MATHS
CBSE - 12

2024

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MATHS

DETERMINANTS

1 DETERMINANTS

1.1 EVALUATION OF DETERMINANTS

Determinants of second order: The symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ consisting of 2^2 numbers called elements, arranged in two rows and two columns, is called a determinant of second order. The elements a_1 and b_2 are said to lie along the principal diagonal; the elements a_2 and b_1 are said to lie along the secondary diagonal.

The value of the determinant is obtained by forming the product of the elements along the principal diagonal and subtracting from it the product of the elements along the secondary diagonal.

Thus $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$ ... (i)

Determinants of third order: The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of 3^2 elements arranged in three rows and three columns, is called a determinant of third order. Its value is

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

This may be written as $a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$

or $\quad a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

We can therefore write

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad \text{... (ii)}$$

Note that each term of a second order determinant is the product of two quantities and each term of a third order determinant is the product of three quantities.

1.2 MINORS

The minor of a given element of a determinant is the determinant of the elements which remain after deleting the row and the column in which the given element occurs.

For example, the minor of a_1 in (ii) is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, and the minor of b_2 in (ii) is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

The minor of a_1 in (i) is b_2 and b_2 may be considered a determinant of first order. Similarly, the minor of a_2 is b_1.

1.3 COFACTORS

In (ii), the elements a_1, b_1, c_1 are multiplied by

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

These expressions are called the cofactors of the elements a_1, b_1, c_1.

Generally, the cofactor of an element is its minor with its sign or opposite sign prefixed in accordance with the following rule:

For any determinant if a_{ij} be the element at the intersection of the ith row and jth column, then the cofactor of a_{ij} has positive sign or negative sign before minor of a_{ij} according as $i + j$ is even or odd. The determinant may be expanded along any chosen row or column.

The cofactors of the elements a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 will be denoted by A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3 respectively.

For example, element b_3 in (ii) lies at the intersection of the third row and the second column. Since 3 + 2 = 5 is an odd number, we have

$$B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The cofactor B_2 of the element b_2 is $+\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$, because element b_2 lies at the intersection of the second row and the second column, and 2 + 2 = 4 is an even number.

Let the determinant (ii) be denoted by Δ. When the cofactors are used, the expansion of the determinant takes the compact form:

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1 = a_2A_2 + b_2B_2 + c_2C_2 = a_3A_3 + b_3B_3 + c_3C_3.$$

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 = b_1B_1 + b_2B_2 + b_3B_3 = c_1C_1 + c_2C_2 + c_3C_3$$

and $a_2A_1 + b_2B_1 + c_2C_1 = 0 = a_2A_3 + b_2B_3 + c_2C_3$ etc.

Illustration 1

Question: **Evaluate the determinant**

$$\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 5 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix}.$$

Solution: Expanding along the second row, we have

$$\Delta = -5\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} -2\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} -1\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$= -5(9 - 8) - 2(6 - 4) - 1(4 - 3)$

$= -5 - 4 - 1 = -10.$

Expanding along the third column, we have

$$\Delta = 4\begin{vmatrix} 5 & -2 \\ 1 & 2 \end{vmatrix} -1\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} +3\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}$$

$= 4(10 + 2) - 1(4 - 3) + 3(-4 - 15)$

$= 48 - 1 - 57 = -10.$

Illustration 2

Question: **Show that**

$$\begin{vmatrix} \mathbf{1} & \boldsymbol{a} & \boldsymbol{a^2} \\ \mathbf{1} & \boldsymbol{b} & \boldsymbol{b^2} \\ \mathbf{1} & \boldsymbol{c} & \boldsymbol{c^2} \end{vmatrix} = \mathbf{(a-b)\,(b-c)\,(c-a).}$$

Solution: Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

If b is put equal to a, two rows are exactly alike.

$\therefore \Delta = 0$ when $b = a$

$\therefore$ $(a-b)$ is a factor of Δ (This follows from the factor theorem which states that for $f(x)$, if $f(a) = 0$, then $(x- a)$ is a factor of $f(x)$).

Similarly $(b-c)$ and $(c- a)$ are factors.

Again, Δ is of the third degree in a, b and c.

And we know already three linear factors $(a-b)$, $(b-c)$ and $(c-a)$. If there is another factor, it must be a mere number.

Thus $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = N\,(a-b)\,(b-c)\,(c-a)$, where N is a number.

By equating coefficients of bc^2 on both sides, $N = 1$

$\therefore \Delta = (a-b)\,(b-c)\,(c-a)$.

Alternative method:

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Subtracting the second row from the first and then the third row from the second, we have

$$\Delta = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)\,(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Now expanding along the first column, we have

$\Delta = (a-b)\,(b-c)\,[(b + c) - (a + b)] = (a-b)\,(b-c)\,(c-a)$.

Illustration 3

Question: **Show that**

$$\Delta = \begin{vmatrix} \boldsymbol{b+c} & \boldsymbol{c} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{c+a} & \boldsymbol{a} \\ \boldsymbol{b} & \boldsymbol{a} & \boldsymbol{b+a} \end{vmatrix} = \mathbf{4abc.}$$

Solution:

$$\Delta = \begin{vmatrix} 2(b+c) & 2(c+a) & 2(a+b) \\ c & c+a & a \\ b & a & a+b \end{vmatrix} \text{ by } R_1 : R_1 + R_2 + R_3$$

Now take 2 as a common factor and then apply $R_2 : R_2-R_1$ and $R_3 : R_3-R_1$

$$\Delta = 2\begin{vmatrix} b+c & c+a & a+b \\ -b & 0 & -b \\ -c & -c & 0 \end{vmatrix}$$

Now apply $C_2 : C_2 - C_1$

$$\Delta = 2\begin{vmatrix} b+c & a-b & a+b \\ -b & b & -b \\ -c & 0 & 0 \end{vmatrix}$$

Now expand through R_3

$\Delta = 2[(-c)\{-ab + b^2 - ab - b^2\}] = 4abc$

Illustration 4

Question: **Show that** $\Delta = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^2-bc} & \mathbf{b^2-ca} & \mathbf{c^2-ab} \end{vmatrix} = \mathbf{0}$.

Solution: $C_1 : C_1 - C_2$ and $C_2 : C_2 - C_3$

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2+c(a-b) & b^2-c^2+a(b-c) & c^2-ab \end{vmatrix}$$

i.e., $(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b+c & b+c+a & c^2-ab \end{vmatrix}$

i.e., $(a-b)(b-c)[(b+c+a)-(a+b+c)] = 0$.

Note: If a determinant can be so transformed that two elements in a row or column are made zero, then the determinant can be expanded in terms of that row or column.

Illustration 5

Question: **Show that** $\begin{vmatrix} \mathbf{1} & \mathbf{a} & \mathbf{a^2} \\ \mathbf{1} & \mathbf{b} & \mathbf{b^2} \\ \mathbf{1} & \mathbf{c} & \mathbf{c^2} \end{vmatrix} = \begin{vmatrix} \mathbf{1} & \mathbf{bc} & \mathbf{b+c} \\ \mathbf{1} & \mathbf{ca} & \mathbf{c+a} \\ \mathbf{1} & \mathbf{ab} & \mathbf{a+b} \end{vmatrix}$.

Solution: We have $\begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix} = \begin{vmatrix} 1 & bc & a+b+c-a \\ 1 & ca & a+b+c-b \\ 1 & ab & a+b+c-c \end{vmatrix}$

$$= \begin{vmatrix} 1 & bc & a+b+c \\ 1 & ca & a+b+c \\ 1 & ab & a+b+c \end{vmatrix} - \begin{vmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} + \begin{vmatrix} bc & 1 & a \\ ca & 1 & b \\ ab & 1 & c \end{vmatrix}$$

$= \begin{vmatrix} bc & 1 & a \\ ca & 1 & b \\ ab & 1 & c \end{vmatrix}$, since the first determinant vanishes

$$= \frac{1}{abc}\begin{vmatrix} abc & a & a^2 \\ abc & b & b^2 \\ abc & c & c^2 \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Illustration 6

Question: Without expanding the determinants, prove that

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

Solution: The determinant on the left is equal to

$$\begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \end{vmatrix}$$

$$=2\begin{vmatrix} a+b+c & b+c & c+a \\ a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \end{vmatrix}$$

$$=2\begin{vmatrix} a & b+c & c+a \\ b & c+a & a+b \\ c & a+b & b+c \end{vmatrix}$$

$$=2\begin{vmatrix} a & b+c & c \\ b & c+a & a \\ c & a+b & b \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Illustration 7

Question: Show that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Solution: Let Δ stand for the determinant on the left.

$$\text{Then } \Delta = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

3 SUM OF DETERMINANTS

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ be two third order determinants in which corresponding second and third columns are identical.

$$\text{Then } \Delta_1 + \Delta_2 = \begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix}$$

This fact is evident if we expand all the three determinants in terms of column 1 and compare the results.

Similarly if $\Delta_3 = \begin{vmatrix} p_1 & q_1 & r_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

then $\Delta_1 + \Delta_3 = \begin{vmatrix} a_1+p_1 & b_1+q_1 & c_1+r_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Here we note that the corresponding second and third rows are identical.

Similarly the determinant $\begin{vmatrix} d_1+e_1+f_1 & d_2+e_2+f_2 & d_3+e_3+f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Can be decomposed into the sum of three determinants

$$\begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It may be observed that the determinant

$\begin{vmatrix} a_1+b_1 & c_1+d_1 & e_1+f_1 \\ a_2+b_2 & c_2+d_2 & e_2+f_2 \\ a_3+b_3 & c_3+d_3 & e_3+f_3 \end{vmatrix}$ can be expressed as sum of $2 \times 2 \times 2 = 8$ determinants.

Illustration 8

Question: **If** $A_k = \begin{vmatrix} 2^{k-1} & x & 2^n-1 \\ 2\,(3^{k-1}) & y & 3^n-1 \\ 3\,(4^{k-1}) & z & 4^n-1 \end{vmatrix}$, **prove that** $\sum_{k=1}^{n} A_k = 0$.

Solution: Observe that all the determinants $A_1, A_2, \ldots, A_n$ have identical second and third column.

Hence $\sum_{k=1}^{n} A_k = \begin{vmatrix} \sum_{k=1}^{n} 2^{k-1} & x & 2^n-1 \\ \sum_{k=1}^{n} 2(3^{k-1}) & y & 3^n-1 \\ \sum_{k=1}^{n} 3(4^{k-1}) & z & 4^n-1 \end{vmatrix}$

Now $\sum_{k=1}^{n} 2^{k-1} = 1 + 2 + 2^2 + \ldots + 2^{n-1} = \frac{2^n-1}{2-1} = 2^n-1$ (sum of n terms of a G.P.)

$$\sum_{k=1}^{n} 2(3^{k-1}) = 2\left\{1 + 3 + 3^2 + \ldots + 3^{n-1}\right\} = \frac{2(3^n-1)}{3-1} = 3^n-1$$

and $\sum_{k=1}^{n} 3(4^{k-1}) = 4^n - 1$

Hence $\sum_{k=1}^{n} A_k = \begin{vmatrix} 2^n-1 & x & 2^n-1 \\ 3^n-1 & y & 3^n-1 \\ 4^n-1 & z & 4^n-1 \end{vmatrix} = 0$ since $C_1 = C_3$.

Illustration 9

Question: If $f(r)=\begin{vmatrix} 2r+1 & {}^nC_r & 1 \\ n^2+2n+1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$, $0 \le r \le n$, then prove that $\sum_{r=0}^{n} f(r) = 0$.

Solution: Since R_2 and R_3 are constants (independent of the variable r)

$$\sum_{r=0}^{n} f(r) = \begin{vmatrix} \sum_{r=0}^{n}(2r+1) & \sum_{r=0}^{n}{}^nC_r & \sum_{r=0}^{n}1 \\ n^2+2n+1 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$$

Now $\sum_{r=0}^{n}(2r+1) = 1+3+......+(2n+1) = (n+1)^2$

$$\sum_{r=0}^{n}{}^nC_r = {}^nC_0 + {}^nC_1 + ...+{}^nC_n = 2^n$$

$$\sum_{r=0}^{n}1 = 1 + 1 + \ldots \ldots (n+1) \text{ times} = n+1$$

$$\therefore \sum_{r=0}^{n} f(r) = \begin{vmatrix} (n+1)^2 & 2^n & n+1 \\ (n+1)^2 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix} = 0 \quad \text{since } R_1 = R_2$$

4 MULTIPLICATION OF DETERMINANTS

Two determinants of the same order i.e. each consisting of the same number of rows and equal number of columns can be multiplied to give a determinant of the same order. Thus if A is a 2 × 2 determinant and *B* is another 2 × 2 determinant, A ×*B* = *C* is also 2 × 2 determinant. The multiplication is done by a method of working the row of A on the columns of *B*.

The method is as follows:

$$A = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}; B = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$$

then AB = $\begin{vmatrix} a_1\alpha_1 + a_2\beta_1 & a_1\alpha_2 + a_2\beta_2 \\ b_1\alpha_1 + b_2\beta_1 & b_1\alpha_2 + b_2\beta_2 \end{vmatrix}$

To cite a numerical example for a 3 × 3 determinant; we have

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 6 \\ 3 & 0 & 2 \end{vmatrix} \times \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1\times2+3\times0+4\times1 & 1\times1+3\times1+4\times2 & 1\times2+3\times3+4\times4 \\ 2\times2+(-1)\times0+6\times1 & 2\times1+(-1)\times1+6\times2 & 2\times2+(-1)\times3+6\times4 \\ 3\times2+0\times0+2\times1 & 3\times1+0\times1+2\times2 & 3\times2+0\times3+2\times4 \end{vmatrix}$$

(The first row is obtained by working the first row elements 1, 3, 4 respectively on 2, 0, 1 the first column; then on 1, 1, 2 the 2nd column; then on 2, 3, 4 the 3rd column. Likewise for the 2nd and the 3rd row.)

$$= \begin{vmatrix} 6 & 12 & 27 \\ 10 & 13 & 25 \\ 8 & 7 & 14 \end{vmatrix}$$

Verification

$$A = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 6 \\ 3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ 0 & -7 & -2 \\ 0 & -9 & -10 \end{vmatrix} \begin{matrix} \to (R_2 - 2R_1) = 70 - 18 = 52 \\ \to (R_5 - 3R_1) \end{matrix}$$

$$B = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -3 & -6 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} \to (R_1 - 2R_3) = -9 + 6 = -3$$

$$C = \begin{vmatrix} 6 & 12 & 27 \\ 10 & 13 & 25 \\ 8 & 7 & 14 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 9 \\ 10 & 13 & 25 \\ 8 & 7 & 14 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 & 9 \\ 0 & 2 & 2 \\ 8 & 7 & 14 \end{vmatrix} \to R_2 - (R_1 + R_3)$$

$$= 3\begin{vmatrix} 2 & 4 & 5 \\ 0 & 2 & 0 \\ 8 & 7 & 7 \end{vmatrix} = 6(14 - 40) = -156$$

$\therefore AB = -156 = C$

The multiplication can also be performed row by row, column by row or column by column.

Illustration 10

Question: **Show that** $\begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$.

Where $r^2 = x^2 + y^2 + z^2$ **and** $u^2 = yz + zx + xy$.

Solution: Consider the determinant $\Delta = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$

Minor of x is $yz - x^2$; of y is $y^2 - zx$, of z is $xy - z^2$

L.H.S. determinant in the problem is therefore

$\begin{vmatrix} X & -Y & Z \\ -Y & Z & -X \\ Z & -X & Y \end{vmatrix}$, where capital letters denote the minor of the corresponding small letters.

$$\therefore LHS = \begin{vmatrix} X & -Y & Z \\ -Y & Z & -X \\ Z & -X & Y \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$= \begin{vmatrix} x^2 + y^2 + z^2 & xy + yz + zx & xz + xy + yz \\ xy + yz + zx & x^2 + y^2 + z^2 & xy + yz + zx \\ xy + yz + zx & xy + yz + zx & x^2 + y^2 + z^2 \end{vmatrix}$$

$= \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}$ in the notation of the problem.

Illustration 11

Question: **For all values of A, *B*, *C* and *P*, *Q*, *R*.**

Show that $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A+R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$.

Solution: The given determinant is the product of

$$\Delta_1 = \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} \cos P & \cos Q & \cos R \\ \sin P & \sin Q & \sin R \\ 0 & 0 & 0 \end{vmatrix}$$

and Δ_1, Δ_2 each = 0 and hence the problem.

(or) alternately

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \cos P \begin{vmatrix} \cos A & \cos(A-Q) & \cos(A-R) \\ \cos B & \cos(B-Q) & \cos(B-R) \\ \cos C & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \sin P \begin{vmatrix} \sin A & \cos(A-Q) & \cos(A-R) \\ \sin B & \cos(B-Q) & \cos(B-R) \\ \sin C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$= (\cos P)\, A_1 + (\sin P)\, B_1$ Where

$$A_1 = \begin{vmatrix} \cos A & \sin A \sin Q & \sin A \sin R \\ \cos B & \sin B \sin Q & \sin B \sin R \\ \cos C & \sin C \sin Q & \sin C \sin R \end{vmatrix}$$

Where C_2 is $C_2 - (\cos Q)\, C_1$
C_3 *is* $C_3 - (\cos R)\, C_1$

$$= \sin Q \sin R \begin{vmatrix} \cos A & \sin A & \sin A \\ \cos B & \sin C & \sin B \\ \cos C & \sin C & \sin C \end{vmatrix} = 0$$

(2^{nd} and 3^{rd} columns are identical)

Similarly it may be proved that $B_1 = 0$.

5 DIFFERENTIATION OF DETERMINANTS

Differentiation of a determinant whose elements are functions of a variable *x*.

Let $F(x) = \begin{vmatrix} f(x) & g(x) \\ h(x) & u(x) \end{vmatrix}$

Then F(*x*) = f(*x*) · u(*x*) – g(*x*) · h(*x*)

and $F'(x) = \dfrac{d}{dx} F(x) = \{f(x) \cdot u'(x) + u(x) \cdot f'(x)\} - \{g(x) \cdot h'(x) + h(x)\, g'(x)\}$

$$= \begin{vmatrix} f'(x) & g'(x) \\ h(x) & u(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ h'(x) & u'(x) \end{vmatrix}$$

Thus F′(*x*) is the sum of two determinants, of which the first one is obtained by differentiating the elements of the first row alone and retaining the second row without any change and the second one is obtained by differentiating the elements of the second row.

Similarly if $F(x) = \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix}$

then $F'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) & h_1'(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2'(x) & g_2'(x) & h_2'(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3'(x) & g_3'(x) & h_3'(x) \end{vmatrix}$

Illustration 12

Question: **If α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degrees 3, 4 and 5 respectively, then show that**

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

is divisible by f(x), where dash denotes the derivatives.

Solution:

Let $g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

Then $g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

Now $g(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

Since two rows are identical, we have $g(\alpha) = 0$

$$g'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

Since two rows are identical, we have $g'(\alpha) = 0$

Since $g(\alpha) = 0$ and also $g'(\alpha) = 0$, α is a repeated root of $g(x) = 0$

$\therefore$ $g(x) = (x-\alpha)^2 h(x)$ … (i)

Since α is a repeated root of $f(x) = 0$, we have

$f(x) = N(x-\alpha)^2$ … (ii)

where N is a some number. From (i) and (ii) we find that $g(x)$, i.e., the given determinant is divisible by $f(x)$.

6 SPECIAL DETERMINANTS

6.1 SYMMETRIC DETERMINANT

If the elements of a determinant are such that $a_{ij} = a_{ji}$, (where a_{ij} is the element of i^{th} row and j^{th} column), then the determinant is said to be a symmetric determinant. The elements situated at equal distances from the diagonal are equal both in magnitude and sign.

e.g.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

6.2 SKEW SYMMETRIC DETERMINANT

If $a_{ij} = -a_{ji}$ (where a_{ij} is the element of i^{th} row and j^{th} column), then the determinant is said to be a skew symmetric determinant, which means that all the diagonal elements are zero and the elements situated at equal distances from the diagonal are equal in magnitude but opposite in sign. The value of a skew symmetric determinant of odd order is zero.

6.3 CIRCULANT DETERMINANTS

The elements of the rows (or columns) are in cyclic arrangement.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc) = -\frac{1}{2}(a+b+c) \times \{(a-b)^2 + (b-c)^2 + (c-a)^2\}$$

Illustration 13

Question: **Evaluate the determinant** $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

and show that it is negative for all positive values of a, *b* and *c*.

Solution: Expanding along the first row, we have

$$\Delta = a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix}$$

$\Delta = a(bc–a^2) –b(b^2–ca) + c(ab–c^2) = 3abc–a^3–b^3–c^3$

$= – (a^3 + b^3 + c^3– 3abc) = – (a + b + c)\ \ \{a^2 + b^2 + c^2–ab–bc–ca\}$

$= -\dfrac{(a+b+c)}{2}\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}$ is negative if *a*, *b* and *c* are positive.

7 SYSTEM OF LINEAR EQUATIONS

7.1 THE SYSTEM OF TWO LINEAR EQUATIONS IN TWO UNKNOWNS

Consider the system of two linear equations in two unknowns:

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

Solving the system we get the answer

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}};\quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Note: The given equations are consistent and independent if and only if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$.

Illustration 14

Question: **Solve the system 4*x* + *y* = 13, 3*x*– 2*y* = 7 using determinants.**

Solution: The solution requires the values of three determinants.

The denominator Δ is formed by writing the coefficients of *x* and *y* in order

$$\Delta = \begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = – 8 – 3 = – 11$$

d_1, the numerator of *x* is formed by replacing the coefficients of *x* by the constant terms.

$$d_1 = \begin{vmatrix} 13 & 1 \\ 7 & -2 \end{vmatrix} = – 26 – 7 = – 33$$

d_2, the numerator of *y* is formed by replacing the coefficients of *y* by the constant terms.

$$d_2 = \begin{vmatrix} 4 & 13 \\ 3 & 7 \end{vmatrix} = 28 – 39 = – 11$$

Then $x = \dfrac{d_1}{\Delta} = \dfrac{-33}{-11} = 3$ and $y = \dfrac{d_2}{\Delta} = \dfrac{-11}{-11} = 1$

7.2 SYSTEM OF THREE EQUATIONS IN TWO UNKNOWNS

The following system of equations

$a_1x + b_1y + c_1 = 0;\ \ a_2x + b_2y + c_2 = 0;\ \ a_3x + b_3y + c_3 = 0$ is consistent

if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Illustration 15

Question: **Find those values of c for which the equation $2x+3y=3;\ (c+2)x+(c+4)y=(c+6)$ and $(c+2)^2x+(c+4)^2y=(c+6)^2$ are consistent. Also solve the equations for those values of c.**

Solution: The condition for consistency is $\begin{vmatrix} 2 & 3 & 3 \\ c+2 & c+4 & c+6 \\ (c+2)^2 & (c+4)^2 & (c+6)^2 \end{vmatrix} = 0$

i.e. $\begin{vmatrix} -1 & 3 & 0 \\ -2 & c+4 & 2 \\ -2(2c+6) & (c+4)^2 & 2(2c+10) \end{vmatrix} = 0$

i.e. $(-1)\{(c+4)(2c+10)-(c+4)^2\}-3\{-2(2c+10)+2(2c+6)\}=0$

i.e. $c^2+8c+16-2c^2-18c-40+12c+60-12c-36=0$

i.e. $-c^2-10c=0 \Rightarrow \quad c=0 \quad$ or $\ c=-10$

For $c = 0$,the 3 equation are 2x + 2y = 3

2x + 4y = 6 ; 4x + 16y = 36

and the solution is $x = -3$; $y = 3$. For $c = -10$, the equation are 2x + 3y = 3

$-8x-6y=-4 \Rightarrow 4x+3y=2$

$64x-36y=16 \Rightarrow 16x+9y=4$ and the corresponding solution is $x=-\frac{1}{2};\ y=\frac{4}{3}$

7.3 THE SYSTEM OF THREE LINEAR EQUATIONS IN THREE UNKNOWNS

Consider the system of three linear equations in three unknowns:

$a_1x + b_1y + c_1z = p$

$a_2x + b_2y + c_2z = q$

$a_3x + b_3y + c_3z = r$

The solution of the system may be expressed as

$$x=\frac{d_1}{\Delta},\ y=\frac{d_2}{\Delta},\ z=\frac{d_3}{\Delta}, \text{ where}$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ d_1 = \begin{vmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{vmatrix},\ d_2 = \begin{vmatrix} a_1 & p & c_1 \\ a_2 & q & c_2 \\ a_3 & r & c_3 \end{vmatrix},\ d_3 = \begin{vmatrix} a_1 & b_1 & p \\ a_2 & b_2 & q \\ a_3 & b_3 & r \end{vmatrix}$$

Note: The determinant Δ is formed by writing the coefficients of x, y, z in order while the determinant appearing in the numerator for any unknown is obtained from Δ by replacing the column of coefficients of that unknown by the column of constants.

7.4 CRAMERS RULE

Consider the system of n linear equations in n unknowns given by

$$a_{11}x_1+a_{12}x_2+\dots\dots+a_{1n}x_n=b_1$$
$$a_{21}x_1+a_{22}x_2+\dots\dots+a_{2n}x_n=b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{n1}x_1+a_{n2}x_2+\dots\dots+a_{nn}x_n=b_n$$

Let $\quad D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ . \\ . \\ . \\ b_n \end{vmatrix}$

Then, if $D \neq 0$, we have $x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, ..., x_n = \frac{D_n}{D}$.

Discuss $D = 0$ cases

Case (i)

If $D = 0$ and the other determinants $D_1 = D_2 = ... = D_n = 0$, then system of equation has infinitely many solutions if all cofactors of $D_1, D_2, ..., D_n$ and D are zero. If any one cofactor of $D_1, D_2, D_3, ..., D_n$ is non zero then system has no solution.

eg. $x + 3y + 2z = 1;\ 2x + 6y + 4z = 5;\ 3x + 9y + 6z = 9$

Here $D_x = D_y = D_z = D = 0$ yet system has no solution where as

$x + 3y + 2z = 1;\ 2x + 6y + 4z = 2;\ 3x + 9y + 6z = 3$ has infinitely many solutions.

Case (ii)

If $D = 0$ but any one of the D_1, D_2 .... or D_n is not equal to zero then the system has no solution, hence is inconsistent.

Illustration 16

Question: **Solve the system**

$x + 4y + 4z = 7$

$3x + 2y + 2z = 6$

$9x + 6y + 2z = 14$ using determinants.

Solution: The solution requires the values of four determinants:

The denominator $\Delta = \begin{vmatrix} 1 & 4 & 4 \\ 3 & 2 & 2 \\ 9 & 6 & 2 \end{vmatrix} = 40$

d_1, the numerator of $x = \begin{vmatrix} 7 & 4 & 4 \\ 6 & 2 & 2 \\ 14 & 6 & 2 \end{vmatrix} = 40$

d_2, the numerator of $y = \begin{vmatrix} 1 & 7 & 4 \\ 3 & 6 & 2 \\ 9 & 14 & 2 \end{vmatrix} = 20$

d_3, the numerator of $z = \begin{vmatrix} 1 & 4 & 7 \\ 3 & 2 & 6 \\ 9 & 6 & 14 \end{vmatrix} = 40$

Then $x = \frac{d_1}{\Delta} = \frac{40}{40} = 1$

$y = \frac{d_2}{\Delta} = \frac{20}{40} = \frac{1}{2}$

$z = \frac{d_3}{\Delta} = \frac{40}{40} = 1$

7.5 THE SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS

Eliminant and non-trivial solution: If the three equations (homogeneous)

$a_1x + b_1y + c_1z = 0; a_2x + b_2y + c_2z = 0$ and $a_3x + b_3y + c_3z = 0$ be considered then there always exists a solution i.e., $x = y = z = 0$. This is called the **Trivial Solution.**

If the three equations are to have a solution other than $x = 0 = y = z$, such a solution is known as **Non-Trivial** solution, then the condition required for the existence of such a solution is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Illustration 17

Question: **Let λ and α be real. Find the set of all values of λ for which the system of linear equations:**

$$\lambda x + (\sin\alpha)\, y + (\cos\alpha)\, z = 0$$

$$x + (\cos\alpha)\, y + (\sin\alpha)\, z = 0$$

$$-x + (\sin\alpha)\, y - (\cos\alpha)\, z = 0$$

has a non-trivial solution. For $\lambda = 1$ find all the values of α.

Solution: The condition for the existence of non-trivial solution (trivial solution is $x = y = z = 0$) is

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0 \quad \text{i.e.,} \quad \begin{vmatrix} 0 & \sin\alpha\,(\lambda+1) & \cos\alpha\,(1-\lambda) \\ 0 & \cos\alpha+\sin\alpha & \sin\alpha-\cos\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

i.e., $(\lambda+1)\sin\alpha\,(\sin\alpha - \cos\alpha) - (1-\lambda)\cos\alpha\,(\cos\alpha + \sin\alpha) = 0$

$$\lambda(\sin^2\alpha + \cos^2\alpha) + \sin^2\alpha - \cos^2\alpha - 2\sin\alpha\cos\alpha = 0$$

$$\lambda = \sin 2\alpha + \cos 2\alpha = \sqrt{2}\sin\left(2\alpha + \frac{\pi}{4}\right)$$

$$-1 \le \frac{\lambda}{\sqrt{2}} \le 1$$

$$-\sqrt{2} \le \lambda \le \sqrt{2}$$

for $\lambda = 1$,

$$\sin\left(2\alpha + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

$$\therefore \quad 2\alpha + \frac{\pi}{4} = \frac{\pi}{4}$$

General solution: $2\alpha + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4}$

$$2\alpha = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$$

If n is even, $2\alpha = n\pi$

odd, $2\alpha = n\pi - \frac{\pi}{2}$

PROFICIENCY TEST

The following questions deal with the basic concepts of this section. Answer the following briefly. Do not consult the Study Material while attempting the questions.

1. Evaluate the determinant

$$\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

2. If $\begin{vmatrix} x & 3 & 6 \\ 3 & 6 & x \\ 6 & x & 3 \end{vmatrix} = \begin{vmatrix} 2 & x & 7 \\ x & 7 & 2 \\ 7 & 2 & x \end{vmatrix} = \begin{vmatrix} 4 & 5 & x \\ 5 & x & 3 \\ x & 4 & 5 \end{vmatrix} = 0$

Then what is the value of x?

3. If $\Delta = \begin{vmatrix} my+nz & mq+nr & mb+nc \\ kz-mx & kr-mp & kc-ma \\ mx+ky & np+kq & na+kb \end{vmatrix}$

and Δ is product of two determinants one of which is $\begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$, then find the other one.

Also show that $\Delta = 0$.

4. If the equation $x = ay + z$, $y = az + x$ and $z = ax + y$ are the consistent having non-trivial solution, then prove that $a^3 + 3a = 0$.

5. If $f(x) = \begin{vmatrix} x^2-x & x^3 & x^4-1 \\ 2x-1 & 3x^2 & 4x^3 \\ 2 & 6x & 12x^2 \end{vmatrix}$, then find the coefficient of x in $f(x)$.

6. If the system of equations $3x + 10y + 17z = 0$, $x + 6y + 13z = 0$ and $20x - 13y + \lambda z = 0$ has a non-trivial solution then find the solution.

ANSWERS TO PROFICIENCY TEST

1. zero

2. $x = -9$

5. 6

6. $x = k$, $y = -2k$, $z = k$

SOLVED OBJECTIVE EXAMPLES

Example 1:

If $f(x)=\begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$**, then** $f(2x) - f(x)$ **equals**

(a) $a(2a + 3x)$ **(b)** $ax(2x + 3a)$

(c) $ax(2a + 3x)$ **(d)** $x(2a + 3x)$

Solution:

Applying $R_2 \to R_2 - xR_1$ and $R_3 \to R_3 - xR_2$, we get

$$f(x)=\begin{vmatrix} a & -1 & 0 \\ 0 & (a+x) & -1 \\ 0 & 0 & a+x \end{vmatrix} = a(a+x)^2$$

$\therefore$ $f(2x) - f(x) = a(a + 2x)^2 - a(a + x)^2 = ax(2a + 3x)$

Hence (c) is the correct answer.

Example 2:

The largest value of a third order determinant whose elements are 0 or 1, is

(a) 0 **(b)** 0

(c) 2 **(d)** 3

Solution:

Let $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant of order 3.

Then, $\Delta = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$

$= (a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)$

Since each element of Δ is either 1 or 0, therefore the value of the determinant cannot exceed 3. Clearly, the value of Δ is maximum when the value of each term in first bracket is 1 and the value of each term in the second bracket is zero. But $a_1b_2c_3 = a_3b_1c_2 = a_2b_3c_1 = 1$ implies that every element of the determinant Δ is 1 and in that case $\Delta = 0$. Thus, we may have

$$\Delta=\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2.$$

Hence (c) is the correct answer.

Example 3:

If $A + B + C = \pi$**, then find the value of** $\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$

(a) 0 **(b)** π

(c) $-\pi$ **(d)** none of these

Solution:

Above is skew symmetric determinant of odd order because $\cos(A+B) = -\cos C$

Hence (a) is the correct answer.

Example 4:

If $\begin{vmatrix} x^n & x^{n+2} & x^{n+3} \\ y^n & y^{n+2} & y^{n+3} \\ z^n & z^{n+2} & z^{n+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$**, then** n **equals**

(a) 1 **(b)** –1 **(c)** 2 **(d)** –2

Solution:

The degree of the determinant is $n + (n + 2) + (n + 3) = 3n + 5$ and the degree of the expression on RHS is 2.

$\therefore \quad 3n + 5 = 2 \Rightarrow n = -1$.

Hence (b) is the correct answer.

{Equating degree alone does not sufficient to evaluate n, the determinant might not be equal however since all options have values we conclude for $n = -1$.}

Example 5:

If α, β, γ **are the roots** $x^3 + ax^2 + b = 0$**, then the value of** $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ **is**

(a) $-a^3$ **(b)** $a^3 - 3b$ **(c)** a^3 **(d)** $a^2 - 3b$

Solution:

Since α, β, γ are the roots of the given equation, therefore $\alpha + \beta + \gamma = -a$, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$ and $\alpha\beta\gamma = -b$. [By theory of equations]

Now, $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = -(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$

$= -(\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma)^2 - 3(\alpha\beta+\beta\gamma+\gamma\alpha)\} = -(-a)\{(a^2 - 0)\} = a^3$.

Hence (c) is the correct answer.

Example 6:

If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x+1) & x(x-1) \end{vmatrix}$**, then** $f(100)$ **is equal to**

(a) 0 **(b)** 1

(c) 100 **(d)** –100

Solution:

We have,

$$f(x)=\begin{vmatrix}1 & x & x+1\\ 2x & x(x-1) & (x+1)x\\ 3x(x-1) & x(x-1)(x+1) & x(x-1)\end{vmatrix}$$

Applying $C_3 \to C_3 - C_2$ we get

$$f(x)=\begin{vmatrix}1 & x & 1\\ 2x & x(x-1) & 2x\\ 3x(x-1) & x(x-1)(x-2) & 3x(x-1)\end{vmatrix}=0$$

∴ $f(100) = 0$.

Hence (a) is the correct answer.

Example 7:

If $f(\theta)=\begin{vmatrix}1 & 1 & -1\\ 1 & e^{i\theta} & 1\\ 1 & -1 & -e^{-i\theta}\end{vmatrix}$, then

(a) $\int_{-\pi/2}^{\pi/2} f(\theta)\,d\theta = 2\int_{0}^{\pi/2} f(\theta)\,d\theta$ **(b) $f(\theta)$ is purely imaginary**

(c) $f(\pi/2) = 0$ **(d) none of these**

Solution:

On operating [$R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$]

$$f(\theta)=\begin{vmatrix}0 & 1-e^{i\theta} & -2\\ 1 & e^{i\theta} & 1\\ 0 & -1-e^{-i\theta} & -1-e^{-i\theta}\end{vmatrix}$$

$= (-1)\,[(1 - e^{i\theta})(-1 - e^{-i\theta}) - 2(-1 - e^{-i\theta})] = 2(1 + \cos\theta)$

Now $f(-\theta)=f(\theta)\Rightarrow f(\theta)$ is an even function.

Hence (a) is the correct answer.

Example 8:

Let $ax^7 + bx^6 + cx^5 + ex^3 + fx^2 + gx + h = \begin{vmatrix}(x+1) & (x^2+2) & (x^2+x)\\ (x^2+x) & (x+1) & (x^2+x)\\ (x^2+2) & (x^2+x) & x+1\end{vmatrix}$. Then

(a) $g = 3$ and $h = -5$ **(b) $g = -3$ and $h = -5$**

(c) $g = -3$ and $h = -9$ **(d) none of these**

Solution:

By putting $x = 0$ on both sides of the equation we have $h=\begin{vmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 2 & 0 & 1\end{vmatrix}=9$

Differentiating both sides and then putting $x = 0$, we get $g = -5$.

Hence (d) is the correct answer.

Example 9:

$$\Delta = \begin{vmatrix} p & 2-i & i+1 \\ 2+i & q & 3+i \\ 1-i & 3-i & r \end{vmatrix}$$ **is always**

(a) real **(b) imaginary** **(c) zero** **(d) none of these**

Solution:

Since $\Delta = \overline{\Delta}$ $\therefore \Delta$ is real only

Hence (a) is the correct answer.

Example 10:

If a, b and c are pth, qth and rth terms of an H.P., then $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} =$

(a) term containing a, b, c, p, q, r **(b) a constant**

(c) zero **(d) none of these**

Solution:

If A is the first term and d is the common difference of the corresponding A.P. then

$$\frac{1}{a} = A + (p-1)d \ ; \ \frac{1}{b} = A + (q-1)d \ ; \ \frac{1}{c} = A + (r-1)\,d$$

Now $$\Delta = abc \begin{vmatrix} 1/a & 1/b & 1/c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$ Operating $R_1 \to R_1 - d(R_2 - R_3) - A$

$$\Delta = abc \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Hence (c) is the correct answer.

SOLVED SUBJECTIVE EXAMPLES

Example 1:

Solve for *x*:

$$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0 .$$

Solution:

Let $\Delta = \begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix}$

Subtracting twice the first row from the third row, we have

$$\Delta = \begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 1 & -4 & 0 \end{vmatrix}$$

Now subtracting $\frac{3}{2}$ times the first row from the second row, we have

$$\Delta = \begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 2 & 0 & -\frac{3}{2} \\ 1 & -4 & 0 \end{vmatrix}$$

Now adding 4 times the first column to the second column, we have

$$\Delta = \begin{vmatrix} 4x & 22x+2 & 8x+1 \\ 2 & 8 & -\frac{3}{2} \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along the third row, we have

$$\Delta = -\frac{3}{2}(22x + 2) - 8(8x + 1)$$

$$= -33x - 3 - 64x - 8 = -97x - 11$$

The given equation now becomes $-97x - 11 = 0$ or $x = -\frac{11}{97}$

Example 2:

Show that

$$\begin{vmatrix} {}^mC_r & {}^mC_{r+1} & {}^mC_{r+2} \\ {}^nC_r & {}^nC_{r+1} & {}^nC_{r+2} \\ {}^pC_r & {}^pC_{r+1} & {}^pC_{r+2} \end{vmatrix} = \begin{vmatrix} {}^mC_r & {}^{m+1}C_{r+1} & {}^{m+2}C_{r+2} \\ {}^nC_r & {}^{n+1}C_{r+1} & {}^{n+2}C_{r+2} \\ {}^pC_r & {}^{p+1}C_{r+1} & {}^{p+2}C_{r+2} \end{vmatrix} .$$

Solution:

Let $\Delta = \begin{vmatrix} {}^mC_r & {}^mC_{r+1} & {}^mC_{r+2} \\ {}^nC_r & {}^nC_{r+1} & {}^nC_{r+2} \\ {}^pC_r & {}^pC_{r+1} & {}^pC_{r+2} \end{vmatrix}$

Adding the second column to the third, and then the first column to the second, and using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, we have $\Delta = \begin{vmatrix} {}^{m}C_r & {}^{m+1}C_{r+1} & {}^{m+1}C_{r+2} \\ {}^{n}C_r & {}^{n+1}C_{r+1} & {}^{n+1}C_{r+2} \\ {}^{p}C_r & {}^{p+1}C_{r+1} & {}^{p+1}C_{r+2} \end{vmatrix}$

Now adding the second column to the third, and again utilizing

$^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$,

we have $\Delta = \begin{vmatrix} {}^{m}C_r & {}^{m+1}C_{r+1} & {}^{m+2}C_{r+2} \\ {}^{n}C_r & {}^{n+1}C_{r+1} & {}^{n+2}C_{r+2} \\ {}^{p}C_r & {}^{p+1}C_{r+1} & {}^{p+2}C_{r+2} \end{vmatrix}$

Example 3:

Show that

$$\begin{vmatrix} bc & bc'+b'c & b'c' \\ ca & ca'+c'a & c'a' \\ ab & ab'+a'b & a'b' \end{vmatrix} = (ab'-a'b)(bc'-b'c)(ca'-c'a)$$

***Solution*:**

Let $\Delta = \begin{vmatrix} bc & bc'+b'c & b'c' \\ ca & ca'+c'a & c'a' \\ ab & ab'+a'b & a'b' \end{vmatrix}$

Then $\Delta = \frac{1}{abc}\begin{vmatrix} abc & a(bc'+b'c) & ab'c' \\ abc & b(ca'+c'a) & bc'a' \\ abc & c(ab'+a'b) & ca'b' \end{vmatrix}$

$= \frac{abc}{abc}\begin{vmatrix} 1 & a(bc'+b'c) & ab'c' \\ 1 & b(ca'+c'a) & bc'a' \\ 1 & c(ab'+a'b) & ca'b' \end{vmatrix}$

Now subtracting the second row from the first and then the third row from the second, we have

$\Delta = \begin{vmatrix} 0 & c(ab'-a'b) & c'(ab'-a'b) \\ 0 & a(bc'-b'c) & a'(bc'-b'c) \\ 1 & c(ab'+a'b) & ca'b' \end{vmatrix}$

$= (ab'-a'b)(bc'-b'c)\begin{vmatrix} 0 & c & c' \\ 0 & a & a' \\ 1 & c(ab'+a'b) & ca'b' \end{vmatrix}$

$= (ab'-a'b)(bc'-b'c)(ca'-c'a)$

Example 4:

If $a \neq p$; $b \neq q$; $c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then

find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$.

***Solution*:**

The determinant is

$p(qr-bc) + b(ca-ar) + c(ab-aq) = 0$

$\therefore$ $pqr + 2abc - pbc - qca - rab = 0$

add the following to both sides

$2pqr - 2abc + 2pbc + 2qca + 2rab - 2aqr - 2bpr - 2cpq$

Then

LHS $= 3pqr + pbc + qca + rab - 2aqr - 2bpr - 2cpq$

$= p(q-b)(r-c) + q(p-a)(r-c) + r(p-a)(q-b)$

RHS $= 2(p-a)(q-b)(r-c)$

$\therefore$ $p(q-b)(r-c) + q(p-a)(r-c) + r(p-q)(q-b) = 2(p-a)(q-b)(r-c)$

Divide by $(p-a)(q-b)(r-c)$

$$\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2.$$

$\therefore$ the value of the required expression is 2.

Example 5:

Consider a determinant of order three whose all elements are 1 or –1, prove that value of determinant will always be a multiple of four.

Solution:

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ be a determinant of order 3.

$= \begin{vmatrix} a_1-b_1 & a_2-b_2 & a_3-b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ Applying $R_1 \to R_1 - R_2$

$= (a_1-b_1)\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - (a_2-b_2)\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + (a_3-b_3)\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

now each of (a_1-b_1), (a_2-b_2) and (a_3-b_3) will either be –2, 0 or 2 and so will be determinant of order two.

$\Rightarrow$ Δ will either be 4, 0 or –4.

Example 6:

Prove the identity $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ **= 2*abc*(*a* + *b* + *c*)^3**

Solution:

It is easy to show that the determinant vanishes when $a = 0$, $b = 0$, $c = 0$; hence abc is a factor of the determinant.

Again, subtracting the second column from the first, and then the third from the second, the

determinant = $\begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ b^2-(c+a)^2 & (c+a)^2-b^2 & b^2 \\ 0 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$

Here both the first and second columns contain (a + b + c) as a factor. Hence $(a+b+c)^2$ must be a factor of the determinant, and since it is of six dimensions, theremaining factor must be of the form $N(a+b+c)$, because of cyclic symmetry where$N$ is a mere number. Thus the given determinant must be equal to $Nabc\,(a+b+c)^3$.

By putting $a = b = c = 1$, we have $\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix}$ = 27N

i.e., 27N = $\begin{vmatrix} 3 & -3 & 0 \\ 0 & 3 & -3 \\ 1 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & -3 & 0 \\ 0 & 0 & -3 \\ 1 & 5 & 4 \end{vmatrix}$ = 3(15 + 3) = 54.

$\therefore$ we have $N = 2$

Thus the given determinant = $2abc\,(a+b+c)^3$.

Example 7:

If D = $\begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$**, then show that** $\left(\frac{D}{(n!)^3}-4\right)$ **is divisible by *n*.**

Solution:

$$\frac{D}{(n!)^3} = \begin{vmatrix} 1 & n+1 & (n+1)(n+2) \\ n+1 & (n+1)(n+2) & (n+1)(n+2)(n+3) \\ (n+1)(n+2) & (n+1)(n+2)(n+3) & (n+1)(n+2)(n+3)(n+4) \end{vmatrix}$$

$$= (n+1)(n+1)(n+2)\begin{vmatrix} 1 & 1 & 1 \\ n+1 & n+2 & n+3 \\ (n+1)(n+2) & (n+2)(n+3) & (n+3)(n+4) \end{vmatrix}$$

$$= (n+1)^2 (n+2)\begin{vmatrix} 1 & 0 & 0 \\ n+1 & 1 & 2 \\ (n+1)(n+2) & 2n+4 & 4n+10 \end{vmatrix}$$

$$= (n+1)^2 (n+2) \cdot 2 = (n^2+2n+1)(n+2)\cdot 2 = 2n(n^2+2n+1) + 4n(n+2) + 4$$

$$\therefore \frac{D}{(n!)^3} - 4 = 2n(n^2+2n+1) + 4n(n+2)$$

and the R.H.S. contains n as a factor. Hence the problem.

***Example* 8:**

Let $A_k = \begin{vmatrix} k-1 & n & 6 \\ (k-1)^2 & 2n^2 & 4n-2 \\ (k-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$, show that $\sum_{k=1}^{n} A_k = 0$.

***Solution*:**

$$\sum_{k=1}^{n} A_k = \begin{vmatrix} \sum_{k=1}^{n}(k-1) & n & 6 \\ \sum_{k=1}^{n}(k-1)^2 & 2n^2 & 4n-2 \\ \sum_{k=1}^{n}(k-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

Now $\sum_{k=1}^{n}(k-1) = 0+1+2+\ldots+(n-1) = \frac{(n-1)n}{2}$

$$\sum_{k=1}^{n}(k-1)^2 = 0^2+1^2+\ldots+(n-1)^2 = \frac{(n-1)n(2n-1)}{6}$$

$$\sum_{k=1}^{n}(k-1)^3 = 0^3+1^3+\ldots+(n-1)^3 = \frac{(n-1)^2 n^2}{4}$$

$$\sum_{k=1}^{n} A_k = \begin{vmatrix} \frac{(n-1)n}{2} & n & 6 \\ \frac{(n-1)n(2n-1)}{6} & 2n^2 & 4n-2 \\ \frac{(n-1)^2 n^2}{4} & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n(n-1)}{12}\begin{vmatrix} 6 & n & 6 \\ 2(2n-1) & 2n^2 & 2(2n-1) \\ 3n(n-1) & 3n^3 & 3n(n-1) \end{vmatrix}$$

$= 0.$

***Example* 9:**

Let three digit numbers $A28$, $3B9$, $62C$ where A, B, C are integers between 0 and 9, be divisible by a fixed integer k. Show that the determinant

$\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k.

Solution:

Given condition $\Rightarrow \left.\begin{array}{l} A28 = A \times 100 + 2 \times 10 + 8 \times 1 = mk \\ 3B9 = 3 \times 100 + B \times 10 + 9 \times 1 = nk \\ 62C = 6 \times 100 + 2 \times 10 + C \times 1 = pk \end{array}\right\}$ … (i)

where m, n, p are some integers.

$$D = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$$ Apply R_2: $R_2 + R_3 \times 10 + R_1 \times 100$ and using (i)

$$= \begin{vmatrix} A & 3 & 6 \\ mk & nk & pk \\ 2 & B & 2 \end{vmatrix} = k \begin{vmatrix} A & 3 & 6 \\ m & n & p \\ 2 & B & 2 \end{vmatrix} \Rightarrow$$ D is divisible by k.

Example 10:

Prove that $\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = a_1a_2a_3\left(1 + \dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3}\right)$.

Solution:

$$\Delta = a_1a_2a_3 \begin{vmatrix} \frac{1}{a_1}+1 & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{a_1} & \frac{1}{a_2}+1 & \frac{1}{a_3} \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3}+1 \end{vmatrix}$$ by $C_1 : C_1 \to \frac{1}{a_1} C_1$ etc.

$$= a_1a_2a_3 \begin{vmatrix} \frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1 & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1 & \frac{1}{a_2}+1 & \frac{1}{a_3} \\ \frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1 & \frac{1}{a_2} & \frac{1}{a_3}+1 \end{vmatrix}$$ by $C_1 : C_1 + C_2 + C_3$

$$= a_1 a_2 a_3 \left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1\right) \begin{vmatrix} 1 & \frac{1}{a_2} & \frac{1}{a_3} \\ 1 & \frac{1}{a_2}+1 & \frac{1}{a_3} \\ 1 & \frac{1}{a_2} & \frac{1}{a_3}+1 \end{vmatrix}$$ by taking out a common factor from C_1

$$= a_1a_2a_3\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1\right) \begin{vmatrix} 1 & \frac{1}{a_2} & \frac{1}{a_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ by R_2: $R_2 - R_1$ and R_3: $R_3 - R_1$

$$= a_1a_2a_3\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+1\right)$$

MIND MAP

DETERMINANTS

DETERMINANT OF ORDER THREE

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

where a_{ij} means element of ith row and jth column.

and $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$.

MINOR

Minor of a_{ij} is the determinant of elements which is obtained after deleting ith row and jth column.

COFACTOR

Cofactor of a_{ij} is the product of minor of a_{ij} with $(-1)^{i+j}$

PROPERTIES OF DETERMINANTS

- If two rows/columns in a determinant are interchanged, the sign of the determinant changes
- If the numbers in one row/column are added, m times the number in another row/column, the value of the determinant remains unaltered.
- If rows and columns are interchanged, the value of the determinant remains unaltered.
- If all the numbers in any row/column are zeros, the value of the determinant is zero.
- If two rows/columns are identical, the value of the determinant is zero.
- If the elements of a row/column are multiplied by any number *m*, the determinant is multiplied by *m*.

SPECIAL DETERMINANTS

- **Symmetric determinants:** The elements situated at equal distances from the diagonal are equal both in magnitude and sign, i.e. $a_{ij} = a_{ji}$

e.g. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

- **Skew Symmetric determinants:** The diagonal elements are zero and the elements situated at equal distances from the diagonal are equal in magnitude but opposite in sign. The value of a skew symmetric determinant of odd order is zero.

SUM OF DETERMINANTS

- If each of the elements of a row is expressed as the sum of two numbers, then the determinant may be written as the sum of two determinants.

$$\begin{vmatrix} a_1+a_3 & b_1+b_3 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 \\ a_2 & b_2 \end{vmatrix}$$

PRODUCT OF DETERMINANTS

- Two determinants can be multiplied together by the following rule.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{vmatrix}$$

SYSTEM OF LINEAR EQUATIONS (CRAMERS RULE)

- Consider the system of *n* linear equations in *n* unknowns given by

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{vmatrix}$

Let D_j be the determinant obtained from D after replacing the j^{th} column by constants.

Then, if $D \neq 0$, we have

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \ldots\ldots, x_n = \frac{D_n}{D}.$$

DIFFERENTIATION OF DETERMINANTS

- $F(x) = \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix}$ then $F'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) & h_1'(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2'(x) & g_2'(x) & h_2'(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3'(x) & g_3'(x) & h_3'(x) \end{vmatrix}$

EXERCISE – I

CBSE PROBLEMS

1. If $x=-9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}=0$, find other roots.

2. Without expanding show that the value of each determinant is zero.

(i) $\begin{vmatrix} 9 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix}$ (ii) $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$

3. Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}=2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

4. Find the value of the determinant $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$.

5. Show that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}=4a^2b^2c^2$.

6. Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}=abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)$.

7. Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}=2(a+b+c)^3$.

8. Show that $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}=(a^3-1)^2$.

9. Show that $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}=x^3$.

10. Show that $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix} = (x-y)(y-z)(z-x)$.

11. Using Cramer's rule, solve the following system of equations:

$x+y+z=1,\ ax+by+cz=p,\ a^2x+b^2y+c^2z=p^2$.

12. Solve the following equations, using Cramer's rule.

$x-y+3z=6$

$x+3y-3z=-4$

$5x+3y+3z=10$.

13. The sum of three numbers is 6. If we multiply third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number, we get 12, use determinant to find the numbers.

14. Given that $x=cy+bz,\ y=az+cx,\ z=bx+ay$; where *x, y, z* are not all zero, show that $a^2+b^2+c^2+2abc=1$.

15. If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ be the vertices of an equilateral triangle whose each side is equal to *k*, then show that $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = 3k^4$.

EXERCISE – II

IIT-SINGLE CHOICE CORRECT

1. The number of roots of $\begin{vmatrix} 1+x & 2 & 3 \\ 1 & 2+x & 3 \\ 1 & 2 & 3+x \end{vmatrix} = 0$ is

(a) one
(b) two
(c) three
(d) more than three

2. The roots of $\begin{vmatrix} 2x+4 & 3+4x & 11 \\ 16 & 12+2x & 4x+2 \\ 7 & 6 & 2x \end{vmatrix} = 0$ are

(a) $\left(\frac{9}{2}, 1, \frac{7}{2}\right)$
(b) $\left(\frac{9}{2}, 1, -\frac{7}{2}\right)$
(c) $\left(-\frac{9}{2}, 1, \frac{7}{2}\right)$
(d) $\left(\frac{9}{2}, -1, \frac{7}{2}\right)$

3. The determinant $\Delta = \begin{vmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & \cos 2\beta \\ \sin\alpha & \cos\alpha & \sin\beta \\ -\cos\alpha & \sin\alpha & \cos\beta \end{vmatrix}$ is independent of

(a) α
(b) β
(c) α and β
(d) neither α nor β

4. The value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ {}^{m}C_1 & {}^{(m+1)}C_1 & {}^{(m+2)}C_1 \\ {}^{m}C_2 & {}^{(m+1)}C_2 & {}^{(m+2)}C_2 \end{vmatrix}$ is equal to

(a) 1
(b) –1
(c) 0
(d) none of these

5. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then

(a) $x = 3, y = 1$
(b) $x = 1, y = 3$
(c) $x = 0, y = 3$
(d) $x = 0, y = 0$

6. If the system of equations $x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$ has a non-zero solution, then the possible values of k are

(a) – 1, 2
(b) 1, 2
(c) 0, 1
(d) – 1, 1

7. Let $f(x) = x(x-1)$, then $\Delta = \begin{vmatrix} f(0) & f(1) & f(2) \\ f(1) & f(2) & f(3) \\ f(2) & f(3) & f(4) \end{vmatrix}$ is equal to

(a) $-2!$ (b) $-3!-2!$
(c) 0 (d) none of these

8. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, then $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$ is equal to

(a) 0 (b) *abc*
(c) –*abc* (d) none of these

9. If $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $\begin{vmatrix} 2a_1+b_1+c_1 & b_1 & c_1 \\ 2a_2+b_2+c_2 & b_2 & c_2 \\ 2a_3+b_3+c_3 & b_3 & c_3 \end{vmatrix}$ is equal to

(a) Δ (b) 2Δ
(c) $\frac{1}{2}\Delta$ (d) none of these

10. If in a ΔABC, $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix}=0$, then the value of $\sin^2 A+\sin^2 B+\sin^2 C$ is

(a) $\frac{9}{4}$ (b) $\frac{4}{9}$
(c) $\frac{3\sqrt{3}}{2}$ (d) 1

11. If $\Delta=\begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$, then a factor of Δ is

(a) $x^2-(a+b)x+a^2+b^2-ab$ (b) $x^2-(a-b)x+a^2+b^2+ab$
(c) $x^2+(a+b)x+a^2+b^2-ab$ (d) none of these

12. If $\alpha, \beta, \gamma \in R$, then $\Delta=\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$ is equal to

(a) –1 (b) $\cos\alpha\cos\beta\cos\gamma$
(c) $\cos\alpha+\cos\beta+\cos\gamma$ (d) none of these

13. If $\begin{vmatrix} x^n & x^{n+2} & x^{n+3} \\ y^n & y^{n+2} & y^{n+3} \\ z^n & z^{n+2} & z^{n+3} \end{vmatrix} = (x-y)(y-z)(z-x)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, then n equals

(a) 1 (b) –1
(c) 2 (d) –2

14. If A, B and C are the angles of a Δ and $\Delta = \begin{vmatrix} e^{-2iA} & e^{ic} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$, then

(a) $\Delta = -4$ (b) $\Delta = -3$
(c) $\Delta = -2$ (d) none of these

15. Let $a, b, c \in R^+$, then $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ has

(a) unique solution (b) no solution
(c) infinitely many solution (d) finite number of solution

16. If the system of equation $x+2y-3z=1$, $(a+2)z=3$, $(2a+1)y+z=2$ is inconsistent, then the value of a is equal to

(a) 1 (b) 0
(c) –2 (d) $-\frac{1}{2}$

17. If $a > 0$ and discriminant of $ax^2+2bx+c$ is negative, then $\begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$ is

(a) positive (b) negative
(c) 0 (d) $(ac-b)^2(ax^2+2bx+c)$

18. If $f(x)=\begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & 1 \end{vmatrix}$, then $\lim_{x\to 0}\frac{f(x)}{x^2}$ is

(a) 3 (b) –1
(c) 0 (d) 1

19. Let $\{\Delta_1, \Delta_2, ...\Delta_k\}$ be the set of the third order determinants that can be made with the distinct non-zero real numbers $a_1, a_2, ...a_9$ (one or more entries may repeated), then

(a) $k = {}^9C_2$ (b) $k = {}^9P_2$
(c) at least one $\Delta_i = 0$ (d) none of these

20. If x, y, z are integers in A.P., lying 1 and 9, $x51$, $y41$ and $z31$ are three digit numbers, then the value of $\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$ is

(a) $x+y+z$ (b) $x-y+z$
(c) 0 (d) none of these

21. Consider the system of equations

$a_1x+b_1y+c_1z=0$; $a_2x+b_2y+c_2z=0$; $a_3x+b_3y+c_3=0$. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ = 0, then the system has

(a) more than two solutions (b) non-trivial and one trivial solutions
(c) no solution (d) only trivial solution (0, 0, 0)

22. If $a^2+b^2+c^2=-2$ and $f(x)=\begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$, then $f(x)$ is a polynomial of degree

(a) 1 (b) 0
(c) 3 (d) 2

23. The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix}=0$ and $xp^2+2yp+z$ has no real root, then

(a) x, y, z are in A.P. (b) x, y, z are in G.P.
(c) x, y, z are in H.P. (d) xy, yz, zx are in A.P.

24. If *f* (*x*) is a polynomial satisfying $f(x)=\frac{1}{2}\begin{vmatrix} f(x) & f\left(\frac{1}{x}\right)-f(x) \\ 1 & f\left(\frac{1}{x}\right) \end{vmatrix}$ and *f* (2) = 17, then *f* (5) =

(a) 126 (b) 626
(c) 124 (d) 624

25. If T_p, T_q, T_r are the p^{th}, q^{th} and r^{th} terms of an A. P., then $\begin{vmatrix} T_p & T_q & T_r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ is equal to

(a) 1 (b) –1
(c) 0 (d) $p+q+r$

EXERCISE – III

IIT-JEE – SINGLE CHOICE CORRECT

1. If $a+b+c=0$, one root of $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}=0$ is

(a) $x = 1$ (b) $x = 2$

(c) $x = 2a^2+3b^2+4c^2$ (d) $x = 0$

2. If α, β and γ are the roots of the equation $x^3+px+q=0$, then determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is equal to

(a) p (b) q

(c) p^2-2q (d) none of these

3. If $f(x)=\begin{vmatrix} 2\cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$, then $f'\left(\frac{\pi}{3}\right)=$

(a) $-\sqrt{3}$ (b) -4

(c) -3 (d) $-\sqrt{2}$

4. The value of det(A) where $A=\begin{bmatrix} 1 & \sin\theta & 0 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$ lies in the interval

(a) [1, 2] (b) [0, 2]

(c) (1, 2) (d) none of these

5. If $S_r=\begin{vmatrix} 2r & x & n(n+1) \\ 6r^2-1 & y & n^2(2n+3) \\ 4r^3-2nr & z & n^3(n+1) \end{vmatrix}$, then the value $\sum_{r=1}^{n} S_r$ is independent of

(a) x only (b) y only

(c) x, y, z and n (d) n only

6. If p, q, r are negative and distinct, then the determinant $\Delta=\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix}$ is

(a) < 0 (b) ≤ 0

(c) 0 (d) > 0

7. If $A + B + C = \pi$, then the value of $\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$ is

(a) 0 (b) 1
(c) 2 sin$B$ tan$A$ cos$C$ (d) none of these

8. If the capital letters denote the cofactors of the corresponding small letters in the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then the value of $\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

(a) Δ (b) Δ^2
(c) 2Δ (d) 0

9. If [x] denotes the greatest integer less than or equal to x, then for $-1 \le x < 0$, $0 \le y < 1$, $1 \le z < 2$, the value of the determinant $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ is

(a) [x] (b) [y]
(c) [z] (d) [x + 1]

10. If $a_1 a_2$ …. form a G. P. and $a_i > 0$ for all $i \ge 1$, then $\Delta = \begin{vmatrix} \log a_m & \log a_{m+1} & \log a_{m+2} \\ \log a_{m+3} & \log a_{m+4} & \log a_{m+5} \\ \log a_{m+6} & \log a_{m+7} & \log a_{m+8} \end{vmatrix}$ is

(a) $\log a_{m+8} - \log a_m$ (b) $\log a_{m+8} + \log a_m$
(c) zero (d) $\log^2 a_{m+4}$

11. Given $a_i^2 + b_i^2 + c_i^2 = 1$ $(i = 1, 2, 3)$ and $a_i a_j + b_i b_j + c_i c_j = 0$ $(i \ne j \;\; i, j = 1, 2, 3)$, then the value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is

(a) 0 (b) $\frac{1}{2}$
(c) 1 (d) 2

12. If $\begin{vmatrix} e^x & \sin x \\ \cos x & \log_e(1+x^2) \end{vmatrix} = A + Bx + Cx^2 + \ldots\ldots$, then A = …… and B = ……

(a) –1, 1 (b) 0, –1
(c) –2, 2 (d) 0, –2

13. $\Delta = \begin{vmatrix} x & 1 & 1 & \ldots\ldots\ldots \\ 1 & x & 1 & \ldots\ldots\ldots \\ 1 & 1 & x & \ldots\ldots\ldots \\ \ldots & \ldots. & \ldots. & \ldots. \end{vmatrix}_{n\times n}$ is equal to

(a) $(x-1)^{n-1}$ (b) $(x-1)^{n-1}(x+n)$
(c) $(x-1)^{n-1}(x+n-1)$ (d) none of these

14. The value of the determinant $\begin{vmatrix} \log_a\left(\frac{x}{y}\right) & \log_a\left(\frac{y}{z}\right) & \log_a\left(\frac{z}{x}\right) \\ \log_b\left(\frac{y}{z}\right) & \log_b\left(\frac{z}{x}\right) & \log_b\left(\frac{x}{y}\right) \\ \log_c\left(\frac{z}{x}\right) & \log_c\left(\frac{x}{y}\right) & \log_c\left(\frac{y}{z}\right) \end{vmatrix}$ is equal to

(a) 1 (b) –16
(c) $\log_z xyz$ (d) none of these

15. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is a constant. Then the value of $\frac{d^3}{dx^3}\{f(x)\}$ at $x = 0$

(a) p (b) $p + p^2$
(c) $p + p^3$ (d) independent of p

16. If the determinant $\begin{vmatrix} \cos 2x & \sin^2 x & \cos 4x \\ \sin^2 x & \cos 2x & \cos^2 x \\ \cos 4x & \cos^2 x & \cos 2x \end{vmatrix}$ is expanded in powers of sin x, then the constant term in the expansion is

(a) 1 (b) 2
(c) –1 (d) none of these

17. If a, b, c are in A.P. and $f(x) = \begin{vmatrix} x+a & x^2+1 & 1 \\ x+b & 2x^2-1 & 1 \\ x+c & 3x^2-2 & 1 \end{vmatrix}$, then $f'(x)$ is

(a) 0 (b) 1
(c) $a + bc$ (d) $\frac{abc}{a+b+c}$

18. If x_1, x_2 and y_1, y_2 are the roots of the equation $3x^2 - 18x + 9 = 0$ and $y^2 - 4y + 2 = 0$, then the value of the determinant $\begin{vmatrix} x_1x_2 & y_1y_2 & 1 \\ x_1 + x_2 & y_1 + y_2 & 2 \\ \sin(\pi x_1 x_2) & \cos\left(\frac{\pi}{2}x_1 x_2\right) & 1 \end{vmatrix}$ is

(a) 0 (b) 1
(c) c (d) none of these

19. If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 3, then $\begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$ is a polynomial of degree

(a) 0
(b) 1
(c) 2
(d) none of these

20. If $At^4 + Bt^3 + Ct^2 + Dt + E = \begin{vmatrix} t^2+3t & t-1 & t-3 \\ t+1 & 2-t & t-3 \\ t-3 & t+4 & 3t \end{vmatrix}$, then E equals

(a) 33
(b) –39
(c) 27
(d) 24

21. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$, then $f(100)$ is equal to

(a) 0
(b) 1
(c) 100
(d) –100

22. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(a) 0
(b) 2
(c) 1
(d) 3

23. Suppose x, y, z are positive and none of x, y, z is 1. If

$\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \sin(x+y) & -\cos(x+y) & \sin^2 z \end{vmatrix}$ then Δ is independent of

(a) x only
(b) x and y only
(c) y and z only
(d) x, y, and z

24. If $D_K = \begin{vmatrix} 2^{K-1} & \frac{1}{K(K+1)} & \sin K\theta \\ x & y & z \\ 2^n - 1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\frac{n}{2}\theta}{\sin\frac{\theta}{2}} \end{vmatrix}$, then $\sum_{K=1}^{n} D_K$ is equal to

(a) 0
(b) –1
(c) 2
(d) –2

25. If the system of equations $x = a(y + z)$, $y = b(z + x)$; $z = c(x + y)$ $(a, b, c \neq -1)$ has a non-trivial solution, then the value of $\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c}$ is

(a) 1
(b) 0
(c) –1
(d) 2

EXERCISE – IV

ONE OR MORE THAN ONE CHOICE CORRECT

1. The system of equations $ax + by = c$, $a'x + b'y = c'$

(a) has a unique solution if $ab' - a'b \neq 0$
(b) has no solution if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$
(c) has infinite solutions if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$
(d) none of these

2. If $x \in R, a_i, b_i, c_i \in R$ for $i = 1, 2, 3$ and $\begin{vmatrix} a_1 + b_1 x & a_1 x + b_1 & c_1 \\ a_2 + b_2 x & a_2 x + b_2 & c_2 \\ a_3 + b_3 x & a_3 x + b_3 & c_3 \end{vmatrix} = 0$, then

(a) $x = 1$
(b) $x = -1$
(c) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$
(d) none of the above

3. If $0 \le x \le 1$ and $f(x) = \begin{vmatrix} x & 1 & 1 \\ -1 & x & 1 \\ -1 & -1 & x \end{vmatrix}$, then

(a) least value of $f(x)$ is 0
(b) greatest value of $f(x)$ is 4
(c) $f(x)$ has local maxima at $x = \frac{2}{3}$
(d) $f(x)$ has local maxima at $x = \frac{1}{3}$

4. $f(x) = \begin{vmatrix} 2\cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$, then

(a) $\int_0^{\pi} f(x)\, dx = 0$
(b) maximum value of $f(x)$ is 4
(c) $\lim_{x \to \pi/2} f(x) = 0$
(d) $f'(0) = 0$

5. The determinant $\begin{vmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is

(a) non negative
(b) independent of θ
(c) independent of ϕ
(d) independent of θ and ϕ both

6. If $\Delta = \begin{vmatrix} e^x & \sin x & 1 \\ \cos x & \log_e(1 + x^2) & 1 \\ x & x^2 & 1 \end{vmatrix} = a + bx + cx^2$, then

(a) $a = 0$
(b) $a = 1$
(c) $b = -1$
(d) $b = -2$

7. The value of x for which $\begin{vmatrix} x & 2 & 2 \\ 3 & x & 2 \\ 3 & 3 & x \end{vmatrix} + \begin{vmatrix} 1-x & 2 & 4 \\ 2 & 4-x & 8 \\ 4 & 8 & 16-x \end{vmatrix} > 33$ is

(a) $0 < x < 1$ (b) $-\frac{1}{2} < x < \frac{1}{2}$ (c) $x < -\frac{1}{7}$ (d) $x > 1$

8. Let $f(n) = \begin{vmatrix} n & n+1 & n+2 \\ {}^{n}P_{n} & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^{n}C_{n} & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$ where the symbols have their usual meaning. The $f(n)$ is divisible by

(a) $n^2 + n + 1$ (b) $(n+1)!$ (c) $n!$ (d) none of these

9. If $f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$ a, b being positive integers, then

(a) constant term of $f(x)$ is 0 (b) coefficient of x in $f(x)$ is 0

(c) constant term in $f(x)$ is $a-b$ (d) constant term in $f(x)$ is $a+b$

10. If maximum and minimum values of the determinant $\begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$ are α and β, then

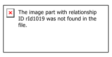
(a) $\alpha + \beta^{99} = 4$

(b) $\alpha^3 - \beta^{17} = 26$

(c) $(\alpha^{2n} - \beta^{2n})$ is always an even integer for $n \in N$

(d) a triangle can be constructed having its sides as $\alpha - \beta, \alpha + \beta$ and $\alpha + 3\beta$.

11. If $\Delta = \begin{vmatrix} x & 2y-z & -z \\ y & 2x-z & -z \\ y & 2y-z & 2x-2y-z \end{vmatrix}$, then

(a) $(x-y)$ is a factor of Δ (b)  is a factor of Δ

(c) $(x-y)^3$ is a factor of Δ (d) Δ is independent of z

12. Let $\Delta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$, then

(a) Δ is independent of θ (b) Δ is independent of ϕ

(c) Δ is a constant (d) $\left.\frac{d\Delta}{d\theta}\right]_{\theta=\pi/2} = 0$

13. Let $\Delta = \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a \\ a+2d & a & a+d \end{vmatrix}$, then

(a) Δ depends on *a* (b) Δ depends on *d* (c) Δ is a constant (d) all above

14. Let $a, b > 0$ and $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$, then

(a) $a+b-x$ is a factor of Δ

(b) $x^2+(a+b)x+a^2+b^2-ab$ is a factor of Δ

(c) $\Delta = 0$ has two real roots if $a = b$

(d) none of these

15. Let $f(x) = \begin{vmatrix} 1/x & \log x & x^n \\ 1 & -1/n & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$, then

(a) *n*th derivative of $f(x)$ at 1 is independent of *a*

(b) *n*th derivative of $f(x)$ at 1 is independent of *n*

(c) *n*th derivative of $f(x)$ at 1 depends upon *a* and *n*

(d) all above are correct

EXERCISE –V

MATCH THE FOLLOWING

Note: Each statement in column – I has one or more than one match in column - II

1.

Column I	Column II
I. If the system of equations $x + ay + az = 0$, $bx + y + bz = 0$ and $cx + cy + z = 0$ where a, b and c are non-zero non unity, has a non-trivial solution, then the value of $\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}$ is	**A.** 9
II. The roots of the equation $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$ are	**B.** –1
III. If the value of a third order determinant is 3, then the value of the square of the determinant formed by the cofactors will be	**C.** 4
IV. The value of the determinant $\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$ is equal to	**D.** 2

Note: Each statement in column – I has one or more than one match in column - II

2.

Column I	Column II
I. If $a \neq b \neq c$, then the value of x satisfying the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$ is	**A.** $\frac{1}{3}(a+b+c)$
II. The equation $\begin{vmatrix} x-a & x-b & x-c \\ x-b & x-c & x-a \\ x-c & x-a & x-b \end{vmatrix} = 0$, where a, b, c are different, is satisfied by	**B.** 0
III. If $a^{-1}+b^{-1}+c^{-1}=0$ such that $\frac{1}{abc}$ $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \lambda + 1$, then the value of λ is	**C.** $-(a + b + c)$
IV. If $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$, then x equals	**D.** –1

Note: Each statement in column – I has one or more than one match in column - II

3. The entries in a 3 × 3 determinant are either 1 or –1, then match the following:

Column I	Column II
I. Total number of such determinants are	**A.** 4
II. The number of determinants whose value is 6 are	**B.** 3
III. The maximum value of such a determinant is	**C.** 512
IV. The maximum value of trace of such determinants	**D.** zero

REASONING TYPE

Directions: Read the following questions and choose

(A) If both the statements are true and statement-2 is the correct explanation of **statement-1.**

(B) If both the statements are true but statement-2 is not the correct explanation of **statement-1.**

(C) If statement-1 is True and **statement-2** is False.

(D) If statement-1 is False and **statement-2** is True.

1. **Statement-1:** $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = 0$.

Statement-2: $\log_b a = \frac{\log a}{\log b}$ and $\Delta = 0$ if rows are identical.

(a) A (b) B (c) C (d) D

2. **Statement-1:** $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} = 0$.

Statement-2: A skew symmetric determinant of odd order is zero.

(a) A (b) B (c) C (d) D

3. **Statement-1:** If $\begin{vmatrix} {}^8C_3 & {}^9C_5 & {}^{10}C_7 \\ {}^8C_4 & {}^9C_6 & {}^{10}C_8 \\ {}^9C_n & {}^{10}C_{n+2} & {}^{11}C_{n+4} \end{vmatrix} = 0$, then $n = 3$.

Statement-2: ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

(a) A (b) B (c) C (d) D

4. **Statement-1:** If each element of a determinant of 3rd order with value A is multiplied by 3, then the value of newly formed determinant is $3A$.

Statement-2: If any row of a determinant has λ a factor with each element of that row then $\Delta' = \lambda A$.

(a) A (b) B (c) C (d) D

5. **Statement-1:** If $F(x), G(x)$ and $H(x)$ are three polynomials of degree 2, then $\phi(x) = \begin{vmatrix} F(x) & G(x) & H(x) \\ F'(x) & G'(x) & H'(x) \\ F''(x) & G''(x) & H''(x) \end{vmatrix}$ is a polynomial of degree 3.

Statement-2: If $y = f(x)$ is of degree 2, then $\frac{d^3y}{dx^3} = 0$.

(a) A (b) B (c) C (d) D

LINKED COMPREHENSION TYPE

Let the given equations be $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Case I: If $\Delta \neq 0$, then system of equations have unique solution.

Case II: If $\Delta = 0$, $\Delta_x = 0, \Delta_y = 0, \Delta_z = 0$, then system has infinite number of solutions.

Case III: If $\Delta = 0$ and at least one of Δ_x, Δ_y or Δ_z non zero, then system has no solution.

If system of equations have solution, then they are consistent, otherwise inconsistent. If in given equations $d_1 = d_2 = d_3 = 0$, then $\Delta = 0$ gives non trivial solution whereas $\Delta \neq 0$ gives trivial solution.

1. If the given system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$ had at least one solution for any real number λ, then the number of solution(s), if $\lambda = -5$ is (are)

(a) infinite solution (b) unique solution

(c) no solution (d) none of these

2. If the given equations $2x + 3y = 3$, $(c + 2)x + (c + 4)y = c + 6$ and $(c + 2)^2x + (c + 4)^2y = (c + 6)^2$ are consistent. Then the value(s) of c is (are)

(a) $c = 0$ (b) $c = -10$

(c) both (a) and (b) (d) none of these

3. If $a \neq b$, then the system of equations $ax + by + bz = 0$; $bx + ay + bz = 0$; $ax + by + az = 0$ will have a non-trivial solution if

(a) $a + b = 0$ (b) $a + 2b = 0$

(c) $2a + b = 0$ (d) $a + 4b = 0$

EXERCISE – VI

SUBJECTIVE PROBLEMS

1. Evaluate $\begin{vmatrix} 2 & a+b+c+d & ab+cd \\ a+b+c+d & 2(a+b)(c+d) & ab(c+d)+cd(a+b) \\ ab+cd & ab(c+d)+cd(a+b) & 2abcd \end{vmatrix}$.

2. Prove that for all values of θ

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0.$$

3. Show that $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a^2+b^2+c^2)(a+b+c)(a-b)(b-c)(c-a)$.

4. For what values of p and q the system of equation

$2x + py + 6z = 8$

$x + 2y + qz = 5$

$x + y + 3z = 4$

has (i) no solution (ii) a unique solution (iii) infinitely many solutions.

5. Prove that $\begin{vmatrix} \cos(\theta+\alpha) & \cos(\theta+\beta) & \cos(\theta+\gamma) \\ \sin(\theta+\alpha) & \sin(\theta+\beta) & \sin(\theta+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$ is independent of θ.

6. Prove that $1 - x^2$ is a factor of $\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix}$

7. If $\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$ where f, g, h are differentiable functions of x and primes denote derivatives, then show that $\frac{d\Delta}{dx} = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$.

8. Find all the values of t for which the system of equations

$(t-1)x + (3t+1)y + 2tz = 0$

$(t-1)x + (4t-2)y + (t+3)z = 0$

$2x + (3t+1)y + 3(t-1)z = 0$ has a non-trivial solution.

9. Let $a > 0$, $d > 0$, show that

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix} = \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

10. Let a, b, c be real numbers, such that $a^2 + b^2 + c^2 = 1$. Show that the equation

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$$ represents a straight line.

ANSWERS

EXERCISE – I

CBSE PROBLEMS

1. 2, 7

4. a^3

11. $x = \frac{(p-b)(c-p)}{(a-b)(c-a)},\ y = \frac{(a-p)(p-c)}{(a-b)(b-c)},\ z = \frac{(b-p)(p-a)}{(b-c)(c-a)}$

12. $x = 1-k,\ y = k;\ z = \frac{2k+5}{3},\ k \in R$

13. 3, 1, 2

EXERCISE – II

IIT-SINGLE CHOICE CORRECT

1. (c)	2. (c)	3. (a)	4. (a)	5. (d)
6. (d)	7. (b)	8. (b)	9. (b)	10. (a)
11. (c)	12. (d)	13. (b)	14. (a)	15. (a)
16. (d)	17. (a)	18. (b)	19. (c)	20. (c)
21. (b)	22. (d)	23. (b)	24. (b)	25. (c)

EXERCISE – III

IIT-JEE – SINGLE CHOICE CORRECT

1. (d)	2. (d)	3. (a)	4. (a)	5. (c)
6. (d)	7. (a)	8. (b)	9. (c)	10. (c)
11. (c)	12. (b)	13. (c)	14. (d)	15. (d)
16. (c)	17. (a)	18. (a)	19. (d)	20. (b)
21. (a)	22. (c)	23. (d)	24. (a)	25. (a)

EXERCISE – IV

ONE OR MORE THAN ONE CHOICE CORRECT

1. (a, b, c)	2. (a, b, c)	3. (a, b)	4. (a, b, c, d)	5. (a, b)
6. (a, c)	7. (c, d)	8. (a, c)	9. (a, b)	10. (a, b, c)
11. (a, b)	12. (b, d)	13. (a, b)	14. (a, b)	15. (a, b)

EXERCISE –V

MATCH THE FOLLOWING

1. I-[B], II-[B], [D], III-[A], IV-[C]

2. I-[B], II-[A], III-[B], IV-[B], [C]

3. I-[C], II-[D], III-[A], IV-[B]

REASONING TYPE

1. (a)	2. (a)	3. (d)	4. (d)	5. (d)

LINKED COMPREHENSION TYPE

1. (a)	2. (c)	3. (c)

EXERCISE – VI

SUBJECTIVE PROBLEMS

1. 0

4. (i) $p \neq 2$, $q = 3$

(ii) $p \neq 2$, $q \neq 3$

(iii) $p = 2$

8. $t = 0, 3$