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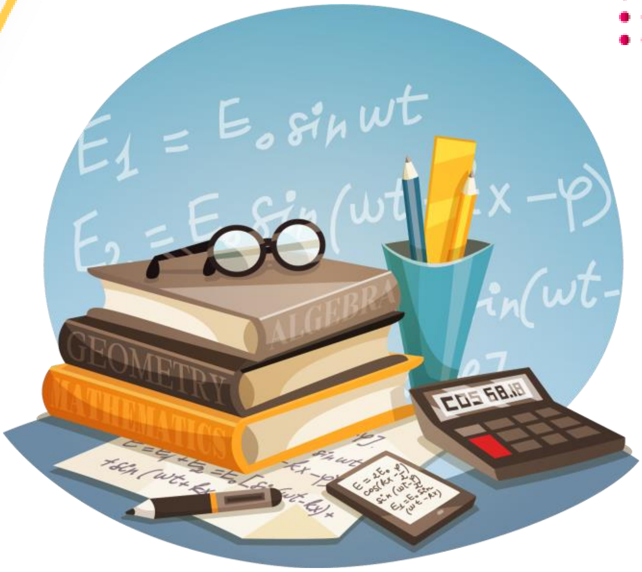
MATHS
NCERT - 11



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MATHS

PERMUTATIONS & COMBINATIONS

1 INTRODUCTION

A branch of mathematics where we count number of objects or number of ways of doing a particular job without actually counting them, is known as combinatorics and in this chapter we will deal with elementary combinatorics.

For example, if in a room there are five rows of chairs and each rows contains seven chairs, then without counting them we can say, total number of chairs is 35. We start this chapter with principle of product or fundamental principle of counting.

2 FUNDAMENTAL PRINCIPLE OF COUNTING

2.1 MULTIPLICATION PRINCIPLE OF COUNTING

If a job can be done in m ways, and when it is done in any one of these ways another job can be done in n, then both the jobs together can be done in mn ways. The rule can be extended to move number of jobs.

Illustration 1

Question: **Find the number of three digit numbers in which all the digits are distinct, odd and number is a multiple of 5.**

Solution: Here it is equivalent to completing three jobs of filling units, tens and hundred place. Now number of ways of filling unit place is only one i.e. 5. Now, four odd digits are left, hence ten's place can be filled in four ways and hundred's place in three ways.

$\therefore$ number of required three digit natural numbers is $1 \times 4 \times 3 = 12$.

Illustration 2

Question: **How many different 7 digit numbers are their sum of whose digits is even?**

Solution: Let us consider 10 successive seven digit numbers

$$a_1\, a_2\, a_3\, a_4\, a_5\, a_6\; 0,$$
$$a_1\, a_2\, a_3\, a_4\, a_5\, a_6\; 1,$$
$$a_1\, a_2\, a_3\, a_4\, a_5\, a_6\; 2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_1\, a_2\, a_3\, a_4\, a_5\, a_6\; 9,$$

where a_1, a_2, a_3, a_4, a_5, a_6 are some digits. We see that half of these 10 numbers i.e., 5 number have an even sum of digits.

The first digit a_1 can assume 9 different values and each of the digits a_2, a_3, a_4, a_5, a_6 can assume 10 different values.

The last digit a_7 can assume only 5 different values of which the sum of all digits is even.

$\therefore$ There are $9 \times 10^5 \times 5 = 45 \times 10^5$ seven digit numbers, the sum of whose digits is even.

2.2 ADDITION PRINCIPLE OF COUNTING

If a job can be done in m ways and another job can be done in n ways then either of these jobs can be done in $m + n$ ways. The rule can be extended to more number of jobs.

Illustration 3

Question: **How many three digit numbers xyz with $x < y$ and $z < y$ can be formed.**

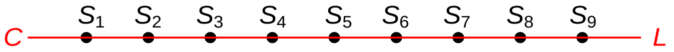
Solution: Obviously $2 \le y \le 9$. If $y = k$, then x can take values from 1 to $k - 1$ and z can take values from 0 to $k - 1$.

Thus required number of numbers $= \sum_{k=2}^{9}(k-1)(k) = 240$.

Illustration 4

Question: **A train is going from Cambridge to London stops at nine intermediate stations. Six persons enter the train during the journey with six different tickets. How many different sets of tickets they have had?**

Solution: For S_1, 9 different tickets available, one for each of the remaining 9 stations; similarly at S_2, 8 different tickets available; and so on.



$\therefore$ Hence, it is clear, that total number of different tickets

$= 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$

Hence, the six different tickets must be any six of these 45 ; and there evidently as many different sets of 6 tickets as there are combinations of 45 things taken 6 at a time tbus we can write if $^{45}C_6$.

3 PERMUTATIONS

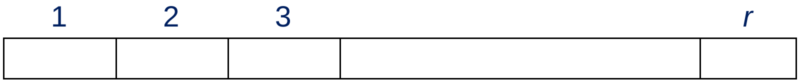
Each of different arrangement which can be made by taking some or all of a number of things is called a ***permutation***. It is assumed that

- the given things let us say there are n of them are all distinct, that is, no two are alike.
- the arrangement is one of placing one thing next to another as in a straight line; hence it is also known as linear permutation.
- In any arrangement any one thing is used only once. In other words, there is no repetition.

3.1 COUNTING FORMULAE FOR PERMUTATION

To find the value of ^nP_r

Suppose there are r blank spaces in a row and n different letters. The number of ways of filling up the blank spaces with n different letters is the number of ways of arranging n things r at a time, i.e. ^nP_r. It must be noted that each space has to be filled up with only one letter.



The first space can be filled in n ways. Having filled it, there are $n - 1$ letters left and therefore the second space can be filled in $n - 1$ ways. Hence the first two spaces can be filled in $n(n - 1)$ ways. When the first two spaces are filled, there are $n - 2$ letters left, so that the third space can be filled in $n - 2$ ways. Therefore the first three spaces can be filled in $n(n - 1)(n - 2)$ ways; proceeding like this, the r spaces can be filled in $n(n - 1)(n - 2) \ldots [n - (r - 1)]$ ways.

The number of permutations of n things taken r at a time is denoted as ^nP_r and its value is equal to: $^nP_r = n(n - 1)(n - 2) \ldots\ldots (n - r + 1)$

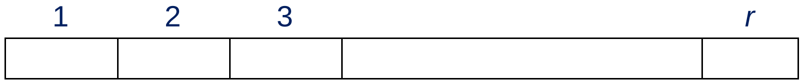
$= \dfrac{n!}{(n-r)!}$ (using factorial notation $n! = n(n - 1) \ldots\ldots$ 3.2.1.) where $0 \le r \le n$.

In particular

- The number of permutations of n different things taken all at a time $= {}^nP_n = n!$
- $^nP_0 = 1$, $^nP_1 = n$ and $^nP_{n-1} = {}^nP_n = n!$
- $^nP_r = n\,({}^{n-1}P_{r-1})$ where $r = 1, 2, \ldots\ldots n$

Illustration 5

Question: **Prove from definition that ${}^nP_r = n\,{}^{n-1}P_{r-1}$ and hence deduce the value of ${}^nP_r$.**



Solution: Suppose there are *n* different letters and a row of *r* blank spaces, each of which has to be filled up with one letter. The number of ways of filling up the blank spaces is the number of ways of arranging *n* things *r* at a time, i.e., ${}^nP_r$.

The first space can be filled in *n* ways. Having filled it, there are *n* – 1 letters left, and *r* – 1 spaces to be filled. By definition, the number of ways of filling up the *r* – 1 spaces with *n* – 1 letters is the number of ways of arranging *n* – 1 things *r* – 1 at a time, i.e. ${}^{n-1}P_{r-1}$.

∴ the number of ways of filling up the *r* blank spaces with *n* different letters is $n\cdot{}^{n-1}P_{r-1}$

i.e., $\quad {}^nP_r = n\cdot{}^{(n-1)}P_{(r-1)}$

$\therefore \quad {}^{n-1}P_{r-1} = (n-1)\cdot{}^{(n-2)}P_{(r-2)}$

$\therefore \quad {}^nP_r = n(n-1)\cdot{}^{(n-2)}P_{(r-2)} = n(n-1)(n-2)\cdot{}^{(n-3)}P_{(r-3)}$

Proceeding like this, we have

$$ {}^nP_r = n(n-1)(n-2)(n-3)\,.....\,(n-r+2)\cdot{}^{(n-r+1)}P_1 $$

Note that the last term is formed as ${}^{n-(r-1)}P_{r-(r-1)} = {}^{(n-r+1)}P_1$

Evidently, ${}^{(n-r+1)}P_1 = n-r+1$

$\therefore \quad {}^nP_r = n(n-1)(n-2)(n-3)\;....\;(n-r+2)(n-r+1).$

In particular,

$$ {}^nP_0 = 1, {}^nP_1 = n $$

$$ {}^nP_n = n(n-1)\;....\;1 = n\,! $$

i.e., the number of permutations of *n* things, taken all at a time is *n* ! .

Illustration 6

Question: **Show that ${}^nP_r = {}^{(n-1)}P_r + r\cdot{}^{(n-1)}P_{(r-1)}$.**

Solution:

$$ {}^{(n-1)}P_r + r\cdot{}^{(n-1)}P_{(r-1)} = \frac{(n-1)\,!}{(n-1-r)\,!} + \frac{r(n-1)\,!}{(n-r)\,!} $$

$$ = (n-1)\,!\left[\frac{1}{(n-1-r)\,!} + \frac{r}{(n-r)\,!}\right] $$

$$ = (n-1)\,!\,\frac{(n-r+r)}{(n-r)\,!} \qquad (\because (n-r)! = (n-r)(n-r-1)!) $$

$$ = \frac{n(n-1)\,!}{(n-r)\,!} = \frac{n\,!}{(n-r)\,!} $$

A common sense interpretation of the identity above is possible. The number of permutations of *r* things which may be made from *n* things, $r\,{}^{(n-1)}P_{r-1}$ contain one specified thing and ${}^{(n-1)}P_r$ do not contain that specified thing and these two together give ${}^nP_r$.

3.2 IMPORTANT RESULTS

Number of permutations under certain conditions:

- Number of permutations of n different things, taken r at a time, when a particular thing is to be always included in each arrangement is $r\,{}^{n-1}P_{r-1}$.
- Number of permutations of n different things, taken r at a time, when a particular thing is never taken in any arrangement is ${}^{n-1}P_r$.
- Number of permutations of n different things, taken r at a time, when m particular things are never taken in any arrangement is ${}^{n-m}P_r$.
- Number of permutations of n different things, taken all at a time, when m specified things always come together is $(m\,!)\,(n-m+1)\,!$.
- Number of permutations of n different things, taken all at a time, when m specified things never come together is $n! - m!\,(n-m+1)\,!$.

Illustration 7

Question: **Find the number of permutations of n different things taken r at a time so that two particular things are always included and are together.**

Solution: The two things can be combined as one unit. The remaining $(n-2)$ things may be permuted $(r-2)$ at a time in ${}^{n-2}P_{r-2}$ ways. In each arrangement of these $(r-2)$ things, are created

$(r-1)$ spaces in which the unit of two things can be placed. Further the two things in the unit may be interchanged. The number of permutations is

$$ {}^{n-2}P_{r-2}\cdot(r-1)\cdot 2 = \frac{2(r-1)\,(n-2)\,!}{(n-r)\,!} $$

Illustration 8

Question: **In how many ways can m persons entering a theatre, be seated in two rows each containing n seats with condition in the first row, no two sit in adjacent seats ($2m \le n$)?**

Solution: Suppose k persons are seated in the first row where $0 \le k \le m$. In the first row, there are n seats of which $n-k$ are vacant; and the k seats between any two of the vacants have positions in $(n-k+1)$ spaces.

$\therefore$ The number of seating arrangements for the 1st row = ${}^{(n-k+1)}P_k$

The number of seating arrangements is

$$ = \sum_{k=0}^{m} {}^{(n-k+1)}P_k\;.\;{}^{n}P_{m-k} $$

providing for none, one or…. All the m persons being seated in the first row.

3.3 PERMUTATION OF *n* DISTINCT OBJECT WHEN REPETITION IS ALLOWED

- The number of permutations of n different things taken r at time when each thing may be repeated any number of times is n^r.

In other words if a job can be completed in n ways and it is to be repeated r number of times then the total number of ways is n^r.

As an example the number of 5 digit numbers, in which digits are not repeated is $9({}^9p_4)$ while when repetition is allowed is 9×10^4.

Also if n distinct things are arrangement at r places when repetition is allowed, then the number of arrangements are $\underbrace{n \times n \times .... \times n}_{r\text{ times}} = n^r$.

Illustration 9

Question: **There are m men and n monkeys ($n > m$). If a man have any number of monkeys. In how many ways may every monkey have a master?**

Solution: The first monkey can select his master by m ways, and after that the second monkey can select his master again by m ways, so can the third. And so on, hence all monkeys can select master by = $m \times m \times m$ …. up to n times = $(m)^n$ ways.

Illustration 10

Question: **Show that the total number of permutations of n different things taken not more than 'r' at a time, each being allowed to repeat any number of times, is $n\dfrac{n^r - 1}{(n-1)}$.**

Solution: Number of a permutations taken one at a time = n

Number of permutations taken two at a time = n^2

.. ..

Number of permutations taken r at a time = n^r

∴ Total number of permutations

$$= n + n^2 + n^3 + \dots\dots\dots + n^r$$

$$= \frac{n(n^r - 1)}{(n-1)}$$

Illustration 11

Question: **In a steamer there are stalls for 12 animals, and there are cows, horses, and calves (not less than 12 of each) ready to be shipped; in how many ways can the shipload be made?**

Solution: The first stall can be occupied by a cow, a horse or a calf, i.e., it can be occupied in 3 ways. The second stall can also be occupied in 3 ways. Therefore the first 2 stalls can be occupied in 3^2 ways. Proceeding like this, it is clear that the 12 stalls can be occupied in 3^{12} ways, i.e., in 531441 ways.

3.4 ARRANGEMENT OF *n* THINGS WHEN ALL ARE NOT DISTINCT

- If given n things are not all distinct, then it is possible that few many be of one kind, and few others may be of second kind, etc. In such case, the number of permutations of n things taken all at a time, where p are alike of one kind, q are alike of second kind and r are alike of third kind and the rest $n - (p + q + r)$ are all distinct is given by

$$\frac{n!}{p!\,q!\,r!} \quad (p + q + r \le n)$$

Illustration 12

Question: **Find the number of ways in which we can arrange four letters of the word MATHEMATICS.**

Solution: The letters of the word MATHEMATICS are (M, M), (A, A), (T, T), H, E, I, C and S, making eight distinct letters. We can choose four out of them in ${}^8C_4$ = 70 ways, and arrange each of these sets of four in 4! = 24 ways, yielding (70) (24) = 1680 arrangements. Second, we can choose one pair from among the three identical letter pairs, and two distinct letters out of the remaining seven in $({}^3C_1)\,({}^7C_2) = (3)\,[(7 \times 6)/2]$ = 63 ways. The letters so obtained can be arranged in 4!/2! = 12 ways, so the number of arrangements in this case is (63) (12) = 756. Finally, we can choose two pairs out of the three identical letter pairs. This can be done in ${}^3C_2$ = 3 ways and the letters obtained can be arranged in 4!/2!2! = 6 ways, so that the number of arrangements in this last case is (3) (6) = 18. Hence the total number of arrangements is 1680 + 756 + 18 = 2454.

4 CIRCULAR PERMUTATIONS

In the event of the given n things arranged in a circular or even elliptical permutation – and in this case the first and the last thing in the arrangement are indistinguishable – the number of permutations is $(n-1)\,!$.

For example is 20 persons are circularly arranged, the number of arrangements is 19 !.

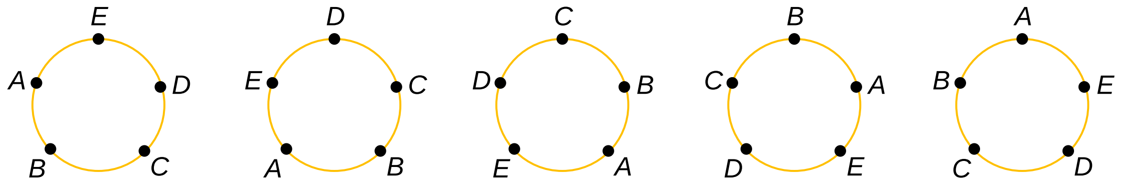
However, in the case of circular permutations wherein clockwise and anticlockwise orders need not be differentiated – as in the case of differently coloured beads or different flowers made into a garland, the number of permutations is $\frac{(n-1)\,!}{2}$ where the number of beads or flowers is taken as n. In details the concept is explained as follows:

4.1 ARRANGEMENTS AROUND A CIRCULAR TABLE

Consider five persons A, B, C, D, E be seated on the circumference of a circular table in order which has no head now, shifting A, B, C, D, E one position in anticlockwise direction we will get arrangements as shown in following figure:

We observe that arrangements in all figures are different.

Thus, the number of circular permutations of n different things taken all at a time is $(n-1)!$, if clockwise and anticlockwise orders are taken as different.

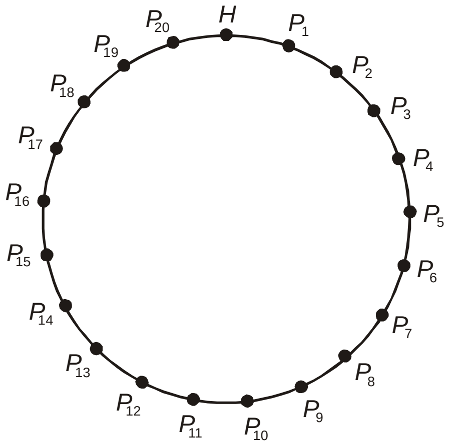


(i) (ii) (iii) (iv) (v)

Illustration 13

Question: **20 persons were invited to a party. In how many ways can they and the host be seated at a circular table? In how many of these ways will two particular persons be seated on either side of the host?**

Solution: **1st part:** Total persons on the circular table = 20 guest + 1 host = 21 they can be seated in (21 – 1)! = 20! ways.



2nd part: After fixing the places of three persons (1 host + 2 persons). Treating (1 host + 2 person) = 1 unit, so we have now {(remaining 18 persons + 1 unit) = 19} and the number of arrangement will be (19 – 1)! = 18! also these two particular person can be seated on either side of the host in 2! ways.

Hence, the number of ways of seating 21 persons on the circular table such that two particular persons be seated on either side of the host = 18! × 2! = 2 × 18!

4.2 ARRANGEMENTS OF BEADS OR FLOWERS (ALL DIFFERENT) AROUND A CIRCULAR NECKLACE OR GARLAND

Consider five beads A, B, C, D, E in a necklace or five flowers A, B, C, D, E in a garland etc. *If the necklace or garland on the left is turned over we obtain the arrangement on the right, i.e., anticlockwise and clockwise order of arrangement is not different we will get arrangements as follows:*

we see that arrangements in above figures are not different.

Then the number of circular permutations of n different things taken all at a time is $\frac{1}{2}(n-1)!$, if clockwise and anticlockwise orders are taken as not different.

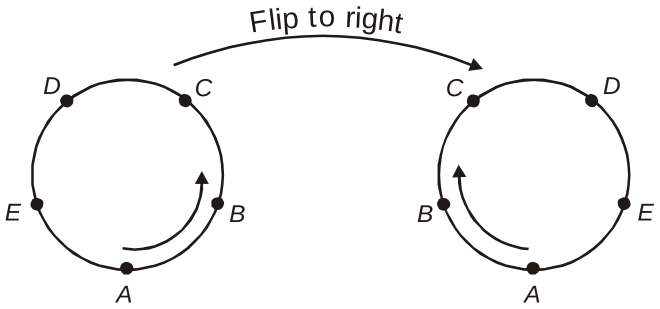


Illustration 14

Question: **Consider 21 different pearls on a necklace. How many ways can the pearls be placed in on this necklace such that 3 specific pearls always remain together?**

Solution: After fixing the places of three pearls. Treating 3 specific pearls = 1 units, so we have now 18 pearls + 1 unit = 19 and the number of arrangement will be (19 – 1)! = 18! also, the number of ways of 3 pearls can be arranged between themselves is 3! = 6. Since, there is no distinction between the clockwise and anticlockwise arrangements. So, the required number of arrangements = $\frac{1}{2}$ 18! . 6 = 3 (18 !).

4.3 NUMBER OF CIRCULAR PERMUTATIONS OF *n* DIFFERENT THINGS TAKEN *r* AT A TIME

CASE I : If clockwise and anticlockwise orders are taken as different, then the required number of circular permutations = $\frac{{}^nP_r}{r}$.

Illustration 15

Question: **In how many ways can 24 persons be seated round a table, if there are 13 seats?**

Solution: In case of circular table the clockwise and anticlockwise order are different, then the required number of circular permutations = $\frac{{}^{24}P_{13}}{13} = \frac{24!}{13 \times 11!}$.

CASE II : If clockwise and anticlockwise orders are taken as not different, then the required number of circular permutations = $\frac{{}^nP_r}{2r}$

Illustration 16

Question: **How many necklace of 12 beads each can be made from 18 beads of various colours?**

Solution: In the case of necklace there is not distinction between the clockwise and anticlockwise arrangements, then the required number of circular permutations

$$= \frac{{}^{18}P_{12}}{2 \times 12} = \frac{18!}{6! \times 24}$$

$$= \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13!}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 24} = \frac{119 \times 13!}{2}$$

Illustration 17

Question: **In how many ways 10 boys and 5 girls can sit around a circular table so that no two girls sit together.**

Solution: 10 boys can be seated in a circle in 9! ways. There are 10 spaces in between the boys, which can be occupied by 5 girls in ${}^{10}P_5$ ways. Hence total numbers of ways are 9! ${}^{10}P_5$.

Illustration 18

Question: **If n distinct objects are arranged in a circle, show that the number of ways of selecting three of these n things so that no two of them is next to each other is $\dfrac{n(n-4)(n-5)}{6}$.**

Solution: Let the n things be $x_1, x_2, \ldots\ldots, x_n$.

The first choice may be one of these n things; and, this is done in ${}^nC_1$ ways.

Suppose x_1 is the one chosen; the next two may be chosen – excluding x_1 and, the two next to x_1, namely x_2, x_n – from the remaining $(n-3)$ in ${}^{n-3}C_2$ ways.

Of these ${}^{n-3}C_2$ there are $(n-4)$ selections when the second two chosen are next to each other, like $x_3x_4, x_4x_5, \ldots., x_{n-2}x_{n-1}$

$\therefore$ the number of ways of selecting the second two after x_1 is chosen, so that the two are not next to each other is

$$ {}^{n-3}C_2 - (n-4) = \frac{(n-3)(n-4)}{1\cdot 2} - (n-4) = \frac{(n-4)(n-5)}{2} $$

The two objects can be relatively interchanged in 2 ways. Further the order of the choice of the three is not to be considered.

Hence the number of ways of choice of the three is

$$ \frac{n(n-4)(n-5)}{2}\,\frac{2!}{3!} = \frac{n(n-4)(n-5)}{6} $$

Illustration 19

Question: **2n persons are to be seated n on each side of a long table. $r(<n)$ particular persons desire to sit on one side; and $s(<n)$ other persons desire to sit on the other side. In how many ways can the persons be seated?**

Solution: For the side where r persons desire to sit, we need $(n-r)$ more persons. This $(n-r)$ may be chosen from $(2n-r-s)$ in ${}^{(2n-r-s)}C_{n-r}$ ways. Automatically the remaining $(n-s)$ person go to the other side where already there are s desirous of seating. Thus there are ${}^{(2n-r-s)}C_{n-r}$ ways of distributing n persons for each side providing for the restriction of r on one side and s on the other side.

n persons on each side can be permuted in n seats in $n!$ ways. The number of ways of seating the $2n$ persons, n on each side, is therefore

$$ {}^{(2n-r-s)}C_{n-r}\,(n!)^2 . $$

5 COMBINATIONS

Each of different grouping or selections that can be made by some or all of a number of given things without considering the order in which things are placed in each group, is called ***combinations***.

5.1 COUNTING FORMULAE FOR COMBINATIONS

The number of combinations (selections or groupings) that can be formed from n different objects taken r at a time is denoted by ${}^nC_r$ and its value is equal to

$$ {}^nC_r = \frac{n!}{(n-r)!\; r!} \qquad (0 \le r \le n) $$

as $\quad {}^nC_r = \dfrac{{}^nP_r}{r\,!}$

as in a permutation the arrangement of r selected objects out of n, is done in $r!$ ways and in combination arrangement in a group is not considered.

In particular

- ${}^nC_0 = {}^nC_n = 1$ i.e. there is only one way to select none or to select all objects out of n distinct objects.
- ${}^nC_1 = n$ There are n ways to select one thing out of n distinct things.
- ${}^nC_r = {}^nC_{n-r}$
  Therefore ${}^nC_x = {}^nC_y \Leftrightarrow x = y$ or $x + y = n$.
- If n is odd then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$.
- If n is even then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$.

Illustration 20

Question: **Prove that product of r consecutive positive integer is divisible by $r\,!$.**

Solution: Let r consecutive positive integers be (m), $(m + 1)$, $(m + 2)$, ….. , $(m + r - 1)$, where $m \in \mathbf{N}$.

$\therefore \quad$ Product $= m(m + 1)\,(m + 2)\,\ldots..\,(m + r - 1)$

$$= \frac{(m-1)!\;m(m+1)\;(m+2)\ldots..(m+r-1)}{(m-1)!}$$

$$= \frac{(m+r-1)!}{(m-1)!} = r\,!\,.\,\frac{(m+r-1)!}{r\,!\,(m-1)!}$$

which is divisible by $r\,!$ $\qquad$ ($\because {}^{m+r-1}C_r$ is natural number)

Illustration 21

Question: **Show that the number of triangles whose angular points are at the vertices of a given polygon of n sides but none of whose sides are the sides of the polygon is $\dfrac{n\,(n-4)\,(n-5)}{6}$.**

Solution: For any triangle to be possible, 3 of the n vertices are to be chosen. This can be done in ${}^nC_3$ ways. Of these there are n triangles with two sides as adjacent sides of the polygon – like side 1 and side 2; side 2 and side 3 etc, the third side of the triangle being the corresponding diagonal; and there are, with one side of the polygon as a side of the triangle, $(n - 4)$ triangles.

$\therefore \quad$ Required number of triangles $= {}^nC_3 - n - n\,(n - 4)$

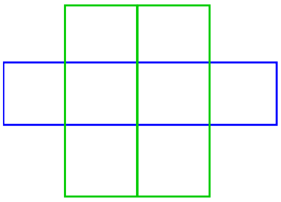
$$= \frac{n(n-1)\,(n-2)}{6} - n - n(n-4)$$

$$= \frac{n}{6}\{n^2 - 3n + 2 - 6 - 6n + 24\}$$

$$= \frac{n}{6}(n^2 - 9n + 20) = \frac{n(n-4)\,(n-5)}{6}$$

Illustration 22

Question: **Six X's have to be placed in spaces on the adjoining Figure so that each row contains at least one X. In how many different ways this can be done?**



Solution: If there be no restriction on the placing of the X's, the number of ways is ${}^8C_6 = {}^8C_2 = 28$. Of these there are two ways in which the X ;s can be placed; one, with the first row empty and the other with the third row empty. These two cases only do not satisfy the condition.

$\therefore$ the number of ways = 28 – 2 = **26**.

5.2 IMPORTANT RESULTS OF COMBINATIONS (SELECTIONS)

- The number of ways in which r objects can be selected from n distinct objects if a particular object is always included is ${}^{n-1}C_{r-1}$.
- The number of ways in which r objects can be selected from n distinct objects if a particular object is always excluded is ${}^{n-1}C_r$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always included is ${}^{n-m}C_{r-m}$.
- The number of ways in which r objects can be selected from n distinct objects if m particular objects are always excluded is ${}^{n-m}C_r$.

Illustration 23

Question: **A lady desires to give a dinner party for 8 guests. In how many ways can the lady select guests for the dinner from her 12 friends, if two of the guests will not attend the party together?**

Solution: The following three methods of approach are indicated.

(i) Number of ways of forming the party

$= {}^{12}C_8 - {}^{10}C_6$ since ${}^{10}C_6$ is the number of ways of making up the party with both the specified guests included.

= 495 – 210 = 285

(OR)

(ii) Number of ways of forming the party

= Number of ways of forming without both of them

\+ Number of ways of forming with one of them and without the other

$= {}^{10}C_8 + 2.{}^{10}C_7 = 45 + 240 = 285$

(OR)

(iii) Split the number of ways of forming the party

= those with one of the two (say A) + those without A

$= {}^{10}C_7 + {}^{11}C_8 = 120 + 165 = 285$

- The number of ways in which r objects can be selected from n objects if m particular objects are identical is $\sum_{r=0}^{r} {}^{n-m}C_r$ or $\sum_{r=r-m}^{r} {}^{n-m}C_r$ according as $r \le m$ or $r > m$.

Illustration 24

Question: **A bag contains 23 balls in which 7 are identical. Then find the number of ways of selecting 12 balls from bag.**

Solution: Here $n = 23$, $p = 7$, $r = 12$ $(r > p)$

Hence, required number of selections = $\sum_{r=5}^{12} {}^{16}C_r$

$= {}^{16}C_5 + {}^{16}C_6 + {}^{16}C_7 + {}^{16}C_8 + {}^{16}C_9 + {}^{16}C_{10} + {}^{16}C_{11} + {}^{16}C_{12}$

$= ({}^{16}C_5 + {}^{16}C_6) + ({}^{16}C_7 + {}^{16}C_8) + ({}^{16}C_9 + {}^{16}C_{10}) + ({}^{16}C_{11} + {}^{16}C_{12})$

$= {}^{17}C_6 + {}^{17}C_8 + {}^{17}C_{10} + {}^{17}C_{12}$ $(\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$

$= {}^{17}C_{11} + {}^{17}C_9 + {}^{17}C_{10} + {}^{17}C_{12}$ $(\because {}^nC_r = {}^nC_{n-r})$

$= ({}^{17}C_{11} + {}^{17}C_{12}) + ({}^{17}C_9 + {}^{17}C_{10})$

$= {}^{18}C_{12} + {}^{18}C_{10} = {}^{18}C_6 + {}^{18}C_8$

6 SELECTION FROM DISTINCT/IDENTICAL OBJECTS

6.1 SELECTION FROM DISTINCT OBJECTS

- The number of ways (or combinations) of selection from *n* distinct objects, taken at least one of them is

$$ {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1 $$

Logically it can be explained in two ways, as one can be selected in ${}^nC_1$ ways, two in ${}^nC_2$ ways and so on ……. and by addition principle of counting the total number of ways of doing either of the job is ${}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n$

Also, for every object, there are two choices, either selection or non-selection. Hence total choices are 2^n. But this also includes the case when none of them is selected. Therefore the number of selections, when atleast one is selected = $2^n - 1$.

Illustration 25

Question: **Find the number of ways in which we can put *n* distinct objects into two identical boxes so that no box remains empty.**

Solution: Let us first label the boxes 1 and 2. We can select at least one or at most (n–1) balls for box 1 in

$${}^nC_1 + {}^nC_2 + \ldots\ldots\ldots\ {}^nC_{n-1} \text{ ways}$$
$$= {}^nC_0 + {}^nC_1 + \ldots\ldots\ldots\ {}^nC_n - {}^nC_0 - {}^nC_n$$
$$= 2^n - 2 = 2(2^{n-1} - 1) \text{ ways.}$$

In this way box 2 is not empty. But since the boxes are identical the number of ways that no box remains empty is $\frac{1}{2} \times 2(2^n - 1) = 2^{n-1} - 1.$

Alternative solutions:

Let us first label the boxes 1 and 2. There are then to choices for each of the *n* objects; we can put it in the first box or in the second box. Therefore the number of choices for *n* distinct objects is $\underbrace{2 \times 2 \times \ldots. \times 2}_{n \text{ times}} = 2^n$.

Two of these choices correspond to either the first or the second box being empty. Thus there are $2^n - 2$ ways in which neither box is empty. If we now remove the labels from the boxes so that they become identical, this number must be divided by 2, yielding the answer $1/2\ (2^n - 2) = 2^{n-1} - 1$.

Illustration 26

Question: **Given five different green dyes, four different blue dyes and three different red dyes, how many combination of dyes can be chosen taking at least one green, one blue dye?**

Solution: Any one dye of a particular colour can be either chosen or not; and, thus there are 2 ways in which each one may be dealt with.

Number of ways of selection so that at least one green dye is included = $2^5 - 1 = 31$
(1 is subtracted to correspond to the case when none of the green dyes is chosen.)
A similar argument may be advanced in respect of other two colours also.
Number of combinations = $(2^5 - 1)(2^4 - 1)(2^3)$.

$$= 31 \times 15 \times 8 = 3720$$

6.2 SELECTION FROM IDENTICAL OBJECTS

1. The number of selections of r ($r \le n$) objects out of n identical objects is 1.
2. The number of ways of selections of at least one object out of n identical object is n.
3. The number of ways of selections of at least one out of $a_1 + a_2$ ……. $+ a_n$ objects, where a_1 are alike of one kind, a_2 are alike of second kind, and so on ……..a_n are alike of n^{th} kind, is
   $(a_1 + 1)(a_2 + 1)$ …….$(a_n + 1) - 1$.
4. The number of ways of selections of atleast one out of $a_1 + a_2 + a_3 +$ ……$+ a_n + k$ objects, where a_1 are alike of one kind, ……….a_n are alike of nth kind and k are distinct is
   $$(a_1 + 1)(a_2 + 1) \ldots\ldots(a_n + 1)\, 2^k - 1.$$

Illustration 27

Question: **Find the number of combinations that can be formed with 5 oranges, 4 mangoes and 3 bananas when it is essential to take**
(i) at least one fruit
(ii) one fruit of each kind.

Solution: Here 5 oranges are alike of one kind, 4 mangoes are alike of second kind and 3 bananas are alike of third kind

(i) The required number of combinations (when at least one fruit)

$$= (5 + 1)(4 + 1)(3 + 1)\,2^0 - 1$$
$$= 120 - 1 = 119$$

(ii) The required number of combinations (when one fruit of each kind)

$$= {}^5C_1 \times {}^4C_1 \times {}^3C_1 = 5 \times 4 \times 3 = 60.$$

7 DIVISORS OF A GIVEN NATURAL NUMBER

Let $n \in \mathbf{N}$ and $n = p_1^{\alpha_1}.\, p_2^{\alpha_2}.\, p_3^{\alpha_3} \ldots\ldots p_k^{\alpha_k}$, where $p_1, p_2, p_3, \ldots\ldots p_k$ are different prime numbers and $\alpha_1, \alpha_2, \alpha_3, \ldots , \alpha_k$ are natural numbers then:

- the total number of divisors of N including 1 and n is
  $$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1)$$
- the total number of divisors of n excluding 1 and n is
  $$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1) - 2$$
- the total number of divisors of n excluding exactly one out of 1 or n is
  $$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1) - 1$$
- the sum of these divisors is
  $$= (p_1^0 + p_1^1 + p_1^2 + \ldots\ldots + p_1^{\alpha_1})(p_2^0 + p_2^1 + p_2^2 + \ldots\ldots + p_2^{\alpha_2}) \ldots\ldots (p_k^0 + p_k^1 + p_k^2 + \ldots\ldots + p_k^{\alpha_k})$$
  (Use sum of G.P. in each bracket)
- the number of ways in which n can be resolved as a product of two factors is
  $$\frac{1}{2}(\alpha_1 + 1)(\alpha_2 + 1) \ldots\ldots (\alpha_k + 1), \text{ if } n \text{ is not a perfect square}$$
  $$\frac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1) \ldots\ldots (\alpha_k + 1) + 1], \text{ if } n \text{ is a perfect square}$$

- the number of ways in which is composite number *n* can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{k-1} where *k* is the number of different factors (or different primes) in *n*.

Illustration 28

Question: **If *n* = 10800, then find the**

(a) total number of divisors of *n*

(*b*) the number of even divisors

(*c*) the number of divisors of the form 4m + 2

(d) the number of divisors which are multiples of 15

Solution: $n = 10800 = 2^4 \times 3^3 \times 5^2$

Any divisor of *n* will be of the form $2^a \times 3^b \times 5^c$ where $0 \le a \le 4$, $0 \le b \le 3$, $0 \le c \le 2$.

For any distinct choices of *a*, *b* and *c*, we get a divisor of *n*

(a) total number of divisors = (4 + 1) (3 + 1) (2 + 1) = 60

(*b*) for a divisor to be even, *a* should be at least one. So total number of even divisors = 4(3 + 1) (2 + 1) = 48.

(*c*) $4m + 2 = 2(2m + 1)$. In any divisor of the form $4m + 2$, *a* should be exactly 1. So number of divisors of the form $4m + 2 = 1\ (3 + 1)\ (2 + 1) = 12$.

(d) A divisor of *n* will be a multiple of 15 if *b* is at least one and *c* is at least one. So number of such divisors = $(4+ 1) \times 3 \times 2 = 30$.

Illustration 29

Question: **Find the number of divisors of 428652000 excluding the number and unity. Find also the sum of the divisors.**

$$\mathbf{428652000 = 2^5 . 3^7 . 5^3 . 7^2}$$

Solution: Any divisor of the given number has to be a combination of the 2's (five); 3's (seven); 5's (three) and 7's (two).

There are 5 + 1 = 6 ways of selecting none or one or two etc., or all the 2's. Similar argument repeats for the other numbers.

The number of divisors = $6 \times 8 \times 4 \times 3 = 576$.

The includes 1 and the given number also.

Excluding these two, the number of divisors = **574**

With regard to the sum of the divisors

Any divisor is of the form $2^p 3^q 5^s 7^t$ where $0 \le p \le 5$; $0 \le q \le 7$; $0 \le s \le 3$ and $0 \le t \le 2$

Thus the sum of the divisors is

$$(1 + 2 + ... + 2^5)\,(1 + 3 + ... + 3^7)\,(1 + 5 + ... + 5^3)\,(1 + 7 + 7^2)$$

$$= (2^6 - 1)\,(3^8 - 1)\,(5^4 - 1)\,(7^3 - 1)/48$$

PROFICIENCY TEST–I

The following questions deal with the basic concepts of this section. Answer the following briefly. Go to the next section only if your score is at least 80%. Do not consult the Study Material while attempting the questions.

1. A conference hall has 10 gates. In how many ways can a person enter the hall through one gate and exit through different gate?

2. How many different signals can be made by 5 flags from 8 flags of different colors.

3. In how many ways selection can be done from three identical pencils and four identical pens?

4. How many words can be formed using letters of word “BRILLIANT”?

5. A child has 5 pockets and 4 marbels. In how many ways can the child put the marbels in his pockets?

6. In how many ways 23 different coloured beads be placed in the necklace so that 3 specific beads always remain together?

7. In how many ways combination of three coloured dyes can be chosen, out of five different green dyes, four different blue dyes and three different red dyes?

8. A compositor drops the letter of word ‘ADVERTISE’, picks them up and replaces them at random. How many possible mistakes can be made?

9. Find the domain of definition of the function $f(x) = {}^{7-x}P_{x-3}$ and the set of its values.

10. How many words can be formed out of the letters of the word COURAGE so that the vowels are in the odd places?

ANSWERS TO PROFICIENCY TEST–I

1. 90 ways

2. $^8P_5$

3. 19

4. $\dfrac{9\,!}{2\,!\;\;2\,!}$

5. 5^4

6. 3(20 !)

7. $(2^5 - 1)\ (2^4 - 1)\ (2^3 - 1)$

8. $\dfrac{9\,!}{2} - 1$

9. Domain {3, 4, 5}
Range {1, 3, 2}

10. 144

8 DIVISION OF DISTINCT OBJECTS INTO GROUPS

In the case of grouping we have the following. If $m + n + p$ things are divided into 3 groups one containing m, the second n and the third p things; number of groupings is ${}^{(m+n+p)}C_m \cdot {}^{(n+p)}C_n \cdot {}^{p}C_p$

$$= \frac{(m+n+p)!}{m!\,n!\,p!}$$ where m, n, p are distinct natural numbers.

If $m = n = p$ (say) then the number of groupings (unmindful of the order of grouping) is

$$\frac{3m!}{(m!)^3\,3!}$$

Thus if 52 cards be divided into four groups of 13 each, the number of groupings is

$$\frac{52!}{(13!)^4 \,.\, 4!}$$

On the other hand, when 52 cards are dealt 13 each to four persons, the number of ways in which this can be done is

$$\frac{52!}{(13!)^4}$$

As another example, if we consider the division of 52 cards into four groups, three groups containing each 16 and the fourth cards, the number of ways in which this can be done is

$$\frac{52!}{3!(16!)^3\,4!}$$

Note the 3 ! factor in the denominator. This is for the reason that there are only 3 equal groups.

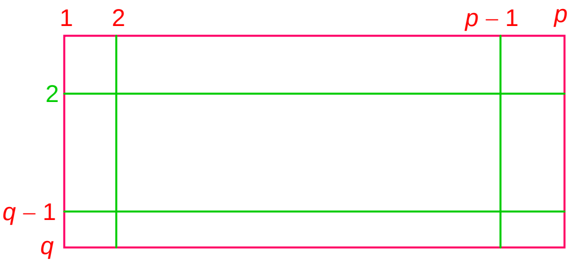
In general, the number of ways in which mn different things can be divided equally into m distinct groups is $\frac{(mn)!}{(n!)^m}$ when order of groups is important.

Whereas when order of groups is not important, then division into m equal group is done in $\frac{(mn)!}{m!\,(n!)^m}$ ways.

In general, we can write formula for grouping as factorial of total number of elements divided by product of factorial of number of elements in each group and product of factorial of number of groups having same number of elements, if any. Also formula for number of distribution is number of grouping multiplied with factorial of number of persons in which objects are distributed, divided by factorial of number of persons who got nothing, if any.

Illustration 30

Question: **A city has 'p' parallel roads running East-West and 'q' parallel roads running North-South. How many rectangles are formed with their sides along these roads? If the distance between every consecutive parallel road is the same, how many shortest possible routes are there to go from one corner of the city to the diagonally opposite corner?**



Solution: To form a rectangle one needs to take two roads from the 'p' parallel roads and two roads from the 'q' parallel roads. The number of rectangles thus formed = ${}^{p}C_2 \,.\, {}^{q}C_2$

$$= \frac{pq(p-1)(q-1)}{4}$$

Let the distance between any two parallel roads be one unit. In going from one corner to the diagonally opposite corner, one has to travel $(p - 1)$ units in the North-South direction and
$(q - 1)$ units in the East-West direction.

Totally therefore one has to travel a distance of $p + q - 2$ units of which $(p - 1)$ units are in one direction and $(q - 1)$ units are in the other direction. These displacements can be taken in any order. As such it is a problem of arranging $p + q - 2$ units of which $(p - 1)$ are of one kind and $(q - 1)$ are of second kind. Hence the number of ways in which this may be done is equal to division of $p + q - 2$ into two groups of $(p - 1)$ and $(q - 1)$, which is equal to

$$\frac{(p+q-2)!}{(p-1)!\,(q-1)!}$$

9 DIVISION OF IDENTICAL OBJECTS INTO GROUPS

The number of ways of division or distribution of n identical things into r different groups is ${}^{n+r-1}C_{r-1}$ or ${}^{n-1}C_{r-1}$ according as empty groups are allowed or not allowed.

Illustration 31

Question: **Find the number of ways in which 11 identical apples can be distributed among 6 children, so that each child receives at least one apple.**

Solution: First give one apple to each child. There are remaining 5 apples to be distributed among 6 children (so that each may receive any number of apples not exceeding five).

The number of ways required

$$= {}^{(11-1)}C_{6-1} = {}^{10}C_5 = 252$$

Alternatively, the number of ways of distribution is given by the coefficient of x^{11} in $(x + x^2 + \ldots + x^{11})^6$.

Illustration 32

Question: **How many positive integral solutions are there for the equation $x + y + z + w = 20$?**

Solution: Taking 20, the number on the right hand side of the equation, mark 20 dots.

$$\begin{array}{cccccccccccccccccccccccc} & & & & A & & & B & & & & & & & & & C & & & & & & \\ 1 & 2 & 3 & 4 & | & 5 & 6 & | & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & | & 15 & 16 & 17 & 18 & 19 & 20 \end{array}$$

Between these 20 dots, 19 spaces are created. Choose any three of them. This choice can be done in ${}^{19}C_3$ ways. One such choice is illustrated above by drawing the lines A, B, C and partitioning the 20 dots into 3 sets.

$x = 4$, number of dots before the line A

$y = 2$, number of dots between line A and line B

$z = 8$, number of dots between line B and line C

$w = 6$, number of dots after C.

Thus 4, 2, 8, 6 is one such.

10 ARRANGEMENT IN GROUPS

The number of ways of distribution and arrangement of n distinct things into r different groups is $n!\ ^{n+r-1}C_{r-1}$ or $n!\ ^{n-1}C_{r-1}$ according as empty groups are allowed or not allowed.

Illustration 33

Question: **In how many ways can three balls of different colours be put in 4 glass cylinders of equal width such that any glass cylinder may have either 0, 1, 2 or 3 balls ?**

Solution: There are four glass cylinders. Consider additionally (4 – 1) = 3 things; and, the number of ways is $^6C_3$ (corresponding to $^{(n+r-1)}C_{r-1}$) multiplied by 3! = 120.

11 METHODS OF INCLUSION EXCLUSION

If $A_1, A_2, \ldots\ldots\ldots, A_m$ are finite sets and $A = A_1 \cup A_2 \cup \ldots. \cup A_m$, then

$$n(A) = a_1 - a_2 + a_3 - a_4 + \ldots\ldots. + (-1)^{m+1} a_m$$

where $a_1 = n(A_1) + n(A_2) + \ldots\ldots + n(A_m)$

$$a_2 = \sum_{1 \le i < j \le m} n(A_i \cap A_j)$$

$$a_3 = \sum_{1 \le i < j < k \le m} n(A_i \cap A_j \cap A_k)$$

and so on.

Corollary (Sieve-Formula)

If A_1, A_2, ….. A_m are m subsets of a set A containing N elements, then $n(A_1' \cap A_2' \cap \ldots.. \cap A_m')$

$$= N - \sum_i n(A_i) + \sum_{1 \le i < j \le m} n(A_i \cap A_j) - \sum_{1 \le i < j < k \le m} n(A_i \cap A_j \cap A_k)$$

$$+ \ldots\ldots + (-1)^m n(A_1 \cap A_2 \cap \ldots\ldots \cap A_m)$$

11.1 DEARRANGEMENTS

It is rearrangement of objects such that no one goes to its original place.

If n things are arranged in a row, the number of ways in which they can be rearranged so that none of them occupies its original placed is $n!\left(1 - \frac{1}{1!} + \frac{1}{2!} \ldots.. + (-1)^n \frac{1}{n!}\right)$

It is denoted by $D_n = n! \sum_{r=0}^{n} (-1)^r \frac{1}{r!}$

Illustration 34

Question: **A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) at least two of them are in the wrong envelopes. (ii) all the letters are in the wrong envelopes.**

Solution: (i) The number of ways in which at least two of them are in the wrong envelopes

$$= \sum_{r=2}^{6} {}^nC_{n-r} D_r$$

$$= {}^nC_{n-2}D_2 + {}^nC_{n-3}D_3 + {}^nC_{n-4}D_4 + {}^nC_{n-5}D_5 + {}^nC_{n-6}D_6$$

Here n = 6

$$= {}^6C_4 . 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3 . 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) + {}^6C_2 . 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$+ {}^6C_1 . 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) + {}^6C_0 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

= 15 + 40 + 135 + 264 + 265 = 719.

(ii) The number of ways in which all letters be placed in wrong envelopes

$$= 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

$$= 720\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}\right)$$

$$= 360 - 120 + 30 - 6 + 1 = 265.$$

12 USE OF MULTINOMIALS

- If there are l objects of one kind, m objects of second kind, n objects of third kind and so on; then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \ldots.$) is the coefficient of x^r in the expansion of

$$(1 + x + x^2 + x^3 + \ldots.\ x^l)\ (1 + x + x^2 + \ldots.. + x^m)\ (1 + x + x^2 + \ldots.. + x^n)$$

Further if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \ldots.$) is the coefficient of x^r in the expansion of

$$(x + x^2 + x^3 + \ldots.\ x^l)\ (x + x^2 + x^3 + \ldots.. + x^m)\ (x + x^2 + x^3 + \ldots.. + x^n)$$

- If there are l objects of one kind, m object of second kind, n object of third kind and so on; then the number of possible arrangements/permutations of r objects out of these objects (i.e., $l + m + n + \ldots..$) is the coefficient of x^r in the expansion of

$$r!\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots.+\frac{x^l}{l!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots.+\frac{x^m}{m!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots.+\frac{x^n}{n!}\right)\ldots$$

Note: For use in problems of the above type the following Binomial expansions may be noted

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + \ldots + x^r + \ldots$$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + \ldots + (r+1)\,x^r + \ldots$$

$$\frac{1}{(1-x)^3} = (1-x)^{-3} = 1 + 3x + 6x^2 + \ldots + \frac{(r+1)(r+2)}{1\cdot 2}x^r + \ldots..$$

$$\frac{1}{(1-x)^4} = (1-x)^{-4} = 1 + 4x + 10x^2 + \ldots + \frac{(r+1)(r+2)(r+3)}{1\cdot 2\cdot 3}x^r + \ldots...$$

and more generally

$$\frac{1}{(1-x)^p} = (1-x)^{-p} = 1 + px + \ldots + \frac{(r+1)(r+2)\ldots(r+p-1)}{1\cdot 2\cdot 3\ \ldots\ (p-1)}\,x^r + \ldots$$

and the coefficient of x^r in this general case is easily seen to be ${}^{(r+p-1)}C_{p-1}$

Illustration 35

Question: **Let us consider the more general problem of distributing n identical things given among r persons, each one whom, can receive 0, 1, 2 or more things ($\le n$).**

Solution: Consider r brackets corresponding to the r persons.

In each bracket take an expression given by $1 + x + x^2 + \ldots + x^n$ (the various powers of x; namely, 0, 1, 2, ..., n correspond to the number of things each person can have in the distribution)

In the continued product $(1 + x + x^2 + \ldots + x^n)$ ( ) ( ) ... repeated r times, collect the coefficient of x^n. This coefficient gives the required number of ways of distribution.

$\therefore$ the number of ways

= Coefficient of x^n in $(1 + x + x^2 + \ldots + x^n)$ ( ) ( ) ... repeated r times

= Coefficient of x^n in $\left(\frac{1-x^{n+1}}{1-x}\right)^r$

$= \text{Coefficient of } x^n \text{ in } (1 - x^{n+1})^r (1 - x)^{-r}$

$= \text{Coefficient of } x^n \text{ in } (1 - x)^{-r}$

$$= \frac{(r+1)(r+2).....(r+n-1)}{1.2.....(r-1)}$$

$= {}^{(n+r-1)}C_{r-1}$

An Alternative method for the general problem

In this case, we consider additionally (r – 1) things (this number is one less than r, the number of persons). It may be seen that the number of ways of dividing the n things among the r persons as per the condition of the problem is ${}^{(n+r-1)}C_{(r-1)}$.

Illustration 36

Question: **An unlimited number of Red, White, Blue and Green balls are given. Ten balls are drawn. Find the number of ways of selection.**

Solution: In this the expression in x to be considered is $1 + x + x^2 + \ldots$ to correspond to the unlimited number available in each colour. There are four such colours.

The required number of ways

$= \text{Coefficient of } x^{10} \text{ in } [1 + x + x^2 + \ldots]^4$

$$= \text{Coefficient of } x^{10} \text{ in } \left(\frac{1}{1-x}\right)^4$$

$$= \frac{11 \times 12 \times 13}{6}$$

$= 286$

13 USE OF MULTINOMIALS IN SOLVING LINEAR EQUATION

Multinomials can be used to solve linear equations. The method is illustrated with the help of following illustrations.

Illustration 37

Question: **Find the number of non negative integral solutions of $x + y + z + w = 20$.**

Solution: Any one of the four variables can take from the value zero to the value 20; and hence we construct a polynomial in a variable (say x) with x raised to different powers which would constitute the values that any one variable can take when the equation is solved in the manner indicated. We, thus, consider the product expression,

$$(1 + x + x^2 + \ldots + x^{20})(1 + x + \ldots + x^{20})(1 + x + \ldots + x^{20})(1 + x + \ldots + x^{20})$$

four factors since there are four variables. If we take x^4 in the first factor; x^5 in the second; x^8 in the third; then we take the term x^3 in the fourth so that the sum of the powers (4 + 5 + 8 + 3 = 20) is 20. It is then we say that there is a solution corresponding to $x = 4$; $y = 5$; $z = 8$;

$w = 3$. Hence the number of solutions in the manner required is the coefficient of x^{20} in $(1 + x + \ldots + x^{20})^4$

i.e. in $\left(\dfrac{1-x^{21}}{1-x}\right)^4$

i.e. in $(1 - x^{21})^4 (1 - x)^{-4}$

i.e. in $(1 - x)^{-4}$ and this is $= {}^{23}C_3$

(**Note:** In $(1 - x)^{-4}$ coefficient of x^n is ${}^{(n+3)}C_3$

Illustration 38

Question: **Let n and k be positive integers such that $n \ge \frac{k(k+1)}{2}$. Find the number of solutions $(x_1, x_2 ...., x_k)$, $x_1 \ge 1, x_2 \ge 2, ..., x_k \ge k$, all integers satisfying $x_1 + x_2 + ... + x_k = n$.**

Solution: The number of solutions is the coefficient of x^n in

$$(x + x^2 + ... + x^n)\,(x^2 + x^3 + ... + x^n)\,...\,(x^k + x^{k+1} + ... + x^n)$$

i.e. in $x^{\frac{k(k+1)}{2}}\dfrac{(1-x^n)}{1-x}\dfrac{(1-x^{n-1})}{1-x}\,...\left(\dfrac{1-x^{n-k+1}}{1-x}\right)$

i.e in $x^{\frac{k(k+1)}{2}}(1-x^n)(1-x^{n-1})\,...\,(1-x^{n-k+1})(1-x)^{-k}$

$= \text{coefficient of } x^{n-\frac{k(k+1)}{2}} \text{ in } (1-x)^{-k}$

$= {}^{\left(n-\frac{k(k+1)}{2}+k-1\right)}C_{k-1}$

(**Note:** Coefficient of x^n *in* $(1-x)^{-k}$ is ${}^{(n+k-1)}C_{k-1}$)

Illustration 39

Question: **How many non-negative integral solutions are there to the equation $x + y + z + w = 29$ gives $x > 0, y > 1, z > 2$ and $w \ge 0$?**

Solution: The number of solutions is the coefficient of x^{29} in

$$(x + x^2 + ... + x^{29})\,(x^2 + ... + x^{29})\,(x^3 + ...x^{29})\,(1 + x + ... + x^{29})$$

i.e. in $x^6\,\dfrac{1-x^{29}}{1-x}\,\dfrac{1-x^{28}}{1-x}\,\dfrac{1-x^{27}}{1-x}\,\dfrac{1-x^{30}}{1-x}$

$= \text{Coefficient of } x^{23} \text{ in } (1-x^{27})(1-x^{28})(1-x^{29})(1-x^{30})(1-x)^{-4}$

$= \text{Coefficient of } x^{23} \text{ in } (1-x)^{-4}$

$= {}^{26}C_3 = \dfrac{26 \cdot 25 \cdot 24}{1 \cdot 2 \cdot 3} = 2600$

14 EXPONENT OF PRIME p IN n!

Since, exponent of prime p in $n!$ is denoted by $E_p(n!)$, where p is a prime number and n is a natural number. Then the last integer amongst 1, 2, 3, ……, $(n-1)$, n which is divisible by p is $[n/p]\,p$, where $[n/p]$ denotes the greatest integer. $(\because [x] \le x)$

$E_p(n\,!) = E_p(1, 2, 3\,.....(n-1)\,.\,n)$

$= E_p(p\,.\,2p\,.\,3p\,.\,.....\,(n-1)\,p\,.\,[n/p]\,p)$

Because the remaining natural numbers from 1 to n are not divisible by p.

Thus, $\left[\dfrac{n}{p}\right] + E_p\left(1.2.3....\left[\dfrac{n}{p}\right]\right)$

Now the last integer among 1, 2, 3, ….. , $\left[\dfrac{n}{p}\right]$ which is divisible by p is

$$\left[\frac{n/p}{p}\right] = \left[\frac{n}{p^2}\right] = \left[\frac{n}{p}\right] + E_p\left(p.2p.3.......\left[\frac{n}{p^2}\right]p\right)$$

Because the remaining natural numbers from 1 to $\left[\dfrac{n}{p}\right]$ are not divisible by p.

Thus, $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+E_p\left(1.2.3...\left[\frac{n}{p^2}\right]\right)$

Similarly, we get $E_p(n!)=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+......+\left[\frac{n}{p^s}\right]$

Where s is the largest natural number such that $p^s \le n < p^{s+1}$

Note : $E_{rp}(n!) = E_r(n!)$ (If $r > p$)

Where r and p are prime numbers.

Illustration 40

Question: **Find the number of zeros at the end of 100 ! .**

Solution: In terms of prime factors 100! can be written as $2^a \,.\, 3^b \,.\, 5^c \,.\, 7^d$ ….

Now $$E_2(100!)=\left[\frac{100}{2}\right]+\left[\frac{100}{2^2}\right]+\left[\frac{100}{2^3}\right]+\left[\frac{100}{2^4}\right]+\left[\frac{100}{2^5}\right]+\left[\frac{100}{2^6}\right]$$

$= 50 + 25 + 12 + 6 + 3 + 1 = 97$

and $$E_5(100!)=\left[\frac{100}{5}\right]+\left[\frac{100}{5^2}\right]=20+4=24$$

$\therefore$ $100! = 2^{97} \,.\, 3^b \,.\, 5^{24} \,.\, 7^d \,..... = 2^{73} \,.\, 3^b \,.\, (2 \times 5)^{24} \,.\, 7^d$ ….

$= 2^{73} \,.\, 3^b \,.\, (10)^{24} \,.\, 7^d$ ….

Hence, number of zeros at the end of 100 ! is 24.

Exponent of 10 in 100 ! = min (97, 24) = 24.

15 PROBLEMS ON FORMATION OF NUMBERS

Formation of numbers using ten digits with or without repetition is also one of the counting techniques or combinatorics some illustrations as follows:

Illustration 41

Question: **How many numbers are there of nine digits with all different digits? What is their sum?**

Solution: In forming the nine digit numbers the first digit, excluding zero, can be one of the 9 and the remaining 8 digits together with zero, making up 9 digits, may be used in forming the next 8 digits in ${}^9P_8$ = 9! ways.

Hence the required number of 9 digit numbers = 9(9 !)

Now regarding their sum;

With 1 in the unit digit there are 8 × ${}^8P_7$ = 8(8!) numbers; and the 1 in all these numbers added up makes up a sum 8(8!). The same is true of numbers with 2 in the unit digit; but their sum is 8(8!) 2.

This way the sum of all the numbers in the unit digit is 8(8!) (1 + 2 + … + 9) = 8(8!) 45. If any digit, instead of being in the unit's place, is in the tenths place the value will be 10.

Illustration 42

Question: **Given the digits 0, 1, 2, 3, 4 and 5**

(i) how many five digit numbers can be formed?

(ii) how many of the five digit numbers and ending with zero?

(iii) how many of the five digit numbers have the odd digits in odd places?

Solution:

(i) Five digit numbers (number of) = 5(5!) = 600

(ii) Number of numbers ending with zero = ${}^5P_4$ = 120

(iii) The 3 odd numbers may form the 3 odd places in 3! ways; and the 3 even numbers may form the two even places in 3! ways. Thus the total number of number is 3! 3! = 36

Illustration 43

Question: **How many number greater than 7×10^5 can be formed from the digits 0, 5, 6, 7, 8 and 9 if**

(i) repetitions are allowed? **(ii) repetitions are not allowed?**

Solution: The first digit can be only 7, 8 or 9, since the number is to be greater than 7×10^5

(i) When repetitions are allowed the number (of numbers) is 3×6^5. Of these the number 7×10^5 is one. The rest is $3 \times 6^5 - 1$

(ii) When repetitions are not allowed the number (of numbers) is $3.{}^5P_5 = 3(5)!$

Illustration 44

Question: **How many four digit numbers are there which contain not more than two different digit?**

Solution: Such four digit numbers can

(i) contain only one digit – numbers like

1111, 2222, … and their number is 9

(ii) contain only two different digits of which

those with zero, ${}^9C_1\,({}^3C_1 + {}^3C_2 + {}^3C_3)$	63
those without zero, ${}^9C_2\,({}^4C_1 + {}^4C_2 + {}^4C_3)$	504
Total	576

Illustration 45

Question: **How many seven digit numbers are there which read the same way from either side?**

Solution: In this case the choice is available only for first four digits because the fifth, the sixth and the seventh digits are respectively the same as (repeat) the third, the second and the first digits.

The first four digits can be had in an arrangement equal to 9×10^3 ways. Hence the required number of numbers is 9×10^3.

PROFICIENCY TEST–II

The following questions deal with the basic concepts of this section. Answer the following briefly. Go to the next section only if your score is at least 80%. Do not consult the Study Material while attempting the questions.

1. In how many ways n things can be distributed among r persons, each one of whom, can receive 0, 1, 2 or more things. ($\leq n$).

2. In how many ways can $2n$ people be divided into n pairs ?

3. In how many ways 3 letters be put into three envelopes such that none of them goes into the right one.

4. What is the exponent of 3 in 100 !.

5. From four unlimited stocks of red, green blue and black balls, in how many ways 25 balls can be selected? (All balls of same colour are identical).

6. In how many ways can 12 oranges be distributed to 5 persons so that each person receives at least one orange?

7. How many divisors are there of the number 115500 excluding the divisor 1 and the number?

8. How many different six digit numbers are there the sum of whose digits is odd?

9. What is the exponent of 2 in 33 !.

10. How many odd divisors are there of the number 1960.

ANSWERS TO PROFICIENCY TEST–II

1. ${}^{n+r-1}C_{r-1}$

2. $\dfrac{2n\,!}{2^n\;n\,!}$

3. 3

4. 48

5. ${}^{28}C_{25}$

6. 330

7. 94

8. 450000

9. 31

10. 6

SOLVED OBJECTIVE EXAMPLES

Example 1:

Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. Then the number of words which have at least one letter repeated is

(a) 69760 (b) 30240 (c) 99748 (d) none of these

Solution:

Suppose we have 5 places, each of which is to be filled by one letter from the 10 letters. The first place may be filled in 10 ways. When repetitions of the letters are allowed, the second place may also be filled in 10 ways.

Proceeding in this way, it is clear that words with five letters are formed in 10^5 ways. These 10^5 ways also include the number of ways of forming words with all different letters without repetition. These are ${}^{10}P_5$ in number.

$\therefore$ the number of words which have at least one letter repeated is

$$10^5 - {}^{10}P_5 = 100000 - 30240 = 69760$$

Hence (a) is the correct answer.

Example 2:

The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{(52-j)}C_3$ is equal to

(a) ${}^{47}C_3$ (b) ${}^{52}C_5$ (c) ${}^{52}C_4$ (d) none of these

Solution:

The given expression = $({}^{47}C_4 + {}^{47}C_3) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 = ({}^{48}C_4 + {}^{48}C_3) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

$= ({}^{49}C_4 + {}^{49}C_3) + {}^{50}C_3 + {}^{51}C_3 = ({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3$

$= {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4$, by using the formula ${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$ repeatedly.

Hence (c) is the correct answer.

Example 3:

In a plane there are 37 straight lines, of which 13 pass through the point *A* and 11 pass through the point *B*. Besides, no three lines pass through one point, no line passes through both points *A* and *B*, and no two are parallel. Then the number of intersection points the lines have is equal to

(a) 535 (b) 601 (c) 728 (d) none of these

Solution:

In the general position, 37 straight lines have ${}^{37}C_2$ points of intersection. But 13 straight lines passing through the point *A* yield one intersection point instead of ${}^{13}C_2$ and 11 straight lines passing through the point *B* yield one intersection point instead of ${}^{11}C_2$.

$\therefore$ the lines have ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2$ points of intersection.

i.e., 666 – 78 – 55 + 2, i.e., 535.

Hence (a) is the correct answer.

Example 4:

A set contains (2*n* + 1) elements. The number of subsets of the set which contain at most *n* elements is

(a) 2^n (b) 2^{n+1} (c) 2^{n-1} (d) 2^{2n}

Solution:

The number of subsets of the set which contain at most *n* elements is

${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n = N$ (say)

We have $2N = 2({}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n)$

$= ({}^{2n+1}C_0 + {}^{2n+1}C_{2n+1}) + ({}^{2n+1}C_1 + {}^{2n+1}C_{2n}) + \ldots + ({}^{2n+1}C_n + {}^{2n+1}C_{n+1})$

$(\because {}^{n}C_r = {}^{n}C_{n-r})$

$= {}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_{2n+1} = 2^{2n+1} \Rightarrow N = 2^{2n}$

Hence (d) is the correct answer.

Example 5:

Let p be a prime number such that $p \geq 3$. Let $n = p! + 1$. The number of primes in the list $n + 1, n + 2, n + 3, \ldots, n + p - 1$ is

(a) $p - 1$ (b) 2 (c) 1 (d) none of these

Solution:

For $1 \leq k \leq p - 1$, $n + k = p! + k + 1$, is clearly divisible by $k + 1$. Therefore, there is no prime number in the given list.

Hence (d) is the correct answer.

Example 6:

Ten persons are arranged in a row. The number of ways of selecting four persons so that no two persons sitting next to each other are selected is

(a) 34 (b) 36 (c) 35 (d) none of these

Solution:

To each selection of 4 persons we associated binary sequence of the form 1001001010 where 1(0) at i^{th} place means the i^{th} person is selected (not selected).

There exists one-to-one correspondence between the set of selections of 4 persons and set of binary sequence containing 6 zeros and 4 ones.

We are interested in the binary sequences in which no 2 ones are consecutive. We first arrange 6 zeros.

0 0 0 0 0 0

This can be done in just one way. Now, 4 ones can be arranged at any of the 4 places marked with a cross in the following arrangement.

× 0 × 0 × 0 × 0 × 0 × 0.

We can arrange 4 1's at 7 places in ${}^7C_4 = 35$ ways.

Hence (c) is the correct answer.

Example 7:

The number of solutions of $x_1 + x_2 + x_3 = 51$ (x_1, x_2, x_3 being odd natural numbers)

(a) 300 (b) 325 (c) 330 (d) 350

Solution:

Let odd natural numbers be $2a - 1$, $2b - 1$, $2c - 1$, where a, b, c are natural numbers $a + b + c = 27$

$a \geq 1, b \geq 1, c \geq 1$

No. of solutions is the coefficient of x^{24} in $(1-x)^{-3}$ … (i)

$= {}^{26}C_2 = 13 \times 25 = 325$

Hence (b) is the correct answer.

Example 8:

In a certain test, there are n questions. In this test 2^{n-i} students gave wrong answers to atleast i questions, where $i = 1, 2, \ldots, n$. If the total number of wrong answers given is 2047, then n is equal to

(a) 10 (b) 11 (c) 12 (d) 13

Solution:

The number of students answering exactly i ($1 \leq i \leq n - 1$) questions wrongly is $2^{n-i} - 2^{n-i-1}$. The number of students answering all n questions wrongly is 2^0. Thus, the total number of wrong answers is $1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3}) + \ldots + (n-1)(2^1 - 2^0) + n(2^0)$

$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^0 = 2^n - 1$

Thus $2^n - 1 = 2047 \Rightarrow 2^n = 2048 = 2^{11} \Rightarrow n = 11$

Hence (b) is the correct answer.

Example 9:

The number of times of the digits 3 will be written when listing the integers from 1 to 1000 is

(a) 269 **(b) 300** **(c) 271** **(d) 302**

Solution:

Since 3 does not occur in 1000, we have to count the number of times 3 occurs when we list the integers from 1 to 999. Any number between 1 and 999 is of the form xyz where $0 \le x, y, z \le 9$. Let us first count the numbers in which 3 occurs exactly once. Since 3 can occur at one place in ${}^3C_1$ ways, there are ${}^3C_1\ (9 \times 9) = 3 \times 9^2$ such numbers. Next, 3 can occur exactly at two places in $({}^3C_2)\ (9) = 3 \times 9$ such numbers.

Lastly, 3 can occur in all three digits in one number only. Hence the number of times 3 occurs is $1 \times (3 \times 9^2) + 2 \times (3 \times 9) + 3 \times 1 = 300$

Hence (b) is the correct answer.

Example 10:

Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five balls. The number of ways in which we can place the balls in the boxes (order is not considered in the box) so that no box remains empty is

(a) 150 **(b) 300** **(c) 200** **(d) none of these**

Solution:

One possible arrangement is

2	2	1

Three such arrangements are possible. Therefore, the number of ways is $({}^5C_2)\ ({}^3C_2)\ ({}^1C_1)\ (3) = 90$

The other possible arrangements

1	1	3

Three such arrangements are possible. In this case, the number of ways is $({}^5C_1)\ ({}^4C_1)\ ({}^3C_3)\ (3) = 60$

Hence, the total number of ways is 90 + 60 = 150.

Hence (a) is the correct answer.

Example 11:

In a plane there are two families of lines $y = x + r$, $y = -x + r$, where $r \in \{0, 1, 2, 3, 4)$. The number of squares of diagonals of length 2 formed by the lines is

(a) 9 **(b) 16** **(c) 25** **(d) none of these**

Solution:

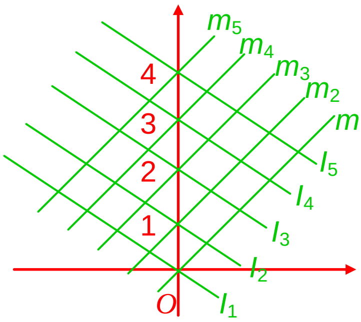
There are two sets of five parallel lines at equal distances. Clearly, lines like l_1, l_3, m_1, m_3 form a squares whose diagonal's length is 2

So, the number of required squares = 3 × 3

{Since choices are (l_1, l_3),

(l_2, l_4) (l_3, l_5) for one set, etc)

Hence (a) is the correct answer.



Example 12:

The number of ways of arranging six persons (having *A*, *B*, *C* and *D* among them) in a row so that *A*, *B*, *C* and *D* are always in order *ABCD* (not necessarily together) is

(a) 4 **(b) 10** **(c) 30** **(d) 720**

Solution:

The number of ways of arranging $ABCD$ is 4!. For each arrangement of $ABCD$, the number of ways of arranging six persons is same.

Hence required number is $\dfrac{6!}{4!} = 30$

Hence (c) is the correct answer.

Example 13:

Let A = (x I x is a prime number and x < 30}. The number of different rational numbers whose numerator and denominator belong to A is

(a) 90 **(b) 180** **(c) 91** **(d) none of these**

Solution:

A = {2, 3, 5, 7, 11, 13, 17, 19, 23 29}. A rational number is made by taking any two in any order.
So, the required number of rational numbers = ${}^{10}P_2 + 1$ (including 1).
Hence (c) is the correct answer.

Example 14:

Let S be the set of all functions from the set A to the set A. If $n(A) = k$, then $n(S)$ is

(a) k ! **(b) k^k** **(c) $2^k - 1$** **(d) 2^k**

Solution:

Each element of the set A can be given the image in the set A in k ways. So, the required number of functions,
i.e., $n(S) = k \times k \times \ldots.$ (k times) $= k^k$.
Hence (b) is the correct answer.

Example 15:

Let A be the set of 4-digit numbers $a_1a_2\, a_3a_4$ where $a_1 > a_2 > a_3 > a_4$, then $n(A)$ is equal to

(a) 126 **(b) 84** **(c) 210** **(d) none of these**

Solution:

Any selection of four digits from the ten digits 0, 1, 2, 3, … 9 gives one such number. So, the required number of numbers = ${}^{10}C_4 = 210$.
Hence (c) is the correct answer.

SOLVED SUBJECTIVE EXAMPLES

Example 1:

15 persons, amongst whom are *A*, *B* and *C* are to speak at a function. Find in how many ways can the speech be done if *A* wants to speak before B and B is to speak before *C* ?

Solution:

We can select three position out of 15 position by ${}^{15}C_3$ ways.

We can provide these position to *A, B, C* by only one ways.
Other 12 persons can speak in 12 ! ways.

Hence total number of ways will be ${}^{15}C_3 \times 1 \times 12\,!$

Example 2:

A committee of twelve is to be formed from 9 women and 8 men. In how many ways can this be done if at least five women have to be included in the committee? In how many of these committees (i) the women are in a majority? (ii) the men are in a majority?

Solution:

The possible ways of formation of the committee are listed:

Constitution of the committee			**Number of ways of formation**
(9) Women	(8) Men		
5	7	$\longrightarrow$	${}^9C_5 \cdot {}^8C_7 = 1008$
6	6	$\longrightarrow$	${}^9C_6 \cdot {}^8C_6 = 2352$
7	5	$\longrightarrow$	${}^9C_7 \cdot {}^8C_5 = 2016$
8	4	$\longrightarrow$	${}^9C_8 \cdot {}^8C_4 = 630$
9	3	$\longrightarrow$	${}^9C_9 \cdot {}^8C_3 = 56$
		Total number of ways	6062

(i) Number of committees with women majority
= 2016 + 630 + 56 = 2702
(ii) Number of committees with men majority = 1008

Example 3:

How many seven digit numbers can be formed using only the three digits 1, 2 and 3; the digit 2 occurring only twice in each number.

Solution:

Any two of the seven digits can be chosen, and in these places, 2 is filled and rest five are filled with 1 or 3.

$\therefore$ The required number is ${}^7C_2 \cdot 2^5 = 672$

Example 4:

A man has 7 relatives in there 4 are Ladies and 3 are Gentlemen. His wife has also 7 relatives in there 3 are Ladies and 4 are Gentlemen. In how many ways can the couple invite for a dinner party, 3 Ladies and 3 Gentlemen so that there are 3 of the man's relatives and 3 of the wife's relatives?

Solution:

The possible number of ways can be listed as follows.

Man		**Wife**		**Number of ways**
Ladies (4)	Gentlemen (3)	Ladies (3)	Gentlemen (4)	
3	0	0	3	${}^4C_3 \cdot {}^4C_3 = 16$
2	1	1	2	$({}^4C_2 \cdot {}^3C_1)^2 = 324$
1	2	2	1	$({}^4C_1 \cdot {}^3C_2)^2 = 144$
0	3	3	0	$({}^3C_3 \cdot {}^3C_3) = 1$
			Total	$= 485$

∴ Total number of ways of inviting for a dinner party of six = 485

Example 5:

There are n points in a plane, no three of which are collinear except '*p*' points all of which are on a line How many (i) straight lines can be formed (ii) triangles can be formed out of these *n* points?

Solution:

(i) To form a line we need two points; and these two points may be chosen in ${}^nC_2$ ways; but, it happens that '*p*' of the '*n*' points are on a line; consequently these points would form only one line instead of ${}^pC_2$.

∴ Number of lines = ${}^nC_2 - {}^pC_2 + 1$

(ii) Number of triangles = ${}^nC_3 - {}^pC_3$

Example 6:

Given the digits 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, how many four digit numbers can be formed?

Solution:

Such four digit numbers can

(i) contain all different digits, any four chosen from 1, 2, 3, 4, 5 and arranged to form four digit number and this is done in ${}^5P_4$ = 120 ways.

(ii) contain one repeated pair, the other two different, numbers like 1123, 3345, 3435, ..– and this is done in ${}^5C_1 \times {}^4C_2 \times \frac{\lfloor\underline{4}}{\lfloor\underline{2}} = 360$ ways

(iii) contain two repeat pairs, numbers like 1122, 1212, 3113, … – and this is done in ${}^5C_2\,\frac{4!}{2!\,2!} = 60$ ways .

∴ Total number of numbers = 540.

Example 7:

If the letters of the word MOTHER be permuted among themselves and the words so formed are arranged as in a dictionary, what is the rank of the word MOTHER?

Solution:

The letters, arranged alphabetically, are
EHMORT

Number of words beginning with E $\underline{|5} = 120$

Number of words beginning with H $\underline{|5} = 120$

Number of words beginning with ME $\underline{|4} = 24$

Number of words beginning with MH $\underline{|4} = 24$

Number of words beginning with MOE $\underline{|3} = 6$

Number of words beginning with MOH $\underline{|3} = 6$

Number of words beginning with MOR $\underline{|3} = 6$

Number of words beginning with MOTE $\underline{|2} = 2$

Number of words beginning with MOTHER 1

$\overline{\underline{309}}$

The position (rank) of the word MOTHER is 309.

Example 8:

Six boys and six girls sit in a row. Find the number of ways in which they can be seated

(i) when the girls are all together

(ii) when the boys and girls are seated alternately

(iii) when the girls are separated

Discuss the above problem in the case when the boys and girls are seated in a circle.

Solution:

(i) Considering the girls as one unit, and the six boys make up 7. These 7 can be seated in $\underline{|7}$ ways. The girls among themselves may be relatively interchanged in $\underline{|6}$ ways.

The number of arrangements is $\underline{|7}\,\underline{|6} = 3{,}628{,}800$

(ii) In this case first arrange the six boys in $\underline{|6}$ ways

Suppose

$$\underset{B_1}{\bullet} \quad \underset{B_2}{\bullet} \quad \underset{B_3}{\bullet} \quad \underset{B_4}{\bullet} \quad \underset{B_5}{\bullet} \quad \underset{B_6}{\bullet}$$

Is one such arrangement

Then the six girls can have their positions between any two of these boys, the arrangement starting with a girl.

$$G_1 \quad \underset{B_1}{\bullet} \quad G_2 \quad \underset{B_2}{\bullet} \quad G_3 \quad \underset{B_3}{\bullet} \quad G_4 \quad \underset{B_4}{\bullet} \quad G_5 \quad \underset{B_5}{\bullet} \quad G_6 \quad \underset{B_6}{\bullet}$$

and the number of arrangements is $(6!)^2$ (or) the positions of the six girls can be (arrangement now starting with a boy)

$$\underset{B_1}{\bullet} \quad G_1 \quad \underset{B_2}{\bullet} \quad G_2 \quad \underset{B_3}{\bullet} \quad G_3 \quad \underset{B_4}{\bullet} \quad G_4 \quad \underset{B_5}{\bullet} \quad G_5 \quad \underset{B_6}{\bullet} \quad G_6$$

and the number of arrangements of six boys and six girls (seated alternately) is $2\,(6!)^2 = 1{,}036{,}800$

(iii) In this case the girls are separated, not necessarily by only one boy, between any two girls. First the boys are arranged in 6! ways.

•	•	•	•	•	•

B_1	B_2	B_3	B_4	B_5	B_6

1 2 3 4 5 6 7

The positions available for the six girls can be chosen from the seven (as indicated). The girls are arranged in $^7P_6 (7!)$ ways.

∴ the total number of arrangement is 6! 7! = 3,628,800

In the case of circular permutation, the arrangements, correspondingly, are

(i) 6! 6! = 518,400 (ii) 5! 6! = 86,400 (iii) 5! 6! = 86,400

Note: In this case whether they sit alternately, or one group (of boys) is separated by the other (of girls) the effect is the same. Hence in (ii), (iii) cases, arrangements are equal.

Example 9:

Three boys picked 5 apples. In how many ways can these 5 apples be distributed among the 3 boys so that each can have any number, of course, not exceeding five? (All the apples are considered the same.)

Solution:

Let us consider, in addition to 5 apples, 2 new things say oranges (this number is one less than the number of boys). Then the number of ways of distribution will correspond to the number of arrangements of five A's and two O's. For example, consider the following arrangements:

(i) I OAAA I O AA; draw line just before O's to partition this arrangement. This arrangement corresponds to 0 + 3 + 2, (the numbers corresponding to the number of A's before the line of demarcation, between consecutive lines and after the last line).

(ii) I O A I O AAAA corresponds to 0 + 1 + 4

(iii) A I OA I O AAA corresponds to 1 + 1 + 3

The number of permutations = $\dfrac{7!}{5!\,2!}$ or $^7C_2$

Alternative solution: If n apples are distributed among r boys, the number of ways of distribution would be $^{n+(r-1)}C_{r-1}$. Hence in this case number of ways are $^7C_2$.

Example 10:

There are n points in a plane which are joined in all possible ways by indefinite straight lines, and no two of these joining lines are parallel and no three of them meet in a point. Find the number of points of intersection, exclusive of the *n* given points.

Solution:

Since two points are required to determine a straight line, we have $N = {^nC_2}$ straight lines by joining the n given points in all possible ways. Since no two are parallel and no three are concurrent, we have $^NC_2$ points of intersection. But each of the given n points is counted as a point of intersection $^{n-1}C_2$ times, since $n - 1$ straight lines pass through each point.

∴ the required number = $^NC_2 - n\cdot{^{n-1}C_2}$

$$= \frac{^nC_2(^nC_2 - 1)}{2} - \frac{n(n-1)(n-2)}{2}$$

$$= \frac{\dfrac{n(n-1)}{2}\left[\dfrac{n(n-1)}{2} - 1\right]}{2} - \frac{n(n-1)(n-2)}{2}$$

$$= \frac{n(n-1)}{2}\left[\frac{n(n-1)-2}{4} - (n-2)\right]$$

$$= \frac{n(n-1)}{2}\cdot\left(\frac{n^2 - 5n + 6}{4}\right) = \frac{1}{8}\cdot n(n-1)(n-2)(n-3).$$

Example 11:

All possible two-factor products are formed from the numbers 1, 2, 3, ..., 100. How many numbers out of the total obtained are multiples of 3?

Solution:

The total number of two-factor products = ${}^{100}C_2$

Out of the numbers 1, 2, 3, ..., 100; the multiples of 3 are 3, 6, 9, ..., 99; i.e., there are 33 multiples of 3, and therefore there are 67 non-multiples of 3.

$\therefore$ the number of two-factor products which are not multiples of 3 = ${}^{67}C_2$.

$\therefore$ the required number = ${}^{100}C_2 - {}^{67}C_2$ = 4950 – 2211 = 2739.

Alternatively, the number of two-factor products formed when both factors are multiples of 3 = ${}^{33}C_2$ and the number of two-factor products formed when one is a multiple of 3 and the other a non-multiple of 3 = ${}^{33}C_1 \times {}^{67}C_1$.

It either case the product is a multiple of 3.

$\therefore$ the required number = ${}^{33}C_2 + 33 \times 67$ = 528 + 2211 = 2739.

Example 12:

Five persons are to address a meeting. If a specified speaker is to speak before another specified speaker, find the number of ways in which this could be arranged. In how many of these arrangements will the first speaker come immediately before the second?

Solution:

Let A, B be the corresponding specified speakers.

(i) Without any restriction the five persons can be arranged among themselves in 5! ways; but the number of ways in which A speaks before B and the number of ways in which B speaks before A together make up 5!. Also the number of ways in which A speaks before B is exactly equal to the number of ways in which B speaks before A.

$\therefore$ the required number of ways = $\frac{1}{2}\cdot$ 5! = 60.

(ii) Regarding AB in that order as a single person, we can arrange them with the remaining three in 4 ways. Each of these arrangements corresponds to a way in which A speaks immediately before B.

$\therefore$ the required number of ways in this case = 4! = 24.

Example 13:

How many even numbers lying between 200 and 500 can be formed from the figures 1, 2, 3, 4, 5, 6, if no figure is to appear more than once in any number.

Solution:

Each even number is to lie between 200 and 500.

Hence the position on the left in $x\,x\,x$ is to be filled with 2, 3 or 4.

When 2 or 4 is filled on the left, the position on the right can be filled with the remaining 2 even figures in 2 ways and the middle position can be filled with the remaining 4 figures in 4 ways.

$\therefore$ the total number of ways of forming the even numbers in this case = $2 \cdot 4 \cdot 2$ = 16.

When 3 is filled on the left, the position on the right can be filled with the 3 even figures in 3 ways, and the middle position can be filled with the remaining 4 figures in 4 ways.

$\therefore$ the total number of ways of forming the even numbers in this case = 1. 4. 3 = 12.

$\therefore$ the required number = 16 + 12 = 28.

Example 14:

Find the number of positive integers which can be formed by any number of digits 0, 1, 2, 3, 4, 5 but using each digit not more than once in each number. How many of these integers are greater than 3000?

Solution:

Zero cannot be a starting digit for any number. Therefore while forming a 6 digit number, we can fill up the first place in 5 ways. The restriction on zero ends with the starting place. Having filled it, there are 5 figures left, and therefore the remaining places can be filled in ${}^5P_5$ ways.

$\therefore$ the number of 6 digit numbers = $5 \cdot {}^5P_5 = 600$

Similarly,

the number of 5 digit numbers = $5 \cdot {}^5P_4 = 600$

the number of 4 digit numbers = $5 \cdot {}^5P_3 = 300$

the number of 3 digit numbers = $5 \cdot {}^5P_2 = 100$

the number of 2 digit numbers = $5 \cdot {}^5P_1 = 25$

the number of single digit numbers = 5

$\therefore$ the total number of positive integers = 1630.

For finding the numbers greater than 3000, we take all the 6 and 5 digit numbers together with 4 digit numbers, starting with 3, 4 or 5. The number of 4 digit numbers starting with 3, 4 or 5 is $3 \cdot {}^5P_3 = 180$.

$\therefore$ the total number of integers greater than 3000 is 600 + 600 + 180 = 1380.

Example 15:

How many different numbers can be formed to satisfy all the conditions given below

(i) the number is less than 2×10^8,

(ii) the number is formed of the digits 0, 1 and 2 and

(iii) the number is divisible by 3?

Solution:

Now $2 \times 10^8 = 200000000$

Numbers less than 2×10^8 are of the form $a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8\, a_9$

For all the nine digit numbers $a_1 = 1\, \textit{and}\, a_2, a_3, \ldots \textit{or}\, a_8$ (at present a_9 is deliberately not considered) can be one of 0, 1 or 2.

For all the 8 digit numbers $a_1 = 0, a_2 \neq 0$; for all the 7 digit numbers $a_1 = 0, a_2 = 0, a_3 \neq 0$. Thus all numbers of 9 digits and less than 9 digits are included when a_1 is chosen of in one of two ways (0 or 1); and $a_2, a_3, \ldots., a_8$ are each chosen in one of 3 ways (0, 1 or 2).

$\therefore$ the choice of $a_1, a_2, \ldots., a_8$ may be done 2×3^7 ways. Suppose one such choice is $a_1, a_2, \ldots., a_8$. If $a_1 + a_2 + \ldots + a_8$ is already divisible by 3, $3, a_9$ which was no considered before).

If $a_1 + a_2 + \ldots. + a_8$ is of the form 3p + 1, leaving a remainder 1 on division by 3, then $a_9 = 2$. If $a_1 + a_2 + \ldots. + a_8$ is of the form 3p + 2, leaving a remainder 2 on division by 3, then $a_9 = 1$. Thus, in any case, there is only one way of choice for a_9. The number of numbers, is there fore, $2 \times 3^7 \times 1$ and this includes 000000000. i.e. zero. The required number of numbers is $2 \times 3^7 - 1 = 4373$.

MIND MAP

FUNDAMENTAL PRINCIPLE OF COUNTING

- **Multiplication principle:**
  If a job can be completed in m ways and another in n ways then both can be completed in mn ways.
- **Addition principle:**
  If a job can be completed in m ways and another in n ways then either of them can be completed in $(m + n)$ ways.

COMBINATIONS

- It is the numbers of ways of selection of r objects from n distinct objects

$$^nC_r = \frac{n!}{(n-r)!\,r!} \qquad (0 \le r \le n)$$

- $^nC_0 = {}^nC_n = 1$
- $^nC_x = {}^nC_y = \Rightarrow x = y$ or $x + y = n$
- The greatest value of ^nC_r is $^nC_{n/2}$ if n is even, $^nC_{(n-1)/2}$ or $^nC_{(n+1)/2}$ if n is odd.
- The number of ways in which r objects can be selected from n distinct objects
  (i) if a particular object is always included is $^{n-1}C_{r-1}$ (ii) if a particular object is always excluded is $^{n-1}C_r$ (iii) if m particular objects are always included is $^{n-m}C_{r-m}$ (iv) if m particular objects are always excluded is $^{n-m}C_r$.

SELECTION FROM IDENTICAL/DISTINCT OBJECTS

- The number of ways of selection of atleast one from n distinct objects is
  $^nC_1 + {}^nC_2 + ….. + {}^nC_n = 2^n - 1$.
- The number of ways of selection of atleast one from $a_1 + a_2 + ……+ a_n$ objects, where a_i ($i = 1, 2, … n$) are identical of some kind is $(a_1 + 1)(a_2 + 1) …… (a_n + 1) - 1$.
- $a_1 + a_2 + …. a_n + k$, where a_i's are alike of some kind and k are distinct is
  $(a_1 + 1)(a_2 + 1) …. (a_n + 1)\,2^k - 1$.

PERMUTATIONS & COMBINATIONS

DIVISORS OF GIVEN NATURAL NUMBERS

- Let $N = p_1^{\alpha_1}.\,p_2^{\alpha_2}.\,p_3^{\alpha_3}.\,......\,p_k^{\alpha_k}$, where $p_1, p_2, p_3, ….. p_k$ are different prime numbers and $\alpha_1, \alpha_2, \alpha_3, …., \alpha_k$ are natural numbers then:

the total number of divisors of N including 1 and N is $= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) …. (\alpha_k + 1)$

- the total number of divisors of N excluding 1 and N is $= (\alpha_1 + 1)(\alpha_2 + 1) …. (\alpha_k + 1) - 2$
- the sum of these divisors is =
  $(p_1^0 + p_1^1 + ... + p_1^{\alpha_1})\,...\,(p_k^0 + p_k^1 + ... + p_k^{\alpha_k})$

  (Use sum of G.P. in each bracket)

EXPONENT OF PRIME p IN n!

- $E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + ... \left[\frac{n}{p^s}\right]$

where s is the largest natural number such that $p^s \le n \le p^{s+1}$.

- $E_{rp}(n!) = E_r(n!)$ (if $r > p$) where r and p both are prime numbers.

PERMUTATIONS

- The number of linear arrangements of r objects out of n distinct objects, is denoted as ^nP_r, where $^nP_r = \frac{n!}{(n-r)!}$.
- The number of permutations of n different things taken r at a time when each thing may be repeated any number of times is n^r.
- The number of arrangements of n things, when p out of them are of one kind, q are of other kind and r of another kind and $p + q + r \le n$ is $\frac{n!}{p!\,q!\,r!}$.

CIRCULAR PERMUTATIONS

- n distinct things can be arranged in a circular in $(n - 1)!$ ways.
- n distinct beads can be arranged in a garland in ½ $[(n - 1)]!$

DIVISION OF DISTINCT OBJECTS INTO GROUPS

- mn different things can be distributed in m equal groups of n objects each, in
  (i) $\frac{mn!}{(n!)^m}$ ways, when order of groups is important .
  (ii) $\frac{(mn)!}{m!\,(n!)^n}$ ways, when order of groups is not important.

DIVISION OF IDENTICAL OBJECTS INTO GROUPS

- n identical objects can be divided into r different groups in $^{n+r-1}C_{r-1}$ or $^{n-1}C_{r-1}$ according as blank groups are not considered.

ARRANGEMENT IN GROUPS

- The number of ways of distribution and arrangement of n distinct things into r different groups is $^{n+r-1}C_{r-1}$ or n ! $^{n-1}C_{r-1}$ as blank groups are not considered.

DEARRANGEMENT

- n things arrangement in a row can be dearranged in n! $\left(1 - \frac{1}{1!} + \frac{1}{2!} + ...(-1)^n \frac{1}{n!}\right)$ ways such that none of them occupies its original place.

USE OF MULTINOMIALS

- If there are l objects of one kind, m objects of second kind, n objects of third kind and so on; then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + ….$) is the coefficient of x^r in $(1 + x + x^2 + x^3 + … x^l)(1 + x + x^2 + ….. + x^m)(1 + x + x^2 + .. + x^n)$. Further if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects is the coefficient of x^r in $(x + x^2 + x^3 + …. x^l)(x + x^2 + x^3 + ….. + x^m)(x + x^2 + x^3 + ….. + x^n)$ and the number of possible permutations of r objects out of these objects is the coefficient of x^r in $r!\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^l}{l!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^m}{m!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!}\right)...$

EXERCISE – I

CBSE PROBLEMS

1. Find n, if $(n + 2)! = 2550 \times n!$.

2. How many three letter words can be formed using a, b, c, d, e if:
(i) repetition is not allowed (ii) repetition is allowed.

3. In how many ways can four jobs I, II, III and IV be assigned to four persons A, B, C and D if one person is assigned only one job and all are capable of doing each job.

4. How many different numbers of six digits each can be formed from the digits 4, 5, 6, 7, 8, 9 when repetition of digits is not allowed ?

5. In how many ways 8 beads can be arranged to form a necklace ?

6. How many words can be formed from the letters of the word 'SERIES' which start with S and end with S ?

7. Among 36 teachers in a college, one principal, one vice-principal and three teacher-incharge are to be appointed. In how many ways can this be done ?

8. The English alphabet has 5 vowels and 21 consonant. How many words with two different vowels and two different consonants can be formed from the alphabets?

9. In how many ways can 6 Hindus and 6 Muslims sit around a round table so that two Muslims may never sit together ?

10. How many different products can be obtained by multiplying two or more of the numbers 3, 5, 7, 11 (without repetition) ?

11. How many natural numbers less than 1000 can be formed from the digits 0, 1, 2, 3, 4, 5 when a digit may be repeated any number of times ?

12. In how many ways can the letters of the word 'STRANGE' be arranged so that
(i) the vowels come together ?
(ii) the vowels never come together ?
(iii) the vowels occupy only the odd places ?

13. There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees,
(i) a particular professor is included. (ii) a particular student is included.
(iii) a particular student is excluded.

14. Find the number of diagonals of (i) a hexagon (ii) a polygon of 16 sides.

15. Three distinct dice are rolled. Find the number of possible outcomes in which at least one die shows 5.

16. In a group there are 4 girls and 6 boys. If all the 4 girls sit together, find number of ways in which they can be arranged around a table.

17. How many numbers can be formed by using the digits 2, 2, 3, 3, 3, 4, 4 all at a time?

18. There are 15 points in a plane, no three of which are collinear. Find the number of triangles formed by joining them.

19. In how many ways can we select a cricket eleven from 17 players, in which only 5 players can bowl and if each cricket eleven must include exactly 4 bowlers ?

20. In how many ways can 5 red and 5 white balls be drawn from an urn containing 8 red and 6 white balls?

EXERCISE – II

IIT-JEE-SINGLE CHOICE CORRECT

1. If ${}^{m+n}P_2 = 90$ and ${}^{m-n}P_2 = 30$, then m and n is

(a) $m = 8, n = 2$ (b) $m = 8, n = 3$

(c) $m = 6, n = 2$ (d) $m = 6, n = 3$

2. If repetition of digits is not allowed how many numbers of four digits divisible by 5 can be formed with the digits 0, 4, 5, 6, 7

(a) 40 (b) 44 (c) 42 (d) 36

3. A servant has to post 5 letters and there are 4 letter boxes. In how many ways can he post the letters is

(a) ${}^5P_4$ (b) $\frac{5\,!}{4\,!}$ (c) 5^4 (d) 4^5

4. In how many ways 5 delegates can be put in 6 hotels of a city if there is no restriction is

(a) 6^5 (b) 5^6 (c) ${}^6P_5$ (d) $\frac{6\,!}{5\,!}$

5. A library has 5 copies of one book, 4 copies of each of 2 books, 6 copies of each of 3 books and single copies of 8 books. In how many ways can all books be arranged so that copies of the same book are always together

(a) 5 ! (b) 14 ! (c) 8 ! (d) 6 !

6. In a dinner party there are 10 Indians, 5 Americans and 5 Englishmen. In how many ways can they be arranged in a row so that all persons of the same nationality sit together

(a) 10 ! 5 ! 5 ! (b) 3 ! 10 ! 5 ! 5 !

(c) 3 ! 10 ! (d) 10 ! 5 !

7. In a class of students there are 4 girls and 6 boys. In how many ways can they be seated in a row so that all the four girls are not together

(a) 10 ! – 7 ! 4 ! (b) ${}^{10}P_4$ (c) ${}^7P_4$ (d) 6 ! 4 !

8. The number of different arrangements (permutations) of the letters of the word 'Banana' is

(a) 40 (b) 120 (c) 60 (d) 50

9. How many different words can be formed with the letters of the word "MATHEMATICS"

(a) $\frac{11\,!}{2\,!\,2\,!\,2\,!}$ (b) 11 !

(c) $\frac{11\,!}{2\,!}$ (d) $\frac{11\,!\,2\,!\,7\,!\,4!}{2\,!\,2\,!\,2\,!\;\;2\,!\,2\,!\,2\,!}$

10. How many words can be formed out of the letters of the word 'Article' so that the vowels occupy the even places

(a) 360 (b) 200 (c) 120 (d) 144

11. The number of arrangements of the letters of the word 'Delhi' if *e* always comes before *i* is

(a) 40 (b) 30 (c) 60 (d) 20

12. In a class of students there are 6 boys and 4 girls. In how many ways can they be seated around a table so that all the 4 girls sit together

(a) 11 ! (b) 6 ! 4 ! (c) 6 ! 2 ! (d) 8 !

13. In How many ways can 7 Englishmen and 6 Indians sit down around a table so that no two Indians are together

(a) 6 ! × 6 ! (b) 7 ! × 7 ! (c) $6\,!\times {}^{7}P_{6}$ (d) 6 !

14. If ${}^{n}C_{6} : {}^{n-3}C_{3} = 33:4$, then n is

(a) 11 (b) 10 (c) 12 (d) 9

15. If ${}^{n-1}C_{3} + {}^{n-1}C_{4} > {}^{n}C_{3}$, then n is greater than

(a) 6 (b) 7 (c) 5 (d) 4

16. There are *n* stations on a railway line. The number of kinds of tickets printed (no return tickets) is 105, then the number of stations are

(a) 15 (b) 10 (c) 12 (d) 9

17. Out of 7 men and 4 ladies a committee of 5 is to be formed. In how many ways can this be done so as to include at least 3 ladies

(a) 90 (b) 120 (c) 91 (d) 360

18. From 8 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done so as to included at least one lady

(a) 720 (b) 366 (c) 736 (d) 422

19. From 5 apples, 4 oranges and 3 mangoes, how many selections of fruits can be made

(a) 119 (b) 120 (c) 59 (d) 60

20. In how many ways 12 different books can be distributed equally among 4 persons

(a) $\frac{126}{4\,!\,(3\,!)^4}$ (b) $\frac{12\,!}{(3\,!)^4}$ (c) 12 ! (d) $\frac{12\,!}{3\,!}$

21. A polygon has 44 diagonals. The number of sides is

(a) 9 (b) 8 (c) 7 (d) 11

22. If ${}^{n}P_{r-1} : {}^{n}P_{r} : {}^{n}P_{r+1} = a:b:c$, then

(a) $b^2 = a(b+c)$ (b) $2b = a + c$ (c) $b^2 = ac$ (d) $b = a + c$

23. The sum of all the 4 digits numbers formed with the digits 1, 3, 3, and 0

(a) 22544 (b) 25544 (c) 22554 (d) 25454

24. The number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together

(a) 24 (b) 35 (c) 30 (d) none of these

25. A student is allowed to select at most *n* books from a collection of (2*n* + 1) books. If the total numbers of ways in which he can select at least one book is 63, then the value of *n*

(a) 3 (b) 6 (c) 7 (d) 5

EXERCISE – III

IIT-JEE – SINGLE CHOICE CORRECT

1. $^{n}C_{k}+\sum_{j=0}^{m}{}^{n+j}C_{k-1}$ is equal to

(a) $^{n+m+1}C_{k}$ (b) $^{m+n}C_{k}$ (c) $^{n+m-1}C_{k}$ (d) $^{m+n}C_{k-1}$

2. How many numbers of 5 digits divisible by 25 can be made with the digits 0, 1, 2, 3, 4, 5, 6 and 7?

(a) 200 (b) 320 (c) 120 (d) 360

3. The number of ways in which the candidates $A_1, A_2, A_3, ...A_{10}$ can be ranked if A_1 is always above A_2 is

(a) 10 ! (b) $^{10}C_2$ (c) $\frac{10\,!}{2\,!}$ (d) none of these

4. How many words can be made with letters of the word INTERMEDIATE if no vowel is between two consonants?

(a) 151200 (b) 164000 (c) 144200 (d) none of these

5. The value of $\frac{n^2\,!}{(n\,!)^{n+2}}$, $\forall n \in N$ is

(a) rational (b) irrational (c) integer (d) natural number

6. The number of integral solutions of $x+y+z=0$ with $x, y, z \geq -5$

(a) 120 (b) 136 (c) 144 (d) none of these

7. There are 4n things of which n are alike and all the rest different. Then the number of permutations of 4n things taken 2n at a time, each permutation containing the n like things

(a) $\frac{4n\,!}{(n\,!)^2}$ (b) $\frac{4n\,!}{2n\,!}$ (c) $\frac{4n\,!}{n!}$ (d) $\frac{3n\,!}{(n\,!)^2}$

8. A committee of 5 can be formed from 8 men and 4 women so as to have at least two women in

(a) 525 ways (b) 520 ways (c) 530 ways (d) 456 ways

9. A question paper consists of two sections having 3 and 5 questions respectively. The following note is given on the paper "It is not necessary to attempt all the question. One question from each section is compulsory". The numbers of ways can a candidate select the questions is

(a) 218 (b) 219 (c) 217 (d) 220

10. There are 4 letters and 4 directed envelopes. The number of ways in which all the letters could be put into the wrong envelopes is

(a) 8 (b) 9 (c) 16 (d) none of these

11. The number of ways in which 6 different toys may be distributed among 4 children so that each child gets at least one toy is

(a) ${}^6P_4 \times 2^4$ (b) ${}^6C_4 \times 2^4$ (c) 1560 (d) none of these

12. There are five balls of different colours and five boxes of same colours as those of the balls. The number of ways in which the balls, one in each box, could be placed such that a ball does not go to a box of its own colour is

(a) 44 (b) 45 (c) 46 (d) none of these

13. The number of subsets of set $A = \{a_1, a_2, ...a_n\}$, which contain even number of elements (null selection included) is

(a) 2^{n-1} (b) $2^n - 1$ (c) 2^{n-2} (d) 2^n

14. The sum $\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$, $\left(\text{where } \binom{p}{q} = 0 \text{ if } p < q\right)$ is maximum when m is

(a) 5 (b) 10 (c) 15 (d) 20

15. The number of positive integral solution of the equation $x_1x_2x_3x_4x_5 = 1050$ is

(a) 1800 (b) 1600 (c) 1400 (d) 1875

16. The number of 5 digit numbers that contain 7 exactly once is

(a) $(41)(9^3)$ (b) $(37)(9^3)$ (c) $(7)(9^4)$ (d) $(41)(9^4)$

17. Total number of divisors of 480, that are of the form $4n + 2$, $n \geq 0$ is equal to

(a) 2 (b) 3 (c) 4 (d) none of these

18. Two players *A* and *B* plays a series of $2n$ games. Each game can result in either a win or a loss for *A*. Total number of ways in which *A* can win the series of these games, is equal to

(a) $\frac{1}{2}\left(2^{2n} - {}^{2n}C_n\right)$ (b) $\frac{1}{2}\left(2^{2n} - 2.\left({}^{2n}C_n\right)\right)$

(c) $\frac{1}{2}\left(2^{n} - {}^{2n}C_n\right)$ (d) $\frac{1}{2}\left(2^{n} - 2.{}^{2n}C_n\right)$

19. A is a set containing *n* elements. A subset *P* of *A* is chosen. The set *A* is reconstructed by replacing the elements of *P*. Then a subset *Q* of *A* is chosen, then the number of ways of choosing *P* and *Q* such that $P \cap Q$ contains exactly 2 elements

(a) 3^n (b) $n.3^{n-2}$ (c) $n.3^{n-1}$ (d) ${}^nC_2 3^{n-2}$

20. Let $E = \{1, 2, 3, 4\}$, $F = \{1, 2\}$, then the number of onto functions from E to F is

(a) 14 (b) 16 (c) 12 (d) 8

21. There are *p* intermediate stations on a railway line from one terminus to another. Number of ways a train can stop at 3 of these intermediate stations if no two of these stopping stations are to be consecutive is

(a) $(p-2)^3$ (b) $(p-2)!$ (c) ${}^{p-2}C_3$ (d) ${}^{p-2}P_3$

22. Total number of positive integral solution of $x_1 + x_2 + x_3 \le 10$ is

(a) ${}^{10}C_3$ (b) ${}^{12}C_3$ (c) ${}^{11}C_3$ (d) ${}^{13}C_3$

23. The number of ways in which three distinct numbers in an increasing A.P., can be selected from the set {1, 2, 3, …… 24} is

(a) 66 (b) 132 (c) 198 (d) none of these

24. If all permutations of the letters of the word AGAIN are arranged as in a dictionary, then 49th word is

(a) NAAGI (b) NAGAI (c) NAAIG (d) none of these

25. If *a, b, c, d* are odd natural numbers such that *a* + *b* + *c* + *d* = 20, then the number of values of ordered quadruplet (*a, b, c, d*) is

(a) 165 (b) 310 (c) 295 (d) 398

EXERCISE – IV

ONE OR MORE THAN ONE CHOICE CORRECT

1. Integral solutions to $x+y+z+t=29$, where $x \ge 1, y > 1, z \ge 3$ and $t \ge 0$, are

(a) ${}^{26}C_3$ (b) ${}^{26}C_4$ (c) 2600 (d) 1300

2. The number of ways 5 identical balls can be distributed into 3 different boxes so that no box remains empty

(a) ${}^{4}C_2$ (b) 6 (c) ${}^{4}C_1$ (d) ${}^{4}C_3$

3. The sum of all five digit numbers that can be formed using the digit 1, 2, 3, 4 and 5 (repetition of digits not allowed)

(a) 24 × 15 × 11111 (b) 5 ! × 15 × 11111
(c) 6 ! (d) 3999960

4. Integers between 1 and 1000000 have the sum of the digits equal to 18

(a) 25927 (b) 33649
(c) 7722 (d) ${}^{23}C_5 - 6 \times {}^{13}C_5$

5. Triangles can be formed by joining the vertices of a decagon

(a) ${}^{10}C_3$ (b) ${}^{10}C_2$ (c) 120 (d) 90

6. Mark the wrong statements among those given below:

(a) the number of arrangements of 10 different flowers in a garland is 9 !
(b) the number of ways in which 13 cards can be given to each of four players from a pack of 52 playing cards is $52\,!/(13\,!)^4$
(c) $\dfrac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \dfrac{n-r}{r}$
(d) if ${}^{n}P_r = 120\ {}^{n}C_r$, then $r = 6$

7. If $N = 37800$

(a) total number of factors of N is 96
(b) total number of factors of N is 92
(c) sum of odd proper divisors of N is 9919
(d) sum of odd proper divisors of N is 9918

8. The number of ways of choosing triplets (x, y, z) such that $z \ge \max\{x, y\}$ and $x, y, z \in \{1, 2, ..., n\}$ is

(a) ${}^{n+1}C_3 + {}^{n+2}C_3$ (b) $\dfrac{1}{6}n(n+1)(2n+1)$
(c) $1^2 + 2^2 + \ldots\ldots + n^2$ (d) $2\left({}^{n+2}C_3\right) - {}^{n+1}C_2$

9. The number of non-negative integral solutions of $2x+y+z=21$, is greater than

(a) 131 (b) 132 (c) 130 (d) 133

10. There are *n* married couples at a party. Each person shakes hand with every person other than her or his spouse. The total number of hand shakes are

(a) ${}^{2n}C_2 - n$ (b) ${}^{2n}C_2 - (n-1)$ (c) $2n(n-1)$ (d) ${}^{2n}C_2$

11. If *n* is the number of positive integral solutions of $x_1x_2x_3x_4 = 210$. Then

(a) *n* must be divisible by 3 distinct primes (b) *n* must be a perfect square

(c) *n* must be a perfect 4th power (d) *n* must be a perfect 8th power

12. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and let $\vec{r}$ be a variable vector such that $\vec{r}.\hat{i}, \vec{r}.\hat{j}$ and $\vec{r}.\hat{k}$ are positive integers. If $\vec{r}.\vec{a} \le 12$, then the number of values of $\vec{r}$ is

(a) ${}^{12}C_4$ (b) ${}^{12}C_3$ (c) ${}^{12}C_9$ (d) ${}^{12}C_8$

13. For the equation $x+y+z+w=19$, the number of positive integral solutions is equal to

(a) the number of ways in which 15 identical things can be distributed among 4 persons

(b) the number of ways in which 19 identical things can be distributed among 4 persons

(c) coefficient of x^{19} in $\left(x^0 + x^1 + x^2 + \ldots\ldots + x^{19}\right)^4$

(d) coefficient of x^{19} in $\left(x + x^2 + x^3 + \ldots\ldots + x^{19}\right)^4$

14. If *N* = 37800

(a) number of odd proper divisors = 23 (b) number of even proper divisors = 71

(c) number of even proper divisors = 72 (d) number of odd proper divisors 24

15. There are 5 mangoes and 4 apples. The number of ways of selecting at least one fruit in which

(a) fruits of the same kind are different = $2^9 - 1$

(b) fruits of the same kind are different = 2^9

(c) fruits of the same kind are identical = 29

(d) fruits of the same kind are identical = 30

EXERCISE – V

MATCH THE FOLLOWING

Note: Each statement in column – I has one or more than one match in column - II

1.

	Column I		Column II
I.	The number of arrangements of the letters of the word EAMCET in which no two vowels are together is	**A.**	24
II.	Total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is	**B.**	3645
III.	The number of odd numbers between 1000 and 10,000 that can be formed with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 is	**C.**	7200
IV.	Number of all four-digit numbers having different digits formed by 1, 2, 3, 4 and 5 and divisible by 4 is	**D.**	72

Note: Each statement in column – I has one or more than one match in column - II

2.

	Column I		Column II
I.	$x+y-1$ straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then the number of parts into which these lines divide the plane is one more than	**A.**	$(y-x-1)$
II.	If $1 \le x < y \le z$, then the number of subset of the set $A=\{1, 2, 3, ..., z\}$ having *x, y* as the least and the greatest elements respectively is 2 raised to the power	**B.**	$\Sigma(x+y-1)$
III.	If $x > y > z$ the number of different selection of $y+z$ things taken x at a time, where y things are identical and z other things are identical is	**C.**	$(x+y-1)$
IV.	The number of arrangements of $(x-1)$ things taken from x different things is y times the number of arrangements of $(x-1)$ things taken from x things in which two things are identical, then the value of y is $\frac{2k}{(x+y-1)}$, then k is	**D.**	$(y+z-x+1)$

Note: Each statement in column – I has one or more than one match in column - II

3.

Column I		Column II	
I.	In a seminar there are n people including A, B and C they have to speak in such a order that A has to speak before B and B has to speak before C	**A.**	The image part with relationship ID rId930 was not found in the file.
II.	There are n different numbers of which three x_1, x_2, x_3 numbers are chosen randomly, then in how many ways $x_1 > x_2 > x_3$	**B.**	$\frac{n\,!}{3\,!}$
III.	There are $(n-3)$ different balls and 3 different balls which are numbered 1, 2, 3 in how many ways can we arrange the balls such that 1 comes before 2 and 2 comes before 3	**C.**	${}^{n}P_{n-3}$
IV.	There are $(n-3)$ different toys that has to be given to n students in such a manner that exactly three students will not receive any number of toys and other students will receive exactly one toy	**D.**	${}^{n}C_3\,(3\,!)$

REASONING TYPE

Directions: Read the following questions and choose

- **(A) If both the statements are true and statement-2** is the correct explanation of **statement-1.**
- **(B) If both the statements are true but statement-2** is not the correct explanation of **statement-1.**
- **(C) If statement-1** is True and **statement-2** is False.
- **(D) If statement-1** is False and **statement-2** is True.

1. **Statement-1:** The number of ways of distributing n identical objects in r distinct boxes is ${}^{n+r-1}C_{r-1}$.

Statement-2: The number of arrangement of n objects of one kind and $r-1$ objects of another kind in a line must be $\frac{(n+r-1)!}{n!\,(r-1)!}$.

(a) A (b) B (c) C (d) D

2. **Statement-1:** The number of selections of r objects from n types of objects where each may be taken as after as we please must be ${}^{n+r-1}C_r$.

Statement-2: The number of non-negative integer solutions of $x_1+x_2+\ldots+x_n=r$ is ${}^{r+n-1}C_{n-1}$.

(a) A (b) B (c) C (d) D

3. **Statement-1:** The number of selections of four letters taken from the word PARALLEL must be 15.

Statement-2: Coefficient of x^4 in the expansion of $(1-x)^{-3}$ is 15.

(a) A (b) B (c) C (d) D

4. **Statement-1:** The number of ways of writing 1400 as a product of two positive integers is 12.

Statement-2: 1400 is divisible by exactly three prime numbers.

(a) A (b) B (c) C (d) D

5. **Statement-1:** The expression $n!(20-n)!$ is minimum when $n=10$.

Statement-2: ${}^{2m}C_r$ is maximum when $r=m$.

(a) A (b) B (c) C (d) D

LINKED COMPREHENSION TYPE

Two persons *A* and *B* are using a set of $(2m + 1)$ successive natural numbers starting from $1\,(m \geq 1)$. '*A*' is asked to speak the sets of all the possible three numbers that are in A.P., with positive common difference and the other is asked to speak sets of the form (1), (1, 2, 3,), (1, 2, 3, 4, 5), …… , $(1, 2, 3, \ldots. 2m+1)$, (2), (2, 3, 4), (2, 3, 4, 5, 6) …… $(2, 3, 4, \ldots. 2m)$ …… and so on i.e. the sets containing odd number of consecutive numbers but starting from every natural number. If *A* speaks such '*P*' number of sets and '*B*' speaks such '*Q*' number of sets then

1. $P.Q$ is equal to

(a) $6\sum_{r=1}^{m} r^2$ (b) $4\sum_{r=1}^{m} r^3$ (c) $2\sum_{r=1}^{m} r$ (d) $3\sum_{r=1}^{m} r^2(r+1)$

2. $|P-Q|$ is

(a) $< 2m+1$ (b) $> 2m+1$ (c) $= 2m+1$ (d) none of these

3. $P + Q$ is

(a) an even number for all *m* (b) an odd number for all *m*

(c) an even number only if *m* is even (d) an even number only if *m* is odd

4. After completing the process (as mentioned in the passage), again if $P + Q$ numbers are given to them and the whole process is repeated, it is noted that the difference of the sets is 841, then initially '*m*' was

(a) 21 (b) 20 (c) 41 (d) 85

EXERCISE – VI

SUBJECTIVE PROBLEMS

1. How many even numbers of 4 digits can be formed with the digits 2, 4, 5, 7, 8, no digit being used more than once in each number?

2. Find the sum of all numbers greater than 10,000 formed by using the digits 0, 2, 4, 6, 8, no digit being repeated in any number.

3. There are 12 intermediate stations on a railway line between 2 stations. In how many ways can a train be made to stop at 4 of these intermediate stations, no two of these halting stations being consecutive?

4. Find the number of seven digit numbers in which every digit is less than the immediately preceding one.

5. Find the number of combinations of the letters occurring in the word “MISSISSIPPI” taken five at a time.

6. A student takes an examination consisting of four papers, each with a maximum of n marks. Show that the number of ways of getting a total of 2n marks in all the papers is $(n + 1)(2n^2 + 4n + 3)/3$.

7. How many different numbers of six digits can be formed of 4, 5, 6, 7, 8, 9 (without repetition)? Find the sum of all these numbers. How many numbers are divisible by 5?

8. How many four digit numbers are there which contain not more than two different digits?

9. In a plane, there are n straight lines of which p pass through the point A and q pass through the point B. Besides, no three lines pass through one point, no line passes through both the points A and B, and no two are parallel. How many intersection points do the lines have?

10. There are m seats in the first row of a theatre, of which n are to be occupied. Find the number of ways of arranging n persons so that
(i) no two persons sit side by side
(ii) each person has exactly one neighbour ; and
(iii) out of any two seats located symmetrically about the middle of the row at least one is empty.

ANSWERS

EXERCISE – I

CBSE PROBLEMS

1. 49

2. (i) 60 (ii) 125

3. 24

4. 720

5. $\frac{1}{2}(7!)$

6. 12

7. 7539840

8. 50400

9. 86400

10. 11

11. 215

12. (i) 1440 (ii) 3600 (iii) 1440

13. 51300 (i) 10260 (ii) 7695 (iii) 43605

14. (i) 9 (ii) 104

15. 91

16. 6! 4!

17. $\frac{7\,!}{2\,!\,3\,!\,2\,!}$

18. 455

19. 3960

20. $^8C_5 \,.\, ^6C_5$

EXERCISE – II

IIT-JEE-SINGLE CHOICE CORRECT

1. (a)	2. (c)	3. (d)	4. (a)	5. (b)
6. (b)	7. (a)	8. (c)	9. (a)	10. (d)
11. (c)	12. (b)	13. (c)	14. (a)	15. (b)
16. (a)	17. (c)	18. (c)	19. (a)	20. (b)
21. (d)	22. (a)	23. (c)	24. (b)	25. (a)

EXERCISE – III

IIT-JEE – SINGLE CHOICE CORRECT

1. (a)	2. (b)	3. (c)	4. (a)	5. (a)
6. (b)	7. (d)	8. (d)	9. (c)	10. (b)
11. (c)	12. (a)	13. (a)	14. (c)	15. (d)
16. (a)	17. (c)	18. (a)	19. (d)	20. (a)
21. (c)	22. (a)	23. (b)	24. (a)	25. (a)

EXERCISE – IV

ONE OR MORE THAN ONE CHOICE CORRECT

1. (a, c)	2. (a, b)	3. (a, d)	4. (a, d)	5. (a, c)
6. (a, c, d)	7. (a, c)	8. (a, b, c, d)	9. (a, c)	10. (a, c)
11. (b, c, d)	12. (b, c)	13. (a, d)	14. (a, b)	15.(a, c)

EXERCISE – V

MATCH THE FOLLOWING

1. I-(D); II-(C); III-(B); IV-(A)

2. I-(B); II-(A); III-(D); IV-(C)

3. I-(A), (B), (C); II-(A), (B), (C); III-(A), (B), (C); IV-(A), (B), (C)

REASONING TYPE

1. (a)	2. (a)	3. (d)	4. (b)	5. (a)

LINKED COMPREHENSION TYPE

1. (b)	2. (c)	3. (b)	4. (b)

EXERCISE – VI

SUBJECTIVE PROBLEMS

1. 72

2. 5199960

3. 126

4. 120

5. 25

7. (i) 720 (ii) 519999480

8. 576

9. ${}^{n}C_2 - {}^{p}C_2 - {}^{q}C_2 + 2$

10. (i) ${}^{m-n+1}P_n$
(ii) ${}^{n}P_{n/2} \, . \, {}^{m-n+1}P_{n/2}$
(iii) ${}^{(m-1)/2}P_n \, (2^n) + n(2^{n-1}) \, {}^{(m-1)/2}P_{n-1}$