$$\frac{n!}{a_1!\,a_2!\,a_3!\ldots a_k}\, x_1^{a_1} \cdot x_2^{a_2} \cdot x_3^{a_3} \ldots x_k^{a_k},$$

where $a_1 + a_2 + a_3 + \ldots + a_k = n$
and $0 \le a_i \le n$, $i = 1, 2, 3, \ldots, k$.
and the number of terms in the expansion are ${}^{n+k-1}C_{k-1}$.

# EXERCISE – I

## IIT-SINGLE CHOICE CORRECT

**1.** If the coefficient of $x^6$ in $\left(x^3+\frac{k}{x}\right)^6$ is 160, then $k$ is equal to

(a) 3 (b) 4
(c) 2 (d) none of these

**2.** The largest coefficient in the expansion of $(1+x)^{24}$ is

(a) ${}^{24}C_2$ (b) ${}^{24}C_{13}$
(c) ${}^{24}C_{12}$ (d) ${}^{24}C_{11}$

**3.** If the coefficients of $r^{th}$ and $(2r+5)th$ terms of the expansion of $(1+x)^{15}$ are equal, then $r$ is equal to

(a) 4 (b) 6
(c) 7 (d) 3

**4.** If the $r$th term in the expansion of $\left(x^2-\frac{1}{x^3}\right)^7$ contains $x^4$, then $r$ is equal to

(a) 3 (b) 4
(c) 2 (d) none of these

**5.** The middle term in the expansion of $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^{10}$ is

(a) ${}^{10}C_5 x^{5/2}$ (b) ${}^{10}C_5 x^{-5/2}$
(c) ${}^{10}C_5$ (d) $-{}^{10}C_5$

**6.** If the coefficient of $x^6$ and $x^5$ in the expansion of $\left(3+\frac{x}{4}\right)^n$ are equal, then $n$ is equal to

(a) 17 (b) 47
(c) 77 (d) 67

**7.** If $\left(1+x-2x^2\right)^4=1+a_1x+a_2x^2+......+a_8x^8$, then the expression $a_2+a_4+a_6+a_8$ is equal to

(a) 8 (b) 7
(c) –8 (d) 16

**8.** In the expansion of $(1+x)^{50}$, the sum of the coefficients of odd powers of $x$ is

(a) 0 (b) $2^{49}$
(c) $2^{50}$ (d) $2^{51}$

**9.** The 5th term from the end in the expansion of $\left(3x - \frac{1}{x^2}\right)^{10}$ is

(a) $\frac{17010}{x^{10}}$ (b) $\frac{17010}{x^8}$

(c) $\frac{210}{x^8}$ (d) none of these

**10.** The sum ${}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + .... + {}^{20}C_{10}$ is equal to

(a) $2^{20} + \frac{20\,!}{(10\,!)^2}$ (b) $2^{19} - \frac{1}{2}.\frac{20\,!}{(10\,!)^2}$

(c) $2^{19} + {}^{20}C_{10}$ (d) none of these

**11.** If in the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of the 7th term from the beginning and the 7th term from the end is $\frac{1}{6}$, then $n$ is equal to

(a) 12 (b) 15
(c) 16 (d) 9

**12.** The greatest term in the expansion of $(2x + 5y)^{13}$, when $x = 10$, $y = 2$, is

(a) ${}^{13}C_5.20^8.10^5$ (b) ${}^{13}C_6.20^7.10^4$
(c) ${}^{13}C_4.20^9.10^4$ (d) none of these

**13.** The greatest integer less than or equal to $(\sqrt{2} + 1)^6$ is

(a) 196 (b) 197
(c) 198 (d) 199

**14.** The coefficient of the term independent of $x$ in the expansion of $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$

(a) $\frac{5}{4}$ (b) $\frac{7}{4}$

(c) $\frac{9}{4}$ (d) none of these

**15.** The sum to $(n + 1)$ terms of the series $\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \frac{C_3}{5} + ......$ is

(a) $\frac{1}{n+1}$ (b) $\frac{1}{n+2}$

(c) $\frac{1}{n(n+1)}$ (d) none of these

**16.** The coefficient of $x^{10}$ in the expansion of $(1 + x^2 - x^3)^8$ is

(a) 476 (b) 496
(c) 506 (d) 528

**17.** The total number of terms in $(x + y)^{200} + (x - y)^{200}$ after simplification is

(a) 202 (b) 101
(c) 100 (d) 201

**18.** The term independent of $x$ in $\left(x^3 - \frac{1}{x^2}\right)^5$ is

(a) –10 (b) 10
(c) –20 (d) 20

**19.** If the binomial coefficients of the second, third and fourth terms in the expansion of $(1+x)^n$ are in A.P., then $n$ is equal to

(a) 8 (b) 7
(c) 6 (d) 3

**20.** The number of terms in the expansions of $(1+2x+x^2)^{20}$ is

(a) 21 (b) 20
(c) 41 (d) 40

**21.** The greatest term in the expansion of $\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$ is

(a) ${}^{20}C_7 \frac{1}{27}$ (b) $-{}^{20}C_7 \frac{1}{27}$
(c) $\frac{1}{27}$ (d) none of these

**22.** The coefficient of the term independent of $x$ in the expansion of $\left(x^{1/6} - x^{-1/3}\right)^9$ is

(a) $-84$ (b) 8.4
(c) 48 (d) 84

**23.** The sum of the coefficients in the expansion of $(1-2x+5x^2)^n$ is '$a$' and the sum of coefficients in the expansion of $(1+x)^{2n}$ is '$b$', then

(a) $a = b$ (b) $a = b^2$
(c) $a^2 = b$ (d) $ab = 1$

**24.** The sum of the last ten coefficients in the expansion of $(1+x)^{19}$ is equal to

(a) $2^{18}$ (b) $2^{19}$
(c) $2^{18} - {}^{19}C_{10}$ (d) none of these

**25.** The sum $\frac{1}{2}\,{}^{10}C_0 - {}^{10}C_1 + 2.\,{}^{10}C_2 - 2^2.\,{}^{10}C_3 + \ldots\ldots + 2^9.\,{}^{10}C_{10}$ is

(a) $\frac{1}{2}$ (b) 0
(c) $\frac{1}{2}.3^{10}$ (d) none of these

# EXERCISE – II

## IIT-JEE – SINGLE CHOICE CORRECT

**1.** The value of $\frac{C_1}{2}+\frac{C_3}{4}+\frac{C_5}{6}+......$ is equal to

(a) $\frac{2^n+1}{n-1}$ (b) $\frac{2^n}{n+1}$

(c) $\frac{2^n+1}{n+1}$ (d) $\frac{2^n-1}{n+1}$

**2.** The coefficient of $x^{33}$ in the expansion of $\sum_{r=0}^{50} {}^{50}C_r (x-4)^{50-r} 3^r$ is

(a) ${}^{50}C_{33}$ (b) ${}^{50}C_{17}$

(c) $-{}^{50}C_{17}$ (d) none of these

**3.** The greatest coefficient in the expansion of $(1+x)^{10}$ is

(a) $\frac{10\,!}{5\,!\;6\,!}$ (b) $\frac{10\,!}{(5\,!)^2}$

(c) $\frac{10\,!}{5\,!\,7\,!}$ (d) none of these

**4.** The coefficient of $a_1^2a_2a_3$ in the expansion of $(a_1+a_2+a_3)^4$ is

(a) 21 (b) 12

(c) 24 (d) 42

**5.** $3^{51}$ when divided by 9 leaves the remainder

(a) 3 (b) 6

(c) 5 (d) 0

**6.** If the sum of the binomial coefficients in the expansion of $(x+y)^n$ is 1024, then the greatest binomial coefficient occurs in the $r^{th}$ term where $r$ is equal to

(a) 3 (b) 6

(c) 7 (d) 4

**7.** If $a_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n}\frac{r}{{}^nC_r}$ equals

(a) $(n-1)\,a_n$ (b) $n\,a_n$

(c) $\frac{1}{2}n\,a_n$ (d) none of these

**8.** The expression $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$ is a polynomial of degree

(a) 5 (b) 6

(c) 7 (d) 8

**9.** If $C_r$ stands for ${}^nC_r$ and $p$ and $q$ are any two numbers, then $p \cdot C_0 - (p+q)C_1 + (p+2q)C_2 - \ldots$ to $(n+1)$ terms is equal to

(a) $p/2^n$ (b) $pn$

(c) 0 (d) $q/2^n$

**10.** If ${}^nC_r$ is denoted as $\binom{n}{r}$ then for $2 \le r \le n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$

(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$

(c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$

**11.** The value of term independent of $x$ in the expansion of

$$(1+x+2x^3)\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9 \text{ is}$$

(a) $\frac{54}{17}$ (b) $\frac{12}{7}$

(c) $\frac{17}{54}$ (d) none of these.

**12.** If $\sum_{r=0}^{n} \left(\frac{r+2}{r+1}\right) {}^nC_r = \frac{2^8 - 1}{6}$, then $n$ is

(a) 8 (b) 4

(c) 6 (d) 5

**13.** The value of $\frac{1}{n!} + \frac{1}{2!(n-2)!} + \frac{1}{4!(n-4)!} + \ldots\ldots$ is

(a) $\frac{2^{n-2}}{(n-1)!}$ (b) $\frac{2^{n-1}}{n!}$

(c) $\frac{2^n}{n!}$ (d) none of these

**14.** The number of rational terms in the expansion of $\left(1+\sqrt{2}+\sqrt[3]{3}\right)^6$ is

(a) 6 (b) 7

(c) 5 (d) 8

**15.** Let $f(n) = 10^n + 3.4^{n+2} + 5$, $n \in N$. The greatest value of the integer which divides $f(n)$ for all $n$ is

(a) 27 (b) 9

(c) 3 (d) none of these

**16.** The value of ${}^{30}C_0\ {}^{30}C_{10} - {}^{30}C_1\ {}^{30}C_{11} + \ldots\ldots + {}^{30}C_{20}\ {}^{30}C_{30}$ is

(a) ${}^{60}C_{20}$ (b) ${}^{30}C_{10}$

(c) ${}^{60}C_{30}$ (d) ${}^{40}C_{30}$

**17.** If $x=\left(7+4\sqrt{3}\right)^{2n}=[x]+f$, then $x(1-f)$ is equal to

(a) 1 (b) 0
(c) –1 (d) even integer

**18.** If ${}^{n-1}C_r={}^{n}C_{r+1}(k^2-3)$, then $k$ lies in the interval

(a) $[-\sqrt{3},\ \sqrt{3}]$ (b) $(\sqrt{3},\ 2]$
(c) $[0,\ \sqrt{3}]$ (d) $(\sqrt{3},\ 2)$

**19.** The coefficient of $t^{24}$ in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is

(a) ${}^{12}C_6+3$ (b) ${}^{12}C_6+1$
(c) ${}^{12}C_6$ (d) ${}^{12}C_6+2$

**20.** $\sum_{r=0}^{n} r(n-r)\left({}^{n}C_r\right)^2$ is equal to

(a) $n^2\,.\,{}^{2n-1}C_n$ (b) $n^2\,.\,{}^{2n}C_{n-1}$
(c) $n^2\,.\,{}^{2n-1}C_{n+1}$ (d) $n^2\,.\,{}^{2n-2}C_n$

## ONE OR MORE THAN ONE CHOICE CORRECT

**1.** The term independent of $x$ in the expansion of $(1+x)^n\,.\left(1-\frac{1}{x}\right)^n$ is

(a) 0, if $n$ is odd (b) $(-1)^{\frac{n-1}{2}}\,.\,{}^{n}C_{\frac{n-1}{2}}$, if $n$ is odd
(c) $(-1)^{n/2}\,.\,{}^{n}C_{n/2}$, if $n$ is even (d) none of these

**2.** Let $n\in N$. If $(1+x)^n=a_0+a_1x+a_2x^2+......+a_nx^n$ and $a_{n-3},a_{n-2},a_{n-1}$ are in A.P., then

(a) $a_1, a_2, a_3$ are in A.P. (b) $a_1, a_2, a_3$ are in H.P.
(c) $n = 7$ (d) $n = 14$

**3.** In the expansion of $(x+y+z)^{25}$

(a) every term is of the form ${}^{25}C_r\,.\,{}^{r}C_k\,.x^{25-r}.y^{r-k}.z^k$
(b) the coefficient of $x^8y^9z^9$ is 0
(c) the number of terms is 325
(d) none of these

**4.** If $(1+2x+3x^2)^{10}=a_0+a_1x+a_2x^2+....+a_{20}x^{20}$, then

(a) $a_1=20$ (b) $a_2=210$
(c) $a_1=10$ (d) $a_2=105$

**5.** If the second, third and fourth terms in the expansion of $(a+b)^n$ are 135, 30 and $\frac{10}{3}$ respectively, then

(a) $a=3$ (b) $b=\frac{1}{3}$

(c) $n=5$ (d) $n=7$

**6.** If $(1+x+x^2)^{50}=a_0+a_1x+a_2x^2+....+a_{100}x^{100}$, then

(a) $a_0+a_2+....+a_{100}=\frac{3^{50}+1}{2}$ (b) $a_1+a_3+...+a_{99}=\frac{3^{50}-1}{2}$

(c) ${}^{50}C_0a_0-{}^{50}C_1a_1+{}^{50}C_2a_2-.....+{}^{50}C_{50}=0$ (d) $a_1=1$

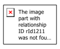
**7.** The coefficient of $x$ in the expansion of $E=1+(1+x)+(1+x)^2+....+(1+x)^n$ is

(a) ${}^{n+2}C_2$ (b) ${}^{n+2}C_n$

(c) ${}^{n+1}C_2$ (d) ${}^{n+1}C_{n-1}$

**8.** If $m$ is a positive integer, then $[(\sqrt{3}+1)^{2m}]+1$, where $[x]$ denotes the greatest integer $\le x$, is divisible by

(a)  (b) $2^{m+2}$

(c) $2^{m+1}$ (d) 

**9.** $(1+2x+2x^2)^n=a_0+a_1x+a_2x^2+....+a_{2n}x^{2n}$, then $a_0a_{2n}-a_1a_{2n-1}+....+a_{2n}a_0$ is equal to

(a) 0 if $n$ is odd (b) ${}^{2n}C_n$

(c) $2^n\ {}^nC_{\frac{n}{2}}$ if $n$ is even (d) $2^n\ {}^{2n}C_n$ if $n$ is even

**10.** The largest coefficient term in the expansion of $(3+2x)^{50}$, where $x=\frac{1}{5}$ is

(a) 5th (b) 51st

(c) 7th (d) 6th

# EXERCISE – III

## MATCH THE FOLLOWING

**Note:** Each statement in column – I has only one match in column - II

**1.**

| Column I | Column II |
|---|---|
| **I.** The sum of the coefficient of $x^{2r}, r = 1, 2, 3, ......$ in the expansion of $(1+x)^n$ is | **A.** $n.2^{2n-1}$ |
| **II.** $1.{}^nC_1 + 2.{}^nC_2 + 3.{}^nC_3 + ...... + n.{}^nC_n$ is equal to | **B.** $2^{2n+2}$ |
| **III.** If $3.{}^nC_0 + 3^2.\frac{{}^nC_1}{2} + 3^3.\frac{{}^nC_2}{3} + ...... + 3^{n+1}.\frac{{}^nC_n}{n+1} = \frac{\lambda - 1}{n+1}$, then $\lambda$ is | **C.** $n.2^{n-1}$ |
| **IV.** The sum $\sum_{r=1}^{n} r\ {}^{2n}C_r$ is equal to | **D.** $2^{n-1} - 1$ |

## REASONING TYPE

***Directions: Read the following questions and choose***

***(A) If both the statements are true and statement-2** is the correct explanation of* ***statement-1.***

***(B) If both the statements are true but statement-2** is not the correct explanation of* ***statement-1.***

***(C) If statement-1** is True and* ***statement-2** is False.*

***(D) If statement-1** is False and* ***statement-2** is True.*

**1.** **Statement-1:** The term independent of $x$ in the expansion of $\left(x + \frac{1}{x} + 2\right)^{2m}$ is $\frac{(4m)!}{(2m!)^2}$.

**Statement-2:** The coefficient of $x^b$ in the expansion of $(1+x)^n$ is .

(a) A (b) B (c) C (d) D

**2.** **Statement-1:** If $n$ is an odd prime then integral part of $[(\sqrt{5}+2)^n] - 2^{n+1}$ (where $[x]$ represents greatest integer $\leq x$) is divisible by 20 $n$.

**Statement-2:** If $n$ is prime then ${}^nC_1, {}^nC_2, {}^nC_3, ...... {}^nC_{n-1}$ must be divisible by $n$.

(a) A (b) B (c) C (d) D

**3.** **Statement-1:** Any positive integral power of $(\sqrt{2} - 1)$ can be expressed as $\sqrt{N} - \sqrt{N-1}$ for some natural number $N \geq 1$.

**Statement-2:** Any positive integral power of $\sqrt{2} - 1$ can be expressed as $A + B\sqrt{2}$ where $A$ and $B$ are integers.

(a) A (b) B (c) C (d) D

**4.** **Statement-1:** If $S_n = {}^nC_1 - \frac{{}^nC_2}{2} + \frac{{}^nC_3}{3} - ...... + \frac{(-1)^{n-1}\ {}^nC_n}{n}$ and $T_n = 1 + \frac{1}{2} + \frac{1}{3} + ..... + \frac{1}{n}$,

then $S_n = T_n$ for all $n$.

**Statement-2:** $S_{n+1} - S_n = T_{n+1} - T_n$.

(a) A (b) B (c) C (d) D

**5.** **Statement-1:** If $n$ is even, then ${}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + ........ + {}^{2n}C_{n-1} = 2^{2n-1}$.

**Statement-2:** ${}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + ........ + {}^{2n}C_{n-1} = 2^{2n-2}$.

(a) A (b) B (c) C (d) D

## LINKED COMPREHENSION TYPE

For $k, n \in N$, we define

$B(k,n) = 1.2.3.....k + 2.3.4...(k+1) + ... + n(n+1)...(n+k-1), S_0(n) = n$

and $S_k(n) = 1^k + 2^k + ... + n^k$

To obtain value of $B(k,n)$, we rewrite $B(k,n)$ as follows:

$$B(k,n) = k!\left[\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + ... + \binom{n+k-1}{k}\right] = k!\binom{n+k}{k+1} = \frac{n(n+1)........(n+k)}{k+1},$$

where $\binom{p}{r} = {}^pC_r$ and ${}^pC_r = \frac{p!}{r!(p-r)!}$

**1.** $S_2(n) + S_1(n)$ equals

(a) $B(2,n)$ (b) $\frac{1}{2}B(2,n)$ (c) $\frac{1}{6}B(2,n)$ (d) none of these

**2.** $S_3(n) + 3S_2(n)$ equals

(a) $B(3,n)$ (b) $B(3,n) - 2B(2,n)$

(c) $B(3,n) - 2B(1,n)$ (d) $B(3,n) + 2B(1,n)$

**3.** If $(1+x)^p = 1 + {}^pC_1x + {}^pC_2x^2 + ...... + {}^pC_px^p$, $p \in N$, then

$\binom{k+1}{1}S_k(n) + \binom{k+1}{2}S_{k-1}(n) + ... + \binom{k+1}{k}S_1(n) + \binom{k+1}{k+1}S_0(n)$ equals

(a) $(n+1)^k$ (b) $(n+1)^{k+1} - 1$

(c) $n^k - (n-1)^k$ (d) $(n+1)^k - (n-1)^k$

# EXERCISE – IV

## SUBJECTIVE PROBLEMS

**1.** If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^{2n}$ are in A.P., show that $2n^2 - 9n + 7 = 0$.

**2.** If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, then prove that

$$\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots\ldots + n\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}.$$

**3.** Find the value of $x$ such that the sixth term in the expansion of

$$\left\{2^{\log_2 \sqrt{3^{x-1}-2}} + 2^{\log_2 (3^{x-1}-2)^{1/5}}\right\}^7$$ is equal to 21.

**4.** Show that $\left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \left({}^{2n}C_2\right)^2 - \ldots + \left({}^{2n}C_{2n}\right)^2 = (-1)^n\ {}^{2n}C_n$.

**5.** Show that $C_0\ {}^{2n}C_n - C_1\ {}^{2n-2}C_n + C_2\ {}^{2n-4}C_n - \ldots = 2^n$.

**6.** For any positive integers, $(m \le n)$ let $\binom{n}{m} = {}^nC_m$. Prove that

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \ldots + \binom{m}{m} = \binom{n+1}{m+1}.$$ Hence or otherwise, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \ldots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}.$$

**7.** Prove that $C_0^n - C_1^n + C_2^n - \ldots + (-1)^{m-1}\ {}^nC_{m-1} =$
$\{(-1)^{m-1}(n-1)(n-2)\ldots(n-m+1)\}/(m-1)!$.

**8.** Prove that $\dfrac{3!}{2(n+3)} = \displaystyle\sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{{}^{r+3}C_r}$.

**9.** Show that

$$C_1^2 + \frac{1+2}{2}C_2^2 + \frac{1+2+3}{3}C_3^2 + \ldots\ldots\ldots\ldots \text{ upto } n \text{ terms} = \frac{1}{2}\left[n.{}^{2n-1}C_n + {}^{2n}C_n - 1\right]$$

**10.** Prove that $\dfrac{{}^nC_0}{n} - \dfrac{{}^nC_1}{n+1} + \dfrac{{}^nC_2}{n+2} - \ldots\ldots + (-1)^n \dfrac{{}^nC_n}{2n} = \dfrac{1}{n\ .\ {}^{2n}C_n}$.

# ANSWERS

## EXERCISE – I

### IIT-SINGLE CHOICE CORRECT

| | | | | |
|---|---|---|---|---|
| 1. (c) | 2. (c) | 3. (a) | 4. (c) | 5. (d) |
| 6. (c) | 7. (b) | 8. (b) | 9. (b) | 10. (d) |
| 11. (d) | 12. (c) | 13. (b) | 14. (a) | 15. (d) |
| 16. (a) | 17. (b) | 18. (a) | 19. (b) | 20. (c) |
| 21. (a) | 22. (a) | 23. (a) | 24. (a) | 25. (a) |

## EXERCISE – II

### IIT-JEE – SINGLE CHOICE CORRECT

| | | | | |
|---|---|---|---|---|
| 1. (d) | 2. (c) | 3. (b) | 4. (b) | 5. (d) |
| 6. (b) | 7. (c) | 8. (c) | 9. (c) | 10. (d) |
| 11. (c) | 12. (d) | 13. (b) | 14. (b) | 15. (b) |
| 16. (b) | 17. (a) | 18. (b) | 19. (d) | 20. (d) |

### ONE OR MORE THAN ONE CHOICE CORRECT

| | | | | |
|---|---|---|---|---|
| 1. (a, c) | 2. (a, c) | 3. (a, b) | 4. (a, b) | 5. (a, b, c) |
| 6. (a, b, c) | 7. (c, d) | 8. (a, c) | 9. (a, c) | 10. (c, d) |

## EXERCISE – III

### MATCH THE FOLLOWING

**1.** I – (D), II – (C), III – (B), IV – (A)

### REASONING TYPE

| | | | | |
|---|---|---|---|---|
| 1. (a) | 2. (a) | 3. (b) | 4. (a) | 5. (c) |

## LINKED COMPREHENSION TYPE

| | | |
|---|---|---|
| 1. (a) | 2. (c) | 3. (b) |

# EXERCISE – IV

## SUBJECTIVE PROBLEMS

**3.** $x = 2$